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# Solutions to the Calapso Equation Associated with Surfaces of Revolution

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**Abstract**

*We construct explicit solutions of the Calapso equation by applying Ribaucour transformations to isothermic surfaces of revolution. We derive conditions under which such transformations preserve the isothermic structure and generate new solutions of the Calapso equation. The resulting solutions depend on three functions, two depending on one variable and one depending on another variable. As a consequence, we obtain solutions of the Zoomeron and Davey–Stewartson III equations. Explicit examples associated with the cylinder, cone and catenoid are provided.*

**Keywords** . Calapso equation, Zoomeron and Davey–Stewartson III equations, Ribaucour transformations, isothermic surfaces.

**Resumen**

*Construimos soluciones explícitas de la ecuación de Calapso aplicando transformaciones de Ribaucour a superficies isotermas de revolución. Derivamos condiciones bajo las cuales tales transformaciones preservan la estructura isoterma y generan nuevas soluciones de la ecuación de Calapso. Las soluciones resultantes dependen de tres funciones, dos de una variable y una de otra variable. Como consecuencia, obtenemos soluciones de las ecuaciones de Zoomeron y Davey–Stewartson III. Se proporcionan ejemplos explícitos asociados al cilindro, cono y catenoide.*

**Palabras clave**. Ecuación de Calapso, ecuaciones de Zoomeron y Davey–Stewartson III, transformaciones de Ribaucour, superficies isotermas.

**1. Introduction.** The Ribaucour transformations for hypersurfaces parametrized by lines of curvature were classically studied by Bianchi [1]. They can be applied to obtain surfaces of constant Gaussian curvature and surfaces of constant mean curvature, from a given surface, with constant Gaussian curvature and constant mean curvature, respectively. The first application of this method to minimal and cmc surfaces in $\mathbb{R}^3$ was obtained by Corro, Ferreira, and Tenenblat in [2], [3] and [4]. For more applications of this method, see [5], [6], [7], [8], [9] and [10].

A regular surface $M$ is isothermic if, locally, near to each non umbilical point of M, it admits isothermic parameters, whose coordinate curves are curvature lines. Particular classes of isothermic surfaces are the constant mean curvature surfaces, quadrics surfaces, and surfaces whose lines of curvature have constant geodesic curvature, in particular, the cyclides of Dupin. Isothermic surfaces are preserved by isometries, dilations and inversions. The classification of isothermic surfaces is an open problem.

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In [11], the author showed that for each isothermic surface there is a solution of an equation with partial derivatives of fourth order, called the Calapso equation, given by

$$\left(\frac{\omega_{,u_1u_2}}{w}\right)_{,u_1u_1}+\left(\frac{\omega_{,u_1u_2}}{w}\right)_{,u_2u_2}+\left(\omega^2\right)_{,u_1u_2}=0.$$

This solution depends on the metric and on the mean curvature of the surface. The Calapso equation is very difficult to solve and is strongly connected to the Painleve ODEs, some authors have found solutions of this equation associated with constant mean curvature surfaces.

It is noted that the transition $u_2 \to iu_2$ transforms the Calapso equation into the Zoomeron equation,

$$\left(\frac{\omega_{,u_1u_2}}{w}\right)_{,u_1u_1}-\left(\frac{\omega_{,u_1u_2}}{w}\right)_{,u_2u_2}+\left(\omega^2\right)_{,u_1u_2}=0.$$

Another equation that poses significant challenges when it comes to solving. The solitons of this equation are referred to as Zoomerons, where they possess the properties of Trappon solitons because they are like particles trapped in a potential well, changing direction indefinitely. For further details about Zoomeron equation, refer to the book [12].

In [12], the authors showed that for each $\phi(u_1,u_2)$ solution of the Zoomeron equation, associates a two-parameter family of the solutions $u(u_1,u_2,t)$ given by

$$u=e^{i(\nu u_1+\mu u_2+\mu\nu t)}\phi(u_1+\mu t,u_2+\nu t),\quad \rho=\frac{\left|u\right|_{,u_1u_2}}{|u|},$$

to the Davey-Stewartson III equation

$$iu_{,t}=u_{,u_1u_2}-\rho u,\quad \rho_{,u_1u_1}-\rho_{,u_2u_2}+\left(\left|u\right|^2\right)_{,u_1u_2}=0.$$

This equation was introduced in [13]. It describes the evolution of a three-dimensional wave-packet on water of finite depth in fluid dynamics.

