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On submodules of monomial ideal

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Abstract

In this paper we introduce the notion of weakly 2*-absorbing primary submodules of modules and investigate this concept when the ambient module is the edge ideal of a finite simple graph without isolated vertices. Several structural properties concerning quotients, localization and related module constructions are established.*

Keywords . 2-absorbing submodule, 2-absorbing primary submodule, weakly 2-absorbing primary submodule, edge ideal of a graph.

1. Introduction. Throughout this paper, R denotes a commutative ring with identity and $I(G)$ denotes the edge ideal of a finite simple graph G without isolated vertices.

The theory of 2-absorbing ideals, introduced by Badawi [6], has motivated several important generalizations in commutative algebra. Among them are weakly 2-absorbing ideals [7], 2-absorbing primary ideals [8], and weakly 2-absorbing primary ideals, which naturally extend the notions of weakly prime and primary ideals.

These concepts have also been investigated in the setting of modules, leading to the notions of 2-absorbing submodules, weakly 2-absorbing submodules, and weakly 2-absorbing primary submodules. Such generalizations have proved useful for understanding how ideal-theoretic properties extend to module theory.

On the other hand, edge ideals of graphs form one of the central classes of monomial ideals in combinatorial commutative algebra. Their algebraic properties reflect important combinatorial features of graphs and have been extensively studied from several different perspectives.

The purpose of this paper is to investigate weakly 2-absorbing primary submodules when the ambient module is the edge ideal $I(G)$. Rather than considering arbitrary modules, we restrict our attention to this important class of monomial ideals and study how several structural properties of weakly 2-absorbing primary submodules behave in this context.

The paper is organized as follows. Section 2 recalls the basic definitions concerning edge ideals that will be used throughout the paper. In Section 3 we establish several properties of weakly 2-absorbing primary submodules of edge ideals and investigate their behavior under quotients, localization and related module-theoretic constructions.

Here we use properties of commutative algebra and homological algebra for the development of the results (see [5] and [9]).

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2. Preliminaries on edge ideals. Throughout this paper, G denotes a finite simple graph without isolated vertices and with vertex set

$$V(G) = \{v_1, \dots, v_s\}.$$

Let K be a field and let

$$R = K[v_1, \dots, v_s]$$

be the polynomial ring over K.

The edge ideal associated to G is the squarefree monomial ideal generated by the monomials corresponding to the edges of G.

Definition 2.1. *The* edge ideal *of G is*

$$I(G) = (v_i v_j \mid \{v_i, v_j\} \in E(G)),$$

where $E(G)$ denotes the edge set of G.

Throughout the paper, $I(G)$ will be regarded as an R-module, and we shall investigate weakly 2-absorbing primary submodules of $I(G)$. This section is in accordance with [10].

3. Weakly 2-absorbing primary submodules of edge ideals. In this section, we present some results about the modules and submodules which involve the theory of graphs together with the edge ideal of a graph G, which is simple and finite and with no isolated vertices.

Here, we take K a fixed field and we consider $K[v_1, v_2 \dots, v_s]$ the polynomial ring over the field K. Since K is a field, we have that K is a Noetherian ring and then $K[v_1, \dots, v_s]$ is also a Noetherian ring (Theorem of the Hilbert Basis).

Remark 3.1. Since K is a field, the polynomial ring $R = K[v_1, \dots, v_s]$ is Noetherian by Hilbert's Basis Theorem. Consequently, the edge ideal $I(G)$ is a finitely generated Noetherian R-module.

Definition 3.1. Let $R = K[v_1, \dots, v_s]$ be the polynomial ring , $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. A proper submodule N of the R-module $I(G)$ is called a weakly 2-absorbing primary submodule of $I(G)$ if $0 \neq f_1 f_2 v \in N$ for any $f_1, f_2 \in R$ and $v \in I(G)$, then $f_1 f_2 \in (N : I(G))$ or $f_1 v \in I(G) - \text{rad}(N)$ or $f_2 v \in I(G) - \text{rad}(N)$.

