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Some identities for Fibonacci and Lucas polynomials via the Fibonacci Q(x)-matrix

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Abstract

In this paper, we study Fibonacci and Lucas polynomials through the Fibonacci Q(x)-matrix. By analyzing its powers and inverses, we derive several identities for both polynomial families, including extensions to negative indices. We establish that the powers of Q(x) and their inverses have identical determinants and permanents. Furthermore, we obtain an m-term Honsberger-type identity for Fibonacci and Lucas polynomials and provide a matrix-based proof.

Keywords . Fibonacci polynomials, Lucas polynomials, matrices.

1. Introduction. The Fibonacci numbers f_n and the Lucas numbers l_n are defined by the recurrence relations $f_n = f_{n-1} + f_{n-2}$ with initial terms $f_0 = 0$, $f_1 = 1$ and $l_n = l_{n-1} + l_{n-2}$ with $l_0 = 2$, $l_1 = 1$.

Several alternative methods to generate Fibonacci and Lucas numbers have been studied, among them Pascal's triangle [1], determinants [2], binomial coefficients [3], convolutions [4], and other methods [5]. An interesting connection between Fibonacci numbers and matrices is given by the matrix $Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, known as Fibonacci Q-matrix [5], such that

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}.$$

The Fibonacci Q-matrix has been widely studied for its role in generating Fibonacci numbers and their properties [6]. Recently, it has been used in applied fields as image encryption [7, 8], biometric authentication [9], and other applications [10]. Fibonacci polynomials are defined by the recurrence relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \text{ for } n > 1,$$

where $F_0(x) = 0$, $F_1(x) = 1$. Similarly, Lucas polynomials are defined by $L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$, for $n > 1$, where $L_0(x) = 2$ and $L_1(x) = x$, cf. [11]. The first few Fibonacci and Lucas polynomials are:

$$\begin{aligned}
F_2(x) &= x, & L_2(x) &= x^2 + 2,\\
F_3(x) &= x^2 + 1, & L_3(x) &= x^3 + 3x,\\
F_4(x) &= x^3 + 2x, & L_4(x) &= x^4 + 4x^2 + 2,\\
F_5(x) &= x^4 + 3x^2 + 1, & L_5(x) &= x^5 + 5x^3 + 5x,\\
F_6(x) &= x^5 + 4x^3 + 3x, & L_6(x) &= x^6 + 6x^4 + 9x^2 + 2,\\
F_7(x) &= x^6 + 5x^4 + 6x^2 + 1. & L_7(x) &= x^7 + 7x^5 + 14x^3 + 7x.
\end{aligned}$$

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It follows directly that $L_n(x) = F_{n+1}(x) + F_{n-1}(x)$ for $n \geq 1$. Fibonacci and Lucas numbers can be defined for negative indices by $f_{-n} = (-1)^{n-1}f_n$ and $l_{-n} = (-1)^n l_n$ cf. [12]; it can be extended to Fibonacci and Lucas polynomials in the form $F_{-n}(x) = (-1)^{n-1}F_n(x)$ and $L_{-n}(x) = (-1)^n L_n(x)$.

A generalization of the Fibonacci Q-matrix is given by $Q(x) = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$, henceforth called Fibonacci $Q(x)$-matrix.

In section 2 we present several properties that connect the Fibonacci $Q(x)$-matrix with Fibonacci and Lucas polynomials. Our focus is on the less explored matrix $Q^{-1}(x)$ through which we establish explicit connections with Fibonacci and Lucas polynomials of negative indices. Furthermore, we prove that $Q^n(x)$ and $Q^{-n}(x)$ have the same determinants and permanents.

In section 3 we prove some identities of Fibonacci and Lucas polynomials through the Fibonacci $Q(x)$-matrix. In section 4 we present the m-term Honsberger identity of Fibonacci and Lucas polynomials, and we give a proof based on the Fibonacci $Q(x)$-matrix.

2. Fibonacci $Q(x)$-matrix. The following proposition establishes a connection between the Fibonacci $Q(x)$-matrix and its inverse $Q^{-1}(x) = \begin{bmatrix} 0 & 1 \\ 1 & -x \end{bmatrix}$, and the Fibonacci polynomials with positive and negative indices, respectively.