In [14], the authors introduced the class of radial inverse mean curvature surfaces ( RIMC - surfaces ), which are isothermic surfaces. In addition, they show that for each isothermic surface there is another solution to the Calapso equation which depends on the metric and on the skew curvature of the surface. This solution is different from the one presented in [11].

In [15], using Ribaucour transformations, the authors obtained new isothermic surfaces associated with the cylinder and provided explicit examples of solutions for the Calapso and Zoomeron equations.

In [16], using Ribaucour transformations, the authors obtained new isothermic surfaces associated with the flat torus $S^1(r_1)\times S^1(r_2)$ in $S^3$ and provided explicit examples of solutions for the Calapso and Zoomeron equations.

In this work, using Ribaucour transformations, we provide explicit solutions of the Calapso equation obtained from isothermic surfaces locally associated with isothermic surfaces of revolution via Ribaucour transformations. Consequently, we also obtain solutions of the Zoomeron and Davey–Stewartson III equations. These solutions depend on three functions: two depending on one variable and one depending on another variable.

**2. Ribaucour Transformation for Isothermic Surface and Solutions for the Calapso Equation.** In this section, we initially review the theory of Ribaucour transformations for surfaces (refer to [1] and [2] for further details). Posteriorly, we establish a sufficient condition for a Ribaucour transformation to transform one isothermic surface into another. Additionally, we present three new solutions for the Calapso equation corresponding to each isothermic surface.

Let $M$ be an orientable surface of $\mathbb{R}^3$ without umbilic points, whose Gauss map we denote by $N$. We say that $\widetilde{M}$ is associated to $M$ by a Ribaucour transformation, if and only if, there exists a differentiable distance function $h$, defined on $M$ and a diffeomorphism $\psi: M\to\widetilde{M}$

(a) for all $p\in M$, $p+h(p)N(p)=\psi(p)+h(p)\widetilde{N}(\psi(p))$, where $\widetilde{N}$ is the Gauss map of $\widetilde{M}$.

(b)The subset $p+h(p)N(p)$, $p\in M$, is a two-dimensional submanifold.

(c) $\psi$ preserves lines of curvature.

We say that $\widetilde{M}$ is locally associated to $M$ by a *Ribaucour transformation* if, for all $\widetilde{p}\in\widetilde{M}$, there exists a neighborhood of $\widetilde{p}$ in $\widetilde{M}$ which is associated by a Ribaucour transformation to an open subset of $M$.

We observe that, whenever the surface $M$ is parametrized by orthogonal lines of curvature $X(u_1,u_2)$ we get that the distance function $h$ satisfies ( see [2] )

$$h_{,12}-\frac{1+h\lambda_1}{1+h\lambda_2}\Gamma^2_{12}h_{,2}-\frac{1+h\lambda_2}{1+h\lambda_1}\Gamma^1_{12}h_{,1}-\left(\frac{\lambda_2}{1+h\lambda_2}+\frac{\lambda_1}{1+h\lambda_1}\right)h_{,1}\,h_{,2}=0. \tag{2.1}$$

For each nonvanishing function $h$, which is a solution of this equation, there is a differentiable function $\Omega$ ( see Proposition 2.2 in [2]) such that, we define

$$\Omega_i = \frac{\Omega_{,i}}{a_i}, \quad a_i = \sqrt{< X_{,i} \, , X_{,i} >}, \quad 1 \le i \le 2, \qquad W = \frac{\Omega}{h},$$

then the nonlinear equation (2.1) becomes a system of linear differential equations as presented in the following theorem, the proof of which can be found in [2] and [3].

