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COMMUNICATION

Artinianness of local cohomology modules

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Abstract
Let R be a commutative Noetherian ring with non-zero identity and I an ideal of R. Let M be a finitely generated R-module. we some results about Artinianness of local cohomology module involving the concept of ZD-module. As applications, it is shown that the local cohomology module, under certain hypotheses, it has vanishing properties, and finite support properties.

Keywords . local cohomology modules, Artinian modules, Noetherian ring, finitely generated module. 2020 Mathematics Subject Classification: 13D45.

1. Introduction. Throughout this paper, R is a commutative Noetherian ring with non-zero identity. Let I be an ideal of R and let M be an R-module. For notations and terminologies not given in this paper, the reader is referred to [4], [5] and [9], if necessary.

The local cohomology theory has been an significant tool in commutative algebra and algebraic geometry. As a generalization of the ordinary local cohomology modules, in [9], the authors introduced the local cohomology modules with respect to a pair of ideals.

Here, we put results for the local cohomology module defined by an ideal. To be more precise, let

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) : I \subseteq \mathfrak{p}\}.$$

The set of elements x of M such that $\operatorname{Supp}_R(Rx) \subseteq V(I)$ is said to be I-torsion submodule of M and is denoted by $\Gamma_I(M)$. It is easy to see that $\Gamma_I(\bullet)$ is a covariant, R-linear functor from the category of R-modules to itself. For an integer i, the local cohomology functor $\mathrm{H}^i_I(\bullet)$ with respect to I is defined to be the i-th right derived functor of $\Gamma_I(\bullet)$.

Also $\mathrm{H}^i_I(M)$ is called the i-th local cohomology module of M with respect to I. Recently, some authors approached the study of local cohomology modules by means of Serre subcategories and it is noteworthy that their approach enables us to deal with several important problems on local cohomology modules comprehensively; see, for example [1], and, [2].

Moreover, also we have the formal local cohomology module, and some results for such module are found in [10].

In this direction, we study the local cohomology modules with respect to an ideal by the notion of Serre subcategory.

In the Section 2, we some results of Artinianness of local cohomology module, and in the Section 3, we put applications for the theory in question. We finalize the paper, with a conclusion.

Here, we use properties of commutative algebra and homological algebra for the development of the results (see [3] and [8]).

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2. Artinianness of local cohomology module. Recall that R is a Noetherian ring, I is an ideal of R and M is an R-module. Let $\mathrm{Z}_R(M)$ denote the set of zero-divisors of M.

Definition 2.1. An R-module M is said to be zero-divisor module if for any submodule N of M, the set $\mathrm{Z}_R(M/N)$ is a finite union of prime ideals in $\mathrm{Ass}_R(M/N)$.

According to Example 2.2, in [7], the class of zero-divisor modules (ZD-modules), contains finitely generated modules.

Definition 2.2. A full subcategory of the category of R-modules is said to be Serre subcategory, if it is closed under taking submodules, quotients and extensions. A Serre subcategory S is said to be satisfy the condition C_I if for any I-torsion R-module M, $(0 :_M I) \in S$ implies that $M \in S$.

Examples 2.4 and 2.5, in [1], show that the class of zero modules, Artinian modules, I-cofinite Artinian modules, modules with finite support and the class of R-modules M with $\dim_R(M) \leq t$, where t is a non-negative integer, are Serre subcategories of the category of R-modules satisfy the condition C_I.

In the rest of the paper, S denotes a subcategory of the category of R-modules satisfying the condition C_I.

Theorem 2.1. *Let M be a ZD-module such that $\Gamma_I(M/N) \in S$, for any submodule N of M. Then $\mathrm{H}^i_I(M/N) \in S$, for any submodule N of M and all $i \geq 0$.*

Proof: We may assume that I is not zero, this can be done simply because $\Gamma_I(\bullet)$ is identity functor when $I = 0$. We use induction on i. The case $i = 0$ is trivial by assumption. So assume, inductively, that $i > 0$, and we have that shown that $\mathrm{H}^{i-1}_I(M'/N') \in S$, for any ZD-module M' and any submodule N' of M'. Now let M be a ZD-module, N a submodule of M and $X = M/N$. Then, we have that

$$\mathrm{H}^i_I(X/\Gamma_I(X)) \cong \mathrm{H}^i_I(X),$$

by Corollary 1.13(4), in [9].

Also, $X/\Gamma_I(X)$ is a (ZD-module) I-torsion free R-module. We therefore assume in addition that X is an I-torsion free R-module. We now use Lemma 2.4, in [7], to deduce that I contains an element a which is a non zero-divisor on X. The exact sequence

$$0 \to X \xrightarrow{a} X \to X/(a)X \to 0,$$

induces an exact sequence

$$\ldots \to \mathrm{H}^{i-1}_I(X/(a)X) \to \mathrm{H}^i_I(X) \xrightarrow{a} \mathrm{H}^i_I(X) \to \mathrm{H}^i_I(X/(a)X) \to \ \ldots\ ,$$

of local cohomology modules. Since

$$X/(a)X \cong M/((a)M + N)$$

is a ZD-module, it follows from the inductive hypothesis that we have

$$\mathrm{H}^{i-1}_I(X/(a)X) \in S.$$

So, the above exact sequence shows that the R-module

$$(0 :_{\mathrm{H}^i_I(X)} a) \in S.$$

Hence, we have that

$$\mathrm{H}^i_I(X) \in S,$$

by Lemma 2.3, in [1]. This completes the inductive step.