Proposition 2.1. *Let $n > 0$ be an integer, for any variable x we have:*

1. $Q^n(x) = \begin{bmatrix} F_{n+1}(x) & F_n(x) \\ F_n(x) & F_{n-1}(x) \end{bmatrix}$.
2. $Q^{-n}(x) = \begin{bmatrix} F_{-(n-1)}(x) & F_{-n}(x) \\ F_{-n}(x) & F_{-(n+1)}(x) \end{bmatrix}$.

Proof: We prove 1.; 2. follows analogously.

Since $\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F_2(x) & F_1(x) \\ F_1(x) & F_0(x) \end{bmatrix}$, the proposition is true for $n = 1$. Assuming that $\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1}(x) & F_n(x) \\ F_n(x) & F_{n-1}(x) \end{bmatrix}$, then

$$\begin{aligned} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^{n+1} &= \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^n \\ &= \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1}(x) & F_n(x) \\ F_n(x) & F_{n-1}(x) \end{bmatrix} \\ &= \begin{bmatrix} xF_{n+1}(x) + F_n(x) & xF_n(x) + F_{n-1}(x) \\ F_{n+1}(x) & F_n(x) \end{bmatrix}. \end{aligned}$$

Thus, $\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^{n+1} = \begin{bmatrix} F_{n+2}(x) & F_{n+1}(x) \\ F_{n+1}(x) & F_n(x) \end{bmatrix}$. $\square$

The Fibonacci $Q(x)$-matrix can be extended as a 3×3 matrix for generating Tribonacci polynomials cf. [13]. Since $Q^n(x)Q^{-n}(x) = I$ we obtain the following result as consequence of Proposition 2.1.

Corollary 2.1. *For each $n > 0$ we have:*

1. $F_{n+1}(x)F_{-(n-1)}(x) + F_n(x)F_{-n}(x) = 1$.
2. $F_{n+1}(x)F_{-n}(x) + F_n(x)F_{-(n+1)}(x) = 0$.

In Proposition 2.1, for $x = 1$ and $x = -1$ we obtain Fibonacci and negafibonacci numbers respectively cf. [14], if $x = 2$ we get Pell numbers cf. [15]. Although Fibonacci polynomials are better known than Lucas polynomials, the study of Lucas polynomials using various methods is an area of interest [16].

The following proposition collects several immediate identities related to the Fibonacci $Q(x)$-matrix, they can be proved using Proposition 2.1.

Proposition 2.2. *For each $n > 0$ we have:*

1. $Tr\left(Q^{n}(x)\right) = L_{n}(x)$.
2. $Tr\left(Q^{-n}(x)\right) = L_{-n}(x)$.
3. $xQ^{n-1}(x) + Q^{n-2}(x) = Q^{n}(x)$
4. $-xQ^{-(n-1)}(x) + Q^{-(n-2)}(x) = Q^{-n}(x)$.
5. $Q^{n+1}(x) + Q^{n-1}(x) = \begin{bmatrix} L_{n+1}(x) & L_{n}(x) \\ L_{n}(x) & L_{n-1}(x) \end{bmatrix}$.
6. $Q^{-(n+1)}(x) + Q^{-(n-1)}(x) = \begin{bmatrix} L_{-(n-1)}(x) & L_{-n}(x) \\ L_{-n}(x) & L_{-(n+1)}(x) \end{bmatrix}$.
7. $Q^{n}(x) + Q^{-n}(x)$ *is equal to:*
   (a) $\begin{bmatrix} xF_{n}(x) & 2F_{n}(x) \\ 2F_{n}(x) & -xF_{n}(x) \end{bmatrix}$ *for n odd.*
   (b) $\begin{bmatrix} L_{n}(x) & 0 \\ 0 & L_{n}(x) \end{bmatrix}$ *for n even.*
8. *The eigenvalues and eigenvectors of $Q(x)$ are:* $\lambda_1 = \dfrac{x-\sqrt{x^2+4}}{2}$ *and* $\lambda_2 = \dfrac{x+\sqrt{x^2+4}}{2}$ *and* $V_1 = \begin{bmatrix} \frac{x-\sqrt{x^2+4}}{2} \\ 1 \end{bmatrix}$ *and* $V_2 = \begin{bmatrix} \frac{x+\sqrt{x^2+4}}{2} \\ 1 \end{bmatrix}$.