**Theorem 2.1.** *Let $M$ be an orientable surface of $\mathbb{R}^3$ parametrized by $X : U \subseteq \mathbb{R}^2 \to M$, without umbilic points. Assume $e_i = \frac{X_{,i}}{a_i}$, $1 \le i \le 2$ where $a_i = \sqrt{g_{ii}}$ are orthogonal principal directions, $-\lambda_i$ the corresponding principal curvatures, and $N$ is a unit vector field normal to $M$. A surface $\widetilde{M}$ is locally associated to $M$ by a Ribaucour transformation if and only if there is differentiable functions $W, \Omega, \Omega_i : V \subseteq U \to R$ which satisfy*

$$\begin{aligned} \Omega_{i,j} &= \Omega_j \frac{a_{j,i}}{a_j}, \quad for \;\; i \neq j, \\ \Omega_{,i} &= a_i \Omega_i, \\ W_{,i} &= -a_i \Omega_i \lambda_i. \end{aligned} \tag{2.2}$$

*$W(W + \lambda_i \Omega) \neq 0$ and $\widetilde{X} : V \subseteq U \to \widetilde{M}$, is a parametrization of $\widetilde{M}$ given by*

$$\widetilde{X} = X - \frac{2\Omega}{S}\left(\sum_{i=1}^{2} \Omega_i e_i - WN\right), \tag{2.3}$$

*where*

$$S = \sum_{i=1}^{2} (\Omega_i)^2 + W^2. \tag{2.4}$$

*Moreover, the normal map of $\widetilde{X}$ is given by*

$$\widetilde{N} = N + \frac{2W}{S}\left(\sum_{i=1}^{2} \Omega_i e_i - WN\right), \tag{2.5}$$

*and the principal curvatures and coefficients of the first fundamental form of $\widetilde{X}$, are given by*

$$\widetilde{\lambda}_i = \frac{WT_i + \lambda_i S}{S - \Omega T_i}, \qquad \widetilde{g}_{ii} = \left(\frac{S - \Omega T_i}{S}\right)^2 g_{ii} \tag{2.6}$$

*where $\Omega_i$, $\Omega$ and $W$ satisfy (2.2), $S$ is given by (2.4), $g_{ii}$, $1 \le i \le 2$ are coefficients of the first fundamental form of $X$, and*

$$T_1 = \frac{2}{a_1}\left(\Omega_{1,1} + \frac{a_{1,2}}{a_2}\Omega_2 - W\lambda_1 a_1\right), \quad T_2 = \frac{2}{a_2}\left(\Omega_{2,2} + \frac{a_{2,1}}{a_1}\Omega_1 - W\lambda_2 a_2\right). \tag{2.7}$$

**Definition 2.1.** *A surface of $\mathbb{R}^3$ is called isothermic if near each non umbilic point it admits isothermic parameters, whose coordinate curves are curvature lines.*

In [11], the author define the Calapso equation by

$$\left(\frac{\phi_{,u_1u_2}}{w}\right)_{,u_1u_1} + \left(\frac{\phi_{,u_1u_2}}{w}\right)_{,u_2u_2} + \left(\omega^2\right)_{,u_1u_2} = 0. \tag{2.8}$$

Such equation describes isothermic surfaces in $\mathbb{R}^3$, where $\phi_{,12}$ denotes the derivative of $\phi$ with respect to $u_1$ and $u_2$. The Calapso equation is very difficult to solve and is strongly connected to the Painleve ODEs, some authors have found solutions of this equation associated with constant mean curvature surfaces.

The following result provides solutions of the Calapso equation. ( see [11] and [14] for a proof and more details).

**Theorem 2.2.** *Let $X(u_1, u_2)$ be an isothermic surface with the first fundamental form given by*

$$I = e^{2\varphi}(du_1^2 + du_2^2).$$

*Then, the functions $\omega = \sqrt{2}e^{\varphi}H$ and $\Omega = \sqrt{2}e^{\varphi}H'$ are solutions of the Calapso equation, where $H$ is the mean curvature of $X$ and $H'$ is the skew curvature of $M$.*

The next Theorem, we provide a sufficient condition for a Ribaucour transformation to transform an isothermic surface into another such surface and provide solutions of the Calapso equation for each given isothermic surface.