Proposition 3.1. *Let $R = K[v_1, \dots, v_s]$ be the polynomial ring , $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let N, K be submodules of the R-module $I(G)$ with $K \subseteq N$. If N is a weakly 2-absorbing primary submodule of $I(G)$, then N/K is a weakly 2-absorbing primary submodule of $I(G)/K$. The converse is true when K is a weakly 2-absorbing primary submodule.*

Proof: Assume that N is a weakly 2-absorbing primary submodule of $I(G)$. Let f_1, $f_2 \in R$ and $v + K \in I(G)/K$ where $0_{I(G)/K} \neq f_1 f_2(v + K) \in N/K$. Since $f_1 f_2(v + K) \neq 0_{I(G)/K}$, we get $f_1 f_2 v \in N$ and $f_1 f_2 v \in K$. If $f_1 f_2 v = 0$, we obtain $f_1 f_2 v + K = 0_{I(G)/K}$. So $f_1 f_2 v \neq 0$. Thus $f_1 f_2 \in (N : I(G))$ or $f_1 v \in I(G) - \text{rad}(N)$ or $f_2 v \in I(G) - \text{rad}(N)$ since N is weakly 2-absorbing primary. Consequently, we get $f_1 f_2 \in (N/K : I(G)/K)$ or $f_1 v + K = f_1(v + K) \in I(G) - \text{rad}(N)/K = I(G)/K - \text{rad}(N/K)$ or $f_2 v + K = f_2(v + K) \in I(G) - \text{rad}(N)/K = I(G)/K - \text{rad}(N/K)$. Conversely, let K be a weakly 2-absorbing primary submodule.

Assume that N/K is a weakly 2-absorbing primary submodule of $I(G)/K$. Let f_1, $f_2 \in R$ and $v \in I(G)$ where $0 \neq f_1 f_2 v \in N$. Then we have $f_1 f_2 v + K \in N/K$. If $f_1 f_2 v + K = 0_{I(G)/K}$, then $f_1 f_2 v \in K$. Thus $f_1 f_2 \in (K : I(G))$ or $f_1 v \in I(G) - \text{rad}(K)$ or $f_2 v \in I(G) - \text{rad}(K)$, since K is weakly 2-absorbing primary. Therefore, $f_1 f_2 \in (N : I(G))$ or $f_1 v \in I(G) - \text{rad}(N)$ or $f_2 v \in I(G) - \text{rad}(N)$, since $K \subseteq N$. Let $f_1 f_2 v + K = f_1 f_2(v + K) \neq 0_{I(G)/K}$. Then $f_1 f_2 \in (N/K : I(G)/K)$ or $f_1(v + K) \in I(G)/K - \text{rad}(N/K)$ or $f_2(v + K) \in I(G)/K - \text{rad}(N/K)$.

Thus $f_1 f_2 \in (N : I(G))$ or $f_1 v \in I(G) - \text{rad}(N)$ or $f_2 v \in I(G) - \text{rad}(N)$. □

We present now more a result.

Proposition 3.2. *Let $R = K[v_1, \dots, v_s]$ be the polynomial ring , $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. For the R-module $I(G)$, we consider S a multiplicatively closed subset of R. If N is a weakly 2-absorbing primary submodule of $I(G)$ where $(N : I(G)) \cap S = \emptyset$, then $S^{-1}N$ is a weakly 2-absorbing primary submodule of $S^{-1}I(G)$.*

Proof: Let $0 \neq \frac{r}{s} \frac{t}{k} \frac{v}{l} = \frac{rtv}{skl} \in S^{-1}N$ where $r, t \in R$, $s, k, l \in S$ and $v \in I(G)$. Then there exists an element u of S such that $0 \neq urtv \in N$. Hence we get $urt \in (N : I(G))$ or $urv \in I(G) - \text{rad}(N)$ or $tv \in I(G) - \text{rad}(N)$ since N is weakly 2-absorbing primary.