Among matrix-based approaches, the Fibonacci $Q(x)$-matrix provides a particularly effective tool for proving identities involving Fibonacci and Lucas polynomials. In parallel, determinant representations of Fibonacci and Lucas numbers and polynomials via tridiagonal matrices have attracted sustained interest, revealing deep connections between linear recurrences and matrix theory [17].

The permanent of a matrix is a classical invariant in linear algebra defined by a formula analogous to the determinant but without alternating signs [18]. Despite its simple definition, the permanent exhibits a markedly different behavior from the determinant and is known to be computationally intractable in general [19]. The following result introduces an interesting connection between the determinants and permanents of $Q^{n}(x)$ and $Q^{-n}(x)$.

Proposition 2.3. *For each $n > 0$, we have:*

1. $det(Q^{n}(x)) = det(Q^{-n}(x))$.
2. $perm(Q^{n}(x)) = perm(Q^{-n}(x))$.

Proof: We have $Q^{n}(x) = \begin{bmatrix} F_{n+1}(x) & F_{n}(x) \\ F_{n}(x) & F_{n-1}(x) \end{bmatrix}$ and $Q^{-n}(x) = \begin{bmatrix} F_{-(n-1)}(x) & F_{-n}(x) \\ F_{-n}(x) & F_{-(n+1)}(x) \end{bmatrix}$, by Proposition 2.1, therefore:

1. Since $det(Q(x)) = det(Q^{-1}(x)) = -1$, by identities of the determinant we have $det(Q^{n}(x)) = det(Q^{-n}(x)) = (-1)^{n}$.
2. By definition of permanent we have $perm(Q^{n}(x)) = F_{n+1}(x)F_{n-1}(x) + F_{n}^{2}(x)$ and $perm(Q^{-n}(x)) = F_{-(n-1)}(x)F_{-(n+1)}(x) + F_{-n}^{2}(x)$. Since $F_{-n}(x) = (-1)^{n-1}F_{n}(x)$, we get

$$\begin{aligned} perm(Q^{-n}(x)) &= (-1)^{n-2}F_{n-1}(x)(-1)^{n}F_{n+1}(x) + [(-1)^{n-1}F_{n}]^{2}(x) \\ &= F_{n+1}(x)F_{n-1}(x) + F_{n}^{2}(x). \end{aligned}$$

Hence, $perm(Q^{n}(x)) = perm(Q^{-n}(x))$.

3. Some identities of Fibonacci and Lucas polynomials. D'Ocagne's identity for Fibonacci numbers is given by $f_m f_{n+1} - f_{m+1} f_n = (-1)^{n} f_{m-n}$ cf. [20], here we present a matrix proof of D'Ocagne's identity for Fibonacci polynomials.

Proposition 3.1. *For each $m > n > 0$ we have $F_{m}(x)F_{n+1}(x) - F_{m+1}(x)F_{n}(x) = (-1)^{n}F_{m-n}(x)$.*

Proof: Since $Q^{m}(x)Q^{-n}(x) = Q^{m-n}(x)$ we have

$$\begin{bmatrix} F_{m+1}(x) & F_m(x) \\ F_m(x) & F_{m-1}(x) \end{bmatrix} \begin{bmatrix} (-1)^n F_{n-1}(x) & (-1)^{n-1} F_n(x) \\ (-1)^n F_n(x) & (-1)^n F_{n+1}(x) \end{bmatrix}$$
$$= \begin{bmatrix} F_{m-n+1}(x) & F_{m-n}(x) \\ F_{m-n}(x) & F_{m-n-1}(x) \end{bmatrix}.$$

By equating the entries 1,2 we have

$$(-1)^{n-1} F_{m+1}(x) F_n(x) + (-1)^n F_m(x) F_{n+1}(x) = F_{m-n}(x). \tag{3.1}$$

Multiplying both sides by $(-1)^n$, we obtain

$$F_m(x) F_{n+1}(x) - F_{m+1}(x) F_n(x) = (-1)^n F_{m-n}(x).$$

□

By considering $m = n - 1$ in the above proposition, we obtain Cassini's formula for Fibonacci polynomials $F_{n-1}(x)F_{n+1}(x) - (F_n(x))^2 = (-1)^n$. A Cassini-type identity for Lucas polynomials is given by $L_n(x)^2 - L_{n-1}(x)L_n(x) = (-1)^n(x^2 + 4)$ cf. [21] here we present a proof by the Fibonacci $Q(x)$-matrix