**Theorem 2.3.** *Let $M$ be a surface of $\mathbb{R}^3$ parametrized by $X : U \subseteq \mathbb{R}^2 \to M$, without umbilic points and let $\widetilde{M}$ be associated to $M$ by a Ribaucour transformation, such that the normal lines intersect at a distance function $h$. Assume that $h = \frac{\Omega}{W}$ is not constant along the lines of curvature and consider $S$ and $T_i$, $1 \leq i \leq 2$, given by (2.4) and (2.7). In this case, we have*

*(i) If $T_1 + T_2 = \dfrac{2S}{\Omega}$ or $T_1 - T_2 = 0$, then $\widetilde{M}$ is a isothermic surface, if and only if $M$ is isothermic surface.*

*(ii) For each isothermic surface $M$, whose first fundamental form is given by $I = e^{2\varphi}\big(du_1^2 + du_2^2\big)$, if $T_1 + T_2 = \dfrac{2S}{\Omega}$, then the functions*

$$\phi_1 = \frac{\epsilon\sqrt{2}e^{\varphi}\big(2W(S - \Omega T_1) + S\Omega(\lambda_2 - \lambda_1)\big)}{2\Omega S}, \tag{2.9}$$

$$\Phi_1 = \frac{-\epsilon\sqrt{2}e^{\varphi}\big(2W + \Omega(\lambda_2 + \lambda_1)\big)}{2\Omega}, \tag{2.10}$$

*are solutions of the Calapso equation. Moreover, if $T_1 - T_2 = 0$, then the function*

$$\phi_2 = \frac{\epsilon\sqrt{2}e^{\varphi}\big(-2WT_1 - S(\lambda_1 + \lambda_2)\big)}{2S}, \tag{2.11}$$

*is also a solution to the Calapso equation, where $\epsilon^2 = 1$ and $-\lambda_i$ $1 \leq i \leq 2$ are the principal curvatures of the $M$.*

*Proof:*

(i): Suppose that $\widetilde{M}$ is a isothermic surface, then the coefficients of the first fundamental form of $\widetilde{X}$ satisfy, $\widetilde{g}_{11} = \widetilde{g}_{22}$. So, using (2.6), we have

$$\left(\frac{S - \Omega T_1}{S}\right)^2 g_{11} = \left(\frac{S - \Omega T_2}{S}\right)^2 g_{22}, \tag{2.12}$$

where $g_{ii}$, $1 \leq i \leq 2$ are the coefficients of the first fundamental form of $X$.
If $T_1 + T_2 = \frac{2S}{\Omega}$, then isolating $T_1$ and substituting in (2.12), we get $g_{11} = g_{22}$.
On the other hand, if $T_1 - T_2 = 0$, then we have from (2.12) that $g_{11} = g_{22}$. Therefore, $M$ is a isothermic surface.
Conversely, suppose that $M$ is a isothermic surface, then using that $T_1 + T_2 = \dfrac{2S}{\Omega}$ or $T_1 - T_2 = 0$, immediately from (2.6), we obtain that $\widetilde{M}$ is a isothermic surface.

(ii): For each isothermic surface $M$, whose first fundamental form is given by $I = e^{2\varphi}\big(du_1^2 + du_2^2\big)$, let $\widetilde{M}$ be the isothermic surface associated to $M$ by a Ribaucour transformation.
If $T_1 + T_2 = \dfrac{2S}{\Omega}$, then using (2.6) we have that the first fundamental form of $\widetilde{X}$ is given by $\widetilde{I} = \psi^2\big(du_1^2 + du_2^2\big)$, where $\psi = \dfrac{|S - \Omega T_2|e^{\varphi}}{S}$. Additionally, isolating $T_1$, we have

$$T_1 = \frac{2S - \Omega T_2}{\Omega}. \tag{2.13}$$

Using (2.6), we obtain that the mean and skew curvatures of $\widetilde{X}$ are given by

$$\begin{aligned} \widetilde{H} &= \frac{1}{2\Omega(S - \Omega T_2)}\Big(2W(S - \Omega T_2) + S\Omega(\lambda_2 - \lambda_1)\Big), \\ \widetilde{H}' &= \frac{S}{2\Omega(S - \Omega T_2)}\Big(2W + \Omega(\lambda_2 + \lambda_1)\Big). \end{aligned} \tag{2.14}$$

Therefore, by Theorem 2.2, the functions given by (2.9) and (**??**) are solutions of the Calapso equation.