Then $\frac{r}{s}\frac{t}{k} = \frac{urt}{usk} \in S^{-1}(N : I(G)) \subseteq (S^{-1}N : S^{-1}I(G))$ or $\frac{r}{s}\frac{v}{l} = \frac{urv}{usl} \in S^{-1}(I(G) - \text{rad}(N)) = S^{-1}I(G) - \text{rad}(S^{-1}N)$ or $\frac{t}{k}\frac{v}{l} = \frac{tv}{kl} \in S^{-1}(I(G) - \text{rad}(N)) = S^{-1}I(G) - \text{rad}(S^{-1}N)$. □

Definition 3.2. Let $R = K[v_1, \dots, v_s]$ be the polynomial ring , $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let N be a weakly 2-absorbing primary submodule of $I(G)$. (f_1, f_2, v) is called a triple-zero of N if $f_1f_2v = 0$, $f_1f_2 \notin (N : I(G))$, $f_1v \notin I(G) - \text{rad}(N)$ and $f_2v \notin I(G) - \text{rad}(N)$.

Note that if N is a weakly 2-absorbing primary submodule of $I(G)$ and there is no triple-zero of N, then N is a 2-absorbing primary submodule of $I(G)$.

Proposition 3.3. *Let $R = K[v_1, \dots, v_s]$ be the polynomial ring , $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $I(G)$ is a multiplication R-module and N is a weakly 2-absorbing primary submodule of $I(G)$ that is not 2-absorbing primary.*

Then $N^3 = 0$.

Proof: We have that $(N : I(G))I(G) = N$ since, by hypothesis, $I(G)$ is multiplication module. Then

$$N^3 = (N : I(G))^3 I(G) = (N : I(G))^2 N = 0.$$

Consequently, $N^3 = 0$. □

Definition 3.3. Let $R = K[v_1, \dots, v_s]$ be the polynomial ring , $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let N be a weakly 2-absorbing primary submodule of the R-module $I(G)$ and let $0 \neq I_1I_2K \subseteq N$ for some ideals I_1, I_2 of R and some submodule K of $I(G)$. N is called free triple-zero in regard to I_1, I_2, K if (f_1, f_2, v) is not a triple-zero of N for every $f_1 \in I_1$, $f_2 \in I_2$ and $v \in K$.

Lemma 3.1. *Let $R = K[v_1, \dots, v_s]$ be the polynomial ring , $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let N be a weakly 2-absorbing primary submodule of the R-module $I(G)$. Assume that $f_1f_2K \subseteq N$ for some $f_1, f_2 \in R$ and some submodule K of $I(G)$ where (f_1, f_2, v) is not a triple-zero of N for every $v \in K$. If $f_1f_2 \notin (N : I(G))$, then $f_1K \subseteq I(G) - \text{rad}(N)$ or $f_2K \subseteq I(G) - \text{rad}(N)$.*

Proof: Assume that f_1K is not contained in $I(G) - \text{rad}(N)$ and f_2K is not contained in $I(G) - \text{rad}(N)$. Then there are $x, y \in K$ such that $f_1x \notin I(G) - \text{rad}(N)$ and $f_2y \notin I(G) - \text{rad}(N)$. We get $f_2x \in I(G) - \text{rad}(N)$ since N is a weakly 2-absorbing primary submodule, (f_1, f_2, x) is not a triple-zero of N, $f_1f_2 \notin (N : I(G))$ and $f_1x \notin I(G) - \text{rad}(N)$. In a similar way, $f_1y \in I(G) - \text{rad}(N)$. Now, we obtain $f_1(x+y) \in I(G) - \text{rad}(N)$ or $f_2(x+y) \in I(G) - \text{rad}(N)$ since $(f_1, f_2, x+y)$ is not a triple-zero of N, $f_1f_2(x+y) \in N$ and $f_1f_2 \notin (N : I(G))$. Assume that $f_1(x+y) = f_1x + f_1y \in I(G) - \text{rad}(N)$.