Proposition 3.2. *For $n > 0$ we have $L_n(x)^2 - L_{n+1}(x)L_{n-1}(x) = (-1)^n(x^2 + 4)$.*

Proof: Since $Q^{n+1}(x) + Q^{n-1}(x) = Q^{n-1}(x)(Q^2(x) + I)$ we have

$$\begin{aligned} det(Q^{n+1}(x) + Q^{n-1}(x)) &= det(Q^{n-1}(x))det(Q^2(x) + I) \\ &= det(Q(x))^{n-1} det(Q^2(x) + I) \\ &= (-1)^{n-1} det(Q^2(x) + I). \end{aligned}$$

By Propositions 2.1 and 2.2 respectively, we have

$$Q^{n+1}(x) + Q^{n-1}(x) = \begin{bmatrix} L_{n+1}(x) & L_n(x) \\ L_n(x) & L_{n-1}(x) \end{bmatrix}$$
$$Q^2(x) + I = \begin{bmatrix} x^2 + 2 & x \\ x & 2 \end{bmatrix}$$

Thus, we obtain

$$\begin{aligned} det(Q^{n+1}(x) + Q^{n-1}(x)) &= (-1)^{n-1} det(Q^2(x) + I), \\ L_{n+1}(x)L_{n-1}(x) - L_n(x)^2 &= (-1)^{n-1}(x^2 + 4). \end{aligned}$$

Hence $L_n(x)^2 - L_{n+1}(x)L_{n-1}(x) = (-1)^n(x^2 + 4)$. □

4. Honsberger-type identities.

The Honsberger identity for Fibonacci numbers is given by $f_{m+n+1} = f_{m+1}f_{n+1} + f_m f_n$ cf. [22], named after Ross Honsberger (1929-2016). The following result establishes a Honsberger-type identity for Fibonacci and Lucas polynomials.

Lemma 4.1. *For each $m, n \geq 0$ we have*

1. $F_{m+n+1}(x) = F_{m+1}(x)F_{n+1}(x) + F_m(x)F_n(x)$.
2. $L_{m+n+1}(x) = F_{m+1}(x)L_{n+1}(x) + F_m(x)L_n(x)$.

Proof: Here we prove 1.; the proof of 2. is analogous. Since $\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^{m+n} = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^m \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^n$, by Proposition 2.1 we get

$$\begin{bmatrix} F_{m+n+1}(x) & F_{m+n}(x) \\ F_{m+n}(x) & F_{m+n-1}(x) \end{bmatrix}$$
$$= \begin{bmatrix} F_{m+1}(x) & F_m(x) \\ F_m(x) & F_{m-1}(x) \end{bmatrix} \begin{bmatrix} F_{n+1}(x) & F_n(x) \\ F_n(x) & F_{n-1}(x) \end{bmatrix}.$$

By equating the 1,1 entries we deduce $F_{m+n+1}(x) = F_{m+1}(x)F_{n+1}(x) + F_m(x)F_n(x)$. □

Similarly we may deduce the identities $L_m(x)F_n(x) = L_n(x)F_m(x)$ and $F_{2n}(x) = F_n(x)L_n(x)$. Considering the trace in the matrix equation in the proof of Lemma 4.1 we get

$$L_{m+n}(x) = F_{m+1}(x)F_{n+1} + 2F_m(x)F_n(x) + F_{m-1}(x)F_{n-1}(x).$$

In the case $m = n$ we obtain $L_{2n}(x) = F_{n+1}^2(x) + 2F_n^2(x) + F_{n-1}^2(x)$. The Honsberger identity for Fibonacci numbers can be generalized to

$$f_{n_1+\cdots+n_m+1} = \sum_{i_1,\ldots,i_m=0}^{1} f_{n_1+1-i_1}\cdots f_{n_{m-1}+1-(i_{m-2}+i_{m-1})} f_{n_m+1-i_{m-1}}. \tag{4.1}$$

Equation (4.1) can be called the m-term Honsberger identity of Fibonacci numbers, a combinatorial proof was introduced in [23]. Here we establish the m-term Honsberger identity of Fibonacci and Lucas polynomials and we provide a proof based on the Fibonacci $Q(x)$-matrix.