Proceeding in a similar way, if $T_1 - T_2 = 0$, then using using (2.6), we obtain that the mean and skew curvatures of $\widetilde{X}$ are given by

$$\begin{aligned} \widetilde{H} &= \frac{2WT_2 + S(\lambda_1+\lambda_2)}{2(S-\Omega T_2)}, \\ \widetilde{H}' &= \frac{S(\lambda_2-\lambda_1)}{2(S-\Omega T_2)}, \end{aligned} \tag{2.15}$$

Using the Theorem 2.2, we conclude the proof.

Next, we make two remarks that will be useful in the subsequent section.

**Remark 2.1.** *Let $X: U \subseteq \mathbb{R}^2 \to M$ be a parameterization for the isothermic surface $M$. Thus, the first fundamental form of $X$ is given by $I = e^{2\varphi}\left(du_1^2 + du_2^2\right)$. Therefore, the additional relations given by $T_1 + T_2 = \dfrac{2S}{\Omega}$ and $T_1 - T_2 = 0$ are, respectively, equivalent to*

$$\Delta\Omega - e^{2\varphi}(\lambda_1+\lambda_2)W = \frac{Se^{2\varphi}}{\Omega} \tag{2.16}$$

$$\Omega_{,11} - \Omega_{,22} - 2\varphi_{,1}\Omega_{,1} + 2\varphi_{,2}\Omega_{,2} - e^{2\varphi}(\lambda_1-\lambda_2)W = 0 \tag{2.17}$$

*In fact, under these conditions using (2.2), $T_i$ $1 \le i \le 2$ given by (2.7), can be rewritten as*

$$\begin{aligned} T_1 &= \frac{2}{e^{2\varphi}}\Big(\Omega_{,11} - \varphi_{,1}\Omega_{,1} + \varphi_{,2}\Omega_{,2} - W\lambda_1 e^{2\varphi}\Big), \\ T_2 &= \frac{2}{e^{2\varphi}}\Big(\Omega_{,22} - \varphi_{,2}\Omega_{,2} + \varphi_{,1}\Omega_{,1} - W\lambda_2 e^{2\varphi}\Big). \end{aligned} \tag{2.18}$$

*Therefore, $T_1 + T_2 = \frac{2}{e^{2\varphi}}\left(\Delta\Omega - e^{2\varphi}W(\lambda_1+\lambda_2)\right)$ and the additional relation $T_1 + T_2 = \frac{2S}{\Omega}$ is equivalent to (2.16). Finally, by substituting (2.18) in $T_1 - T_2 = 0$, we obtain (2.17).*

**Remark 2.2.** *Let $X$ be as in the previous remark. Then, the parameterization $\widetilde{X}$ of $\widetilde{M}$, locally associated to $X$ by a Ribaucour transformation, given by (2.3), is defined on*

$$V = \{(u_1,u_2) \in U;\ \Omega T_2 - S \neq 0\}.$$

**3. Solutions of the Calapso Equation Obtained from Isothermic Surfaces of Revolution.** In this section, we will provide new solutions for the Calapso equation that depend on two given functions from an isothermic surface of revolution. Consequently providing to new solutions for the Zoomeron and Davey–Stewartson III equations as well.

The following theorem, we apply Theorem 2.3 to the isothermic surfaces associated to a surface of revolution by Ribaucour transformations. This way, we obtain solutions of the Calapso equation that dedepend on three functions, two depending on $u_1$ and and one depending on $u_2$.