As $f_1y \in I(G) - \text{rad}(N)$, we get $f_1x \in I(G) - \text{rad}(N)$, which is a contradiction. Assume that $f_2(x+y) = f_2x + f_2y \in I(G) - \text{rad}(N)$. As $f_2x \in I(G) - \text{rad}(N)$, we get $f_2y \in I(G) - \text{rad}(N)$, a contradiction again. Hence we obtain that $f_1K \subseteq I(G) - \text{rad}(N)$ or $f_2K \subseteq I(G) - \text{rad}(N)$. □

Let N be a weakly 2-absorbing primary submodule of the R-module $I(G)$ and $I_1I_2K \subseteq N$ for some ideals I_1, I_2 of the ring $R = K[v_1, v_2, \dots, v_s]$ and some submodule K of $I(G)$, where N is free triple-zero in regard to I_1, I_2, K. Note that if $f_1 \in I_1$, $f_2 \in I_2$ and $v \in K$, then $f_1f_2 \in (N : I(G))$ or $f_1v \in I(G) - \text{rad}(N)$ or $f_2v \in I(G) - \text{rad}(N)$.

We presented now the following result.

Theorem 3.1. *Let $R = K[v_1, \dots, v_s]$ be the polynomial ring , $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Assume that N is a weakly 2-absorbing primary submodule of the R-module $I(G)$ and $0 \neq I_1I_2K \subseteq N$ for some ideals I_1, I_2 of R and some submodule K of $I(G)$ where N is free triple-zero in regard to I_1, I_2, K. Then $I_1I_2 \subseteq (N : I(G))$ or $I_1K \subseteq I(G) - \text{rad}(N)$ or $I_2K \subseteq I(G) - \text{rad}(N)$.*

Proof: Let N be a weakly 2-absorbing primary submodule of $I(G)$ and $0 \neq I_1I_2K \subseteq N$ for some ideals I_1, I_2 of R and some submodule K of $I(G)$ where N is free triple-zero in regard to I_1, I_2, K. Suppose that I_1I_2 is not contained in $(N : I(G))$. Now, we show that $I_1K \subseteq I(G) - \text{rad}(N)$ or $I_2K \subseteq I(G) - \text{rad}(N)$. Assume that I_1K is not contained in $I(G) - \text{rad}(N)$ and I_2K is not contained in $I(G) - \text{rad}(N)$. Then xK is not contained in $I(G) - \text{rad}(N)$ and yK is not contained in $I(G) - \text{rad}(N)$ where $x \in I_1$ and $y \in I_2$. By Lemma 3.1, we get $xy \in (N : I(G))$ since $xyK \subseteq N$, xK is not contained in $I(G) - \text{rad}(N)$ and yK is not in contained in $I(G) - \text{rad}(N)$. By our assumption, there are $f_1 \in I_1$ and $f_2 \in I_2$ such that $f_1f_2 \notin (N : I(G))$. By Lemma 3.1, we get $f_1K \subseteq I(G) - \text{rad}(N)$ or $f_2K \subseteq I(G) - \text{rad}(N)$ since $f_1f_2K \subseteq N$ and $f_1f_2 \notin (N : I(G))$. We investigate three cases.

First case: Assume that $f_1K \subseteq I(G) - \text{rad}(N)$ and f_2K is not contained in $I(G) - \text{rad}(N)$. Since $xf_2K \subseteq N$, xK is not contained in $I(G) - \text{rad}(N)$ and f_2K is not contained in $I(G) - \text{rad}(N)$, then we get $xf_2 \in (N : I(G))$, by Lemma 3.1. We have $(x + f_1)K$ is not contained in $I(G) - \text{rad}(N)$ since $(x + f_1)f_2K \subseteq N$, xK is not contained in $I(G) - \text{rad}(N)$ and $f_1K \subseteq I(G) - \text{rad}(N)$. Since f_2K is not contained in $I(G) - \text{rad}(N)$ and $(x + f_1)K$ is not contained in $I(G) - \text{rad}(N)$, then we obtain

$f_2(x+f_1) \in (N : I(G))$, by Lemma 3.1. Thus since $f_2(x+f_1) = xf_2 + f_1f_2 \in (N : I(G))$ and $xf_2 \in (N : I(G))$, then $f_1f_2 \in (N : I(G))$, which is a contradiction.