Theorem 4.1. *For $n_1, \ldots, n_m \geq 0$ we have:*

1. $F_{n_1+\cdots+n_m+1}(x) = \sum_{i_1,\ldots,i_{m-1}=0}^{1} F_{n_1+1-i_1}(x)\cdots F_{n_{m-1}+1-(i_{m-2}+i_{m-1})}(x)F_{n_m+1-i_{m-1}}(x)$.
2. $L_{n_1+\cdots+n_m+1}(x) = \sum_{i_1,\ldots,i_{m-1}=0}^{1} F_{n_1+1-i_1}(x)\cdots F_{n_{m-1}+1-(i_{m-2}+i_{m-1})}(x)L_{n_m+1-i_{m-1}}(x)$.

Proof: We prove the identity for Fibonacci polynomials, the result for Lucas polynomials can be proved analogously.

We proceed by induction on m. By Lemma 4.1 we have $F_{n_1+n_2+1}(x) = F_{n_1+1}(x)F_{n_2+1}(x) + F_{n_1}(x)F_{n_2}(x)$, it can be rewritten as

$$F_{n_1+n_2+1}(x) = \sum_{i_1=0}^{1} F_{n_1+1-i_1}(x)F_{n_2+1-i_1}(x).$$

Therefore, the statement holds for $m = 2$. Assuming that $F_{n_1+\cdots+n_m}(x)$ is given by

$$\sum_{i_1,\ldots,i_{m-1}=0}^{1} F_{n_1+1-i_1}(x)\cdots F_{n_{m-1}+1-(i_{m-2}+i_{m-1})}(x)F_{n_m-i_{m-1}}(x),$$

we will prove the identity for $m+1$ terms. Since

$$\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^{n_1+\cdots+n_m+n_{m+1}} = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^{n_1+\cdots+n_m} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^{n_{m+1}}.$$

We have

$$\begin{bmatrix} F_{n_1+\cdots+n_m+n_{m+1}+1}(x) & F_{n_1+\cdots+n_m+n_{m+1}}(x) \\ F_{n_1+\cdots+n_m+n_{m+1}}(x) & F_{n_1+\cdots+n_m+n_{m+1}-1}(x) \end{bmatrix} =$$
$$\begin{bmatrix} F_{n_1+\cdots+n_m+1}(x) & F_{n_1+\cdots+n_m}(x) \\ F_{n_1+\cdots+n_m}(x) & F_{n_1+\cdots+n_m-1}(x) \end{bmatrix} \begin{bmatrix} F_{n_{m+1}+1}(x) & F_{n_{m+1}}(x) \\ F_{n_{m+1}}(x) & F_{n_{m+1}-1}(x) \end{bmatrix}.$$

Thus, by equating 1,1 entries of the matrices above we obtain $F_{n_1+\cdots+n_{m+1}+1}(x)$ is equal to

$$F_{n_1+\cdots+n_m+1}(x)F_{n_{m+1}+1}(x) + F_{n_1+\cdots+n_m}(x)F_{n_{m+1}}(x).$$

By induction hypothesis, $F_{n_1+\cdots+n_{m+1}+1}(x)$ is equal to

$$\sum_{i_1,\ldots,i_{m-1}=0}^{1} F_{n_1+1-i_1}(x)\cdots F_{n_{m-1}+1-(i_{m-2}+i_{m-1})}(x)U(x),$$

where $U(x) = F_{n_m+1-i_{m-1}}(x)F_{n_{m+1}+1}(x) + F_{n_m-i_{m-1}}(x)F_{n_{m+1}}(x)$. Since the first term in $U(x)$ is the case $i_m = 0$ and the second term is the case $i_m = 1$, we conclude that $F_{n_1+\cdots+n_{m+1}+1}(x)$ is given by

$$\sum_{i_1,\ldots,i_m=0}^{1} F_{n_1+1-i_1}(x)\cdots F_{n_m+1-(i_{m-1}+i_m)}(x)F_{n_m+1-i_m}(x).$$