**Theorem 3.1.** *For each function $\varphi = \varphi(u_1)$ satisfying $1 - (\varphi')^2 > 0$. The functions*

$$\phi = \frac{\epsilon\sqrt{2}e^{\varphi}\left(\lambda_2-\lambda_1\right)}{2} + \frac{\epsilon\sqrt{2}(1-c)\left(\lambda_2 e^{\varphi}(f_1+f_2) - g_1\right)}{2b + f_1 + f_2 + c(f_1-f_2) + 2\varphi' f_1' - 2\lambda_2 g_1 e^{\varphi}}, \tag{3.1}$$

$$\Phi = \frac{\epsilon\sqrt{2}g_1}{f_1+f_2} + \frac{\epsilon\sqrt{2}e^{\varphi}\left(\lambda_1-\lambda_2\right)}{2}, \tag{3.2}$$

*are also solutions of the Calapso equation, where $c \neq 1$ is a constant and $f_1 = f_1(u_1)$, $f_2 = f_2(u_2)$ and $g_1 = g_1(u_1)$ satisfy*

$$f_2'' + cf_2 = b, \tag{3.3}$$

$$(\lambda_2-\lambda_1)e^{\varphi}f_1' - g_1' = 0, \tag{3.4}$$

$$f_1'' - cf_1 + (\lambda_2-\lambda_1)e^{\varphi}g_1 = b, \tag{3.5}$$

*with initial conditions satisfying*

$$\left[\left(f_1'\right)^2 - cf_1^2 - 2bf_1 + g_1^2 + \left(f_2'\right)^2 + cf_2^2 - 2bf_2\right](u_1^0,u_2^0) = 0. \tag{3.6}$$

*Moreover, the function*

$$\phi_2 = \frac{2\epsilon\sqrt{2}\left(\lambda_2 e^{\varphi}(f_1+f_2) - g_1\right)\left(b + f_1 - g_1\lambda_2 e^{\varphi}\right)}{c_2 + 2(f_1+f_2)\left(b + f_1 - g_1\lambda_2 e^{\varphi} + \varphi' f_1'\right)} - \frac{\epsilon\sqrt{2}e^{\varphi}\left(\lambda_2+\lambda_1\right)}{2}, \tag{3.7}$$

*is solution of the Calapso equation, where* $\epsilon^2 = 1$, $c_2 = A_1^2 + A_2^2 - b^2 + c_1$ *are real constants,* $f_2 = A_1 \cos(u_2) + A_2 \sin(u_2) + b$, $f_1 = f_1(u_1)$ *and* $g_1 = g_1(u_1)$ *satisfy (3.4) and (3.5) with* $c = 1$ *and the initial conditions satisfy*

$$\left[(f_1')^2 - f_1^2 - 2bf_1 + g_1^2\right](u_1^0, u_2^0) = c_1, \tag{3.8}$$

*and*

$$\lambda_1 = \frac{\varphi''}{B'}, \qquad \lambda_2 = -e^{-2\varphi} B', \tag{3.9}$$

*where* $B = \delta \int e^{\varphi} \sqrt{1 - (\varphi')^2}\, du_1$, *with* $\delta^2 = 1$.

We will prove Theorem 3.1 later in this section using four lemmas. The proof of Theorem 3.1 is quite technical.

The first step, given in Lemma 3.1, consists in proving that there exist three functions that describe a surface $\widetilde{X}$ locally associated to a surface of revolution by Ribaucour transformations. For this, we make use of Theorem 2.1. The second step involves obtaining sufficient conditions for $\widetilde{X}$ to be an isothermic surface. For this purpose, we employ Theorem 2.3, providing Lemmas 3.2 and 3.3. Finally, the third step, as given in Lemma 3.4, consists of providing the principal curvatures and the coefficients of the first fundamental form of the isothermic surface $\widetilde{X}$, locally associated to a surface of revolution by Ribaucour transformations. With these four lemmas in hand, using Theorem 2.3, we prove Theorem 3.1.

Before stating and proving the first lemma, we begin with a brief description of a surface of revolution that is isothermic.