The result follows by induction. □

Now, it follows the following corollary.

Corollary 2.1. *Let $(R, \mathfrak{m})$ be a local ring, $\sqrt{I} = \mathfrak{m}$, S non-zero and M a finitely generated R-module. Then, we have that*

$$\mathrm{H}^i_I(M/N) \in S,$$

for any submodule N of M and all $i \geq 0$.

Proof: In view of the Theorem 2.1, it is enough to show that $\Gamma_I(M/N) \in S$ for any submodule N of M. Assume that N is a submodule of M; Proposition 1.4, in [9], shows that

$$\Gamma_I(M/N) = \Gamma_{\sqrt{I}}(M/N) = \Gamma_{\mathfrak{m}}(M/N).$$

Since, $\Gamma_{\mathfrak{m}}(M/N)$ is a finitely generated R-module and annihilated by a power of $\mathfrak{m}$, hence we have that

$$\Gamma_{\mathfrak{m}}(M/N)$$

has finite length.

So by Lemma 2.11, in [2], we have that

$$\Gamma_{\mathfrak{m}}(M/N) \in S.$$

This finishes the proof. □

The following corollary improves the Corollary 2.1, when S is considered the class of I-cofinite Artinian modules.

Corollary 2.2. *Let* $\dim_R(R/I) = 0$*, and let* M *be a finitely generated* R*-module. Then, we have that*

$$\mathrm{H}^i_I(M)$$

is I*-cofinite Artinian for all* $i \geq 0$*.*

Proof: The proof is similar to that of Corollary 2.1. □

3. More some results and applications. In this section, we some results for M a finitely generated R-module of finite dimension.

Theorem 3.1. *Let* R *be a local ring,* S *non-zero and* M *a finitely generated* R*-module with* $\dim_R(M) = d$*. Then, we have that*

$$\mathrm{H}^d_I(M) \in S.$$

Proof: When $\dim_R(M) = -1$, there is nothing to prove, as then $M = 0$. We argue by induction on $\dim_R(M)$. If $\dim_R(M) = 0$, then M has finite length. Thus, $\Gamma_I(M)$ has finite length. So The result follows by Lemma 2.11, in [2]. Now suppose, inductively, that $\dim_R(M) = n > 0$, and the result has been proved for all R-modules of dimensions smaller than n satisfying the hypothesis. The exact sequence

$$0 \to M \to M \to 0,$$

where $M \to M$ is the R-homomorphism identity, induces the long exact sequence

$$\ldots \to \mathrm{H}^i_I(M) \to \mathrm{H}^i_I(M) \to \mathrm{H}^{i+1}_I(M) \to \ldots .$$

Thus, we have that

$$\mathrm{H}^d_I(M) \in S,$$

by the previous paragraph.

This finishes the proof. □

We have now the following proposition.

Proposition 3.1. *Let* $M \neq 0$ *be a finitely generated* R*-module of finite Krull dimension. If we have* $\dim_R(M) = n \geq 0$*, then* $\mathrm{H}^{n+1}_I(M) = 0$*.*

Proof: We use induction on $\dim_R(M)$. If $\dim_R(M) = 0$, then we have that

$$\mathrm{H}^1_I(M) = 0,$$

by Theorem 4.7(1), in [9]. So assume, inductively, that $\dim_R(M) = n > 0$, and we established the result for R-modules of dimension smaller than n satisfying the hypothesis.

By another using of Theorem 4.7(1), in [9], we have that

$$\mathrm{H}^{n+1}_I(M) = 0,$$

and this finishes the proof. □

In the same context, it follows the following result which involve the theory in question.

Corollary 3.1. *Let* M *be a ZD-module such that* $\Gamma_I(M/N)$ *has finite support for any submodule* N *of* M*. Then, we have that*

$$\mathrm{H}^i_I(M/N),$$

has finite support for any submodule N *of* M *and for all* $i \geq 0$*.*

Proof: Apply Theorem 2.1, and the fact that the class of modules with finite support is a Serre subcategory of the category of R-modules satisfying the condition C_I. □

We conclude the paper with a corollary that involves the previously presented concepts.

Corollary 3.2. *Let R be a local ring and let M be a finitely generated R-module such that for any submodule N of M and for all ideal*

$$\mathfrak{p} \in \operatorname{Supp}_R(\Gamma_I(M/N)),$$

we have that $\dim_R(R/\mathfrak{p}) \leq 1$. Then, we have that $\mathrm{H}^i_I(M)$ has finite support for all $i \geq 0$.

Proof: In view of Corollary 4.3, in [6], we have that $\Gamma_I(M/N)$ has finite support, for any submodules N of M. Now, The result follows by Corollary 3.1. This finishes the proof. □

We finished the article, with the following conclusion.

4. Conclusion. In this article, we can relate the theory of local cohomology, with respect to an ideal, with the theory of Serre subcategories of modules. With the results of the article, we show the importance of local cohomology theory as a study tool within of the commutative algebra theory. Moreover, by making this relationship, we get applications for the local cohomology module in a general theory of modules.

5. Author contributions. This declaration is not applicable in this work.

Conflicts of interest The author declares that there is no conflict of interest.

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