Second case: Assume that f_1K is not contained in $I(G) - \mathrm{rad}(N)$ and $f_2K \subseteq I(G) - \mathrm{rad}(N)$. It is easily shown similarly to the first case.

Third case: Suppose that $f_1K \subseteq I(G) - \mathrm{rad}(N)$ and $f_2K \subseteq I(G) - \mathrm{rad}(N)$. Then $(y+f_2)K$ is not contained in $I(G) - \mathrm{rad}(N)$ since yK is not contained in $I(G) - \mathrm{rad}(N)$ and $f_2K \subseteq I(G) - \mathrm{rad}(N)$. By Lemma 3.1, $x(y+f_2) \in (N : I(G))$ since $x(y+f_2)K \subseteq N$, xK is not contained in $I(G) - \mathrm{rad}(N)$ and $(y+f_2)K$ is not contained in $I(G) - \mathrm{rad}(N)$. Then $xf_2 \in (N : I(G))$ since $x(y+f_2) \in (N : I(G))$ and $xy \in (N : I(G))$. As $f_1K \subseteq I(G) - \mathrm{rad}(N)$ and xK is not contained in $I(G) - \mathrm{rad}(N)$, then $(x+f_1)K$ is not contained in $I(G) - \mathrm{rad}(N)$. As $(x+f_1)yK \subseteq N$, yK is not contained in $I(G) - \mathrm{rad}(N)$ and $(x+f_1)K$ is not contained in $I(G) - \mathrm{rad}(N)$, then $(x+f_1)y = xy + f_1y \in (N : I(G))$ by Lemma 3.1. As $xy \in (N : I(G))$ and $f_1y + xy \in (N : I(G))$, then $f_1y \in (N : I(G))$. By Lemma 3.1, we get $(x+f_1)(y+f_2) = xy + xf_2 + f_1y + f_1f_2 \in (N : I(G))$ since $(x+f_1)(y+f_2)K \subseteq N$, $(x+f_1)K$ is not contained in $I(G) - \mathrm{rad}(N)$ and $(y+f_2)K$ is not contained in $I(G) - \mathrm{rad}(N)$.

As $xf_2, f_1y, xy \in (N : I(G))$, then $xy + xf_2 + f_1y \in (N : I(G))$. Thus, $f_1f_2 \in (N : I(G))$ since $xy + xf_2 + f_1y + f_1f_2 \in (N : I(G))$ and $xy + xf_2 + f_1y \in (N : I(G))$, a contradiction. Hence $I_1K \subseteq I(G) - \mathrm{rad}(N)$ or $I_2K \subseteq I(G) - \mathrm{rad}(N)$. □

Lemma 3.2. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring , $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let N be a weakly 2-absorbing primary submodule of $I(G)$. If $f_1f_2K \subseteq N$ and $0 \neq 2f_1f_2K$ for some submodule K of $I(G)$ and for some $f_1, f_2 \in R$, then $f_1f_2 \in (N : I(G))$ or $f_1K \subseteq I(G) - \mathrm{rad}(N)$ or $f_2K \subseteq I(G) - \mathrm{rad}(N)$.*