□

To illustrate Theorem 4.1, note that the m-term Honsberger identity of Fibonacci polynomials consists of products of m Fibonacci polynomials and expands into 2^{m-1} summands. Analogously, the m-term Honsberger identity of Lucas polynomials consists of products of $m-1$ Fibonacci polynomials and 1 Lucas polynomials and expands into 2^{m-1} summands. For example, the case $m = 3$ is given by the formulas

$$F_{n_1+n_2+n_3+1}(x) = \sum_{i_1,i_2=0}^{1} F_{n_1+1-i_1}(x)F_{n_2+1-(i_1+i_2)}(x)F_{n_3+1-i_2}(x),$$
$$L_{n_1+n_2+n_3+1}(x) = \sum_{i_1,i_2=0}^{1} F_{n_1+1-i_1}(x)F_{n_2+1-(i_1+i_2)}(x)L_{n_3+1-i_2}(x).$$

Expanding the sums, we deduce that $F_{n_1+n_2+n_3+1}(x)$ is equal to

$$F_{n_1+1}(x)F_{n_2+1}(x)F_{n_3+1}(x)+F_{n_1+1}(x)F_{n_2}(x)F_{n_3}(x)+F_{n_1}(x)F_{n_2}(x)F_{n_3+1}(x)+F_{n_1}(x)F_{n_2-1}(x)F_{n_3}(x).$$

Analogously we obtain that $L_{n_1+n_2+n_3+1}(x)$ is equal to

$$F_{n_1+1}(x)F_{n_2+1}(x)L_{n_3+1}(x)+F_{n_1+1}(x)F_{n_2}(x)L_{n_3}(x)+F_{n_1}(x)F_{n_2}(x)L_{n_3+1}(x)+F_{n_1}(x)F_{n_2-1}(x)L_{n_3}(x).$$

As an illustration, choosing $n_1 = 1$, $n_2 = 2$ and $n_3 = 3$ yields

$$F_7(x) = F_2(x)F_3(x)F_4(x) + F_2(x)F_2(x)F_3(x) + F_1(x)F_2(x)F_4(x) + F_1(x)F_1(x)F_3(x),$$
$$L_7(x) = F_2(x)F_3(x)L_4(x) + F_2(x)F_2(x)L_3(x) + F_1(x)F_2(x)L_4(x) + F_1(x)F_1(x)L_3(x).$$

Since $F_1(x) = 1$, $F_2(x) = x$, $F_3(x) = x^2+1$, $F_4(x) = x^3+2x$, $L_3(x) = x^3+3x$ and $L_4(x) = x^4+4x^2+2$, we conclude that $F_7(x) = x^6+5x^4+6x^2+1$ and $L_7(x) = x^7+7x^5+14x^3+7x$.

Since $F_n(2) = p_n$ where p_n is the n-th Pell number, we obtain as a corollary the following formula

$$p_{n_1+\cdots+n_m+1} = \sum_{i_1,\ldots,i_{m-1}=0}^{1} p_{n_1+1-i_1}\cdots p_{n_{m-1}+1-(i_{m-2}+i_{m-1})}p_{n_m+1-i_{m-1}}. \tag{4.2}$$

This formula may be referred to as the m-term Honsberger identity for Pell numbers. Since Equation (4.1) was obtained in [23] through a combinatorial interpretation of Fibonacci numbers in terms of square and domino tilings, and Pell numbers are also related to tiling enumeration [24], Equation (4.2) admits a similar combinatorial interpretation. In particular, the summation indices naturally correspond to the binary choices appearing in the combinatorial tiling interpretation of Honsberger-type identities.

5. Conclusions. In this paper, we presented a matrix approach based on the Fibonacci $Q(x)$-matrix to derive identities for Fibonacci and Lucas polynomials. Through the analysis of its powers and inverse, we obtained several classical identities in the polynomial setting, including extensions to negative indices, D'Ocagne-type and Cassini-type identities. The main contribution of this work is the establishment of an m-term Honsberger-type identity for Fibonacci and Lucas polynomials with a matrix proof.

An additional noteworthy aspect of our study is the existence of a family of matrices for which both the determinant and the permanent coincide with those of their inverses. This dual symmetry reveals a remarkable structural property of the Fibonacci $Q(x)$-matrix and may inspire further exploration of similar properties in other matrix families associated with linear recurrences.

The matrix framework highlights the multiplicative structure underlying these polynomial identities and may also provide a useful perspective for teaching topics in linear algebra and discrete mathematics; for example, by incorporating computational exercises aimed at verifying the identities presented in this work.

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