Proof: Assume that $f_1f_2 \notin (N : I(G))$. Now, we show that $K \subseteq (I(G) - \mathrm{rad}(N) : f_1) \bigcup (I(G) - \mathrm{rad}(N) : f_2)$. Let $x \in K$. If $0 \neq f_1f_2x$, then $f_1x \in I(G) - \mathrm{rad}(N)$ or $f_2x \in I(G) - \mathrm{rad}(N)$ since $f_1f_2 \notin (N : I(G))$. Thus $x \in (I(G) - \mathrm{rad}(N) : f_1) \bigcup (I(G) - \mathrm{rad}(N) : f_2)$. Suppose that $f_1f_2x = 0$. As $0 \neq 2f_1f_2K$, there exists $y \in K$ such that $2f_1f_2y \neq 0$. Then we get $0 \neq f_1f_2y \in N$. So $f_1y \in I(G) - \mathrm{rad}(N)$ or $f_2y \in I(G) - \mathrm{rad}(N)$ since N is weakly 2-absorbing primary. Let $z = x + y$. Then $0 \neq f_1f_2z \in N$. Since $f_1f_2 \notin (N : I(G))$, then $f_1z \in I(G) - \mathrm{rad}(N)$ or $f_2z \in I(G) - \mathrm{rad}(N)$. Now, we consider three cases.

First case: Assume that $f_1y \in I(G) - \mathrm{rad}(N)$ and $f_2y \notin I(G) - \mathrm{rad}(N)$. Let $f_1x \notin I(G) - \mathrm{rad}(N)$. Then $f_1z \notin I(G) - \mathrm{rad}(N)$. Thus $f_2z \in I(G) - \mathrm{rad}(N)$. So $f_1(z+x) \notin I(G) - \mathrm{rad}(N)$ and $f_2(z+x) \notin I(G) - \mathrm{rad}(N)$. Hence $0 = f_1f_2(z+x) = 2f_1f_2z$ since N is a weakly 2-absorbing primary submodule and $f_1f_2 \notin (N : I(G))$. This is a contradiction. Consequently, $f_1x \in I(G) - \mathrm{rad}(N)$.

Second case: Assume that $f_1y \notin I(G) - \mathrm{rad}(N)$ and $f_2y \in I(G) - \mathrm{rad}(N)$. It is proved by a similar way to first case.

Third case: Assume that $f_1y \in I(G) - \mathrm{rad}(N)$ and $f_2y \in I(G) - \mathrm{rad}(N)$. Since $f_1z \in I(G) - \mathrm{rad}(N)$ or $f_2z \in I(G) - \mathrm{rad}(N)$, $f_1x \in I(G) - \mathrm{rad}(N)$ or $f_2x \in I(G) - \mathrm{rad}(N)$. □

We conclude with a proposition that uses the previous lemma.

Proposition 3.4. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring , $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let N be a weakly 2-absorbing primary submodule of the R-module $I(G)$. If $f_1JK \subseteq N$ and $0 \neq 4f_1JK$ for some submodule K of $I(G)$, for any ideal J of R and for some $f_1 \in R$, then $f_1J \subseteq (N : I(G))$ or $f_1K \subseteq I(G) - \mathrm{rad}(N)$ or $JK \subseteq I(G) - \mathrm{rad}(N)$.*

Proof: Suppose that f_1J is not contained in $(N : I(G))$. Then there exists $j \in J$ such that $f_1j \notin (N : I(G))$. Our claim is that there exists $f_2 \in J$ such that $0 \neq 4f_1f_2K$ and $f_1f_2 \notin (N : I(G))$. As $0 \neq 4f_1JK$, then there exists $j' \in J$ such that $0 \neq 4f_1j'K$.

Assume that $f_1j' \notin (N : I(G))$ and $0 \neq 4f_1jK$. Then we have the result for $b = j'$ or $b = j$. Now, let $f_1j' \in (N : I(G))$ and $4f_1jK = 0$. Then $0 \neq 4f_1(j + j')K \subseteq N$ and $f_1(j + j') \notin (N : I(G))$ since $f_1j \notin (N : I(G))$ and $f_1j' \in (N : I(G))$. Hence we get $f_2 = j + j' \in J$ such that $0 \neq 4f_1f_2K \subseteq N$ and $f_1f_2 \notin (N : I(G))$. Thus $0 \neq 2f_1f_2K$ and so $K \subseteq (I(G) - \mathrm{rad}(N) : f_1) \bigcup (I(G) - \mathrm{rad}(N) : f_2)$ by Lemma 3.2. If $K \subseteq (I(G) - \mathrm{rad}(N) : f_1)$, the proof is completed. Suppose that K is not contained in $(I(G) - \mathrm{rad}(N) : f_1)$. Then $K \subseteq (I(G) - \mathrm{rad}(N) : f_2)$, that is, $f_2K \subseteq I(G) - \mathrm{rad}(N)$. Let $c \in J$. Assume that $0 \neq 2f_1cK$. By Lemma 3.2, $f_1c \in (N : I(G))$ or $cK \subseteq I(G) - \mathrm{rad}(N)$ since K is not contained in $(I(G) - \mathrm{rad}(N) : f_1)$. Then $c \in ((N : I(G)) : f_1) \bigcup (I(G) - \mathrm{rad}(N) : K)$. Now, let $2f_1cK = 0$. Then $0 \neq 2f_1(f_2 + c)K \subseteq N$. By Lemma 3.2, we have $f_1(f_2 + c) \in (N : I(G))$ or $(f_2 + c)K \subseteq I(G) - \mathrm{rad}(N)$ since K is not contained in $(I(G) - \mathrm{rad}(N) : f_1)$. Thus we get $f_2 + c \in ((N : I(G)) : f_1) \bigcup (I(G) - \mathrm{rad}(N) : K)$. If $f_2 + c \in (I(G) - \mathrm{rad}(N) : K)$, then $c \in (I(G) - \mathrm{rad}(N) : K)$. Hence $JK \subseteq I(G) - \mathrm{rad}(N)$. Let $f_2 + c \in ((N : I(G)) : f_1) \setminus (I(G) - \mathrm{rad}(N) : K)$.

Note that $2f_1(f_2 + c + f_2)K = 4f_1f_2K \neq 0$ and $2f_1(f_2 + c + f_2)K \subseteq N$. Since $f_1f_2 \notin (N : I(G))$ and $f_1(f_2 + c) \in (N : I(G))$, $f_1(f_2 + c + f_2) \notin (N : I(G))$. Thus by Lemma 3.2, $K \subseteq (I(G) - \mathrm{rad}(N) : f_1) \bigcup (I(G) - \mathrm{rad}(N) : (f_2 + c + f_2))$. As $f_2 + c \notin (I(G) - \mathrm{rad}(N) : K)$ and $f_2 \in (I(G) - \mathrm{rad}(N) : K)$, then we get $(f_2 + c + f_2) \notin (I(G) - \mathrm{rad}(N) : K)$ and so $K \subseteq (I(G) - \mathrm{rad}(N) : f_1)$, a contradiction.

Thus $f_2+c \in (I(G)-\mathrm{rad}(N) : K)$. Since $f_2 \in (I(G)-\mathrm{rad}(N) : K)$, $c \in (I(G)-\mathrm{rad}(N) : K)$. Hence $J \subseteq ((N : I(G)) : f_1) \bigcup (I(G) - \mathrm{rad}(N) : K)$.

Consequently $J \subseteq (I(G) - \mathrm{rad}(N) : K)$ since $f_1 J$ is not contained in $(N : I(G))$. □

3.1. Applications to edge ideals. In this subsection we illustrate how the previous results apply to edge ideals associated with some classical families of graphs. These examples show that the theory developed in the previous sections can be naturally specialized to important classes of monomial ideals arising from graph theory.

Example 3.1. *Let G be a path graph*

$$P_n : v_1 - v_2 - \cdots - v_n.$$

Then

$$I(G) = (v_1v_2, v_2v_3, \ldots, v_{n-1}v_n).$$

Since $I(G)$ is generated by squarefree quadratic monomials, every weakly 2-absorbing primary submodule of $I(G)$ satisfies the properties established in results of the paper.

Therefore the localization, quotient and annihilator properties obtained in Section 3 hold for edge ideals of path graphs.

Example 3.2. *Let $G = K_{1,n}$ be a star graph.*

Its edge ideal is

$$I(G) = (v_0v_1, \ldots, v_0v_n).$$

Since all generators have a common vertex, the previous results describe the behaviour of weakly 2-absorbing primary submodules of this edge ideal under localization and quotient constructions.

Hence, the results of the paper apply directly to this family of edge ideals.

Example 3.3. *Let $G = C_n$ be a cycle.*

Then

$$I(G) = (v_1v_2, v_2v_3, \ldots, v_nv_1).$$

The results obtained in this paper provide structural information about weakly 2-absorbing primary submodules of this edge ideal, independently of the parity of the cycle. In particular, the localization and quotient constructions preserve the weakly 2-absorbing primary property under the hypotheses of results of the paper.

The previous examples show that the general theory developed in this work can be applied to several important classes of edge ideals arising from graph theory. Consequently, weakly 2-absorbing primary submodules may be investigated in concrete combinatorial settings such as paths, cycles and star graphs, providing a natural interaction between module theory and edge ideals.

4. Concluding remarks. The theory of weakly 2-absorbing primary submodules has been mainly developed in the setting of arbitrary modules over commutative rings. In this paper we have considered this theory in the particular context of edge ideals of finite simple graphs.

The obtained results show that several structural properties of weakly 2-absorbing primary submodules remain valid when the ambient module is an edge ideal. In particular, quotient modules, localization, and annihilator conditions preserve important aspects of the theory under appropriate hypotheses.

Since edge ideals constitute one of the fundamental classes of monomial ideals in combinatorial commutative algebra, this approach provides a natural connection between module theory and graph theory.

Several questions remain open. For instance, it would be interesting to study weakly 2-absorbing primary submodules of edge ideals associated with special families of graphs, such as trees, bipartite graphs, chordal graphs and Cohen–Macaulay graphs. Another possible direction is to investigate whether combinatorial invariants of the graph influence the structure of weakly 2-absorbing primary submodules of its edge ideal.

We hope that the present work motivates further investigations on the interaction between the theory of generalized primary submodules and combinatorial properties of monomial ideals arising from graphs.

Final Observation. The results established in this paper provide a framework for studying weakly 2-absorbing primary submodules in an important class of monomial ideals. We expect that further developments may establish stronger connections between the algebraic properties of these submodules and the combinatorial structure of the underlying graphs.

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References

[1] Alilooee A, Banerjee A. Powers of edge ideals of regularity three bipartite graphs. J Commut Algebra. 2017;9:441–454.
[2] Anderson DF, Badawi A. On n-absorbing ideals of commutative rings. Comm Algebra. 2011;39:1646–1672.
[3] Anderson DD, Smith E. Weakly prime ideals. Houston J Math. 2003;29:831–840.
[4] Atani SE, Farzalipour F. On weakly primary ideal. Georgian Math J. 2005;12:423–429.
[5] Atiyah MF, Macdonald IG. Introduction to Commutative Algebra. Reading (MA): Addison-Wesley; 1969.
[6] Badawi A. On 2-absorbing ideals of commutative rings. Bull Aust Math Soc. 2007;75:417–429.
[7] Badawi A, Darani AY. On weakly 2-absorbing ideals of commutative rings. Houston J Math. Forthcoming.
[8] Badawi A, Tekir U, Yetkin E. On 2-absorbing primary ideals in commutative rings. Bull Korean Math Soc. 2014;51:1163–1173.
[9] Rotman JJ. An Introduction to Homological Algebra. New York: Academic Press; 1979.
[10] Simis A, Vasconcelos WV, Villarreal RH. The integral closure of subrings associated to graphs. J Algebra. 1998;199:281–289.