Let $X$ be a surface of revolution parametrized by

$$X(u_1, u_2) = \left(e^{\varphi(u_1)} \cos(u_2),\ e^{\varphi(u_1)} \sin(u_2),\ B(u_1)\right).$$

Suppose that $X$ is an isothermic surface, then the functions $B = B(u_1)$ and $\varphi = \varphi(u_1)$ satisfy

$$(\varphi')^2 + (B')^2 e^{-2\varphi} = 1. \tag{3.10}$$

Moreover the first and second fundamental forms of $X$ are

$$I = e^{2\varphi}\left(du_1^2 + du_2^2\right), \qquad II = e^{2\varphi}\left(\lambda_1 du_1^2 + \lambda_2 du_2^2\right), \tag{3.11}$$

where $\lambda_1$, $\lambda_2$ are the principal curvatures of $X$, given by

$$\lambda_1 = e^{-2\varphi}\Big( - B'(\varphi'' + (\varphi')^2) + \varphi' B''\Big), \qquad \lambda_2 = e^{-2\varphi} B', \tag{3.12}$$

where $\varphi$, $\lambda_1$ and $\lambda_2$ satisfy the Gauss and Codazzi equations

$$\varphi'' + \lambda_1 \lambda_2 e^{2\varphi} = 0, \qquad \lambda_{2,1} = \varphi'(\lambda_1 - \lambda_2). \tag{3.13}$$

**Lemma 3.1.** *Consider the surface of revolution parametrized by*

$$X(u_1, u_2) = \left(e^{\varphi(u_1)} \cos(u_2),\ e^{\varphi(u_1)} \sin(u_2),\ B(u_1)\right), \tag{3.14}$$

*and suppose that* $X$ *is a isothermic surface. Let* $\widetilde{X}$ *be the surface locally associated to* $X$ *by a Ribaucour transformation as in Theorem 2.1. Then, the functions* $\Omega$ *and* $W$ *satisfying (2.2) are given by*

$$\Omega = e^{\varphi}\left(f_1(u_1) + f_2(u_2)\right), \tag{3.15}$$

$$W = -\lambda_2 \Omega + g_1(u_1), \tag{3.16}$$

*with*

$$(\lambda_2 - \lambda_1) e^{\varphi} f_1' - g_1' = 0, \tag{3.17}$$

*where* $\lambda_1$ *and* $\lambda_2$ *are the principal curvatures of* $X$, *given by (3.12).*

*Proof:* Let $\Omega$ and $W$ be functions that satisfy (2.2). We deduce that $W_{,2} = -\lambda_2 \Omega_{,2}$, resulting in $W = -\lambda_2 \Omega + g_1(u_1)$.

Furthermore, note that $\Omega_{2,1} = 0$, and as we obtain the expression for $W$, we get $\Omega = e^{\varphi}\big(f_1(u_1) + f_2(u_2)\big)$.

Finally, differentiating $W = -\lambda_2\Omega + g_1(u_1)$ with respect to $u_1$ and using the third equation of (2.2) for $i = 1$, along with Codazzi equation given by (3.13), we obtain (3.17).

**Lemma 3.2.** *Consider the surface of revolution $X$ parametrized by (3.14) as an isothermic surface. A parametrized surface $\widetilde{X}(u_1, u_2)$ is an isothermic surface locally associated to $X$ by a Ribaucour transformation as in Theorem 2.3 with additional relation given by $T_1 + T_2 = \dfrac{2S}{\Omega}$, if and only if, up to a rigid motion of $\mathbb{R}^3$, it is given by (2.3) where $S = \big(\Omega_1\big)^2 + \big(\Omega_2\big)^2 + W^2$, $\Omega_i = e^{-\varphi}\Omega_{,i}$, $\Omega = e^{\varphi}\big(f_1 + f_2\big)$, $W = -\lambda_2\Omega + g_1$, $\lambda_i$, $1 \le i \le 2$ are given by (3.12) and $f_1 = f_1(u_1)$, $f_2 = f_2(u_2)$ and $g_1 = g_1(u_1)$ satisfy*

$$\begin{aligned} &f_2'' + cf_2 = b, \\ &(\lambda_2 - \lambda_1)e^{\varphi} f_1' - g_1' = 0, \\ &f_1'' - cf_1 + (\lambda_2 - \lambda_1)e^{\varphi} g_1 = b. \end{aligned} \tag{3.18}$$

*$c \neq 1$ and $b$ are real constants. With initial conditions satisfying*

$$\big[\big(f_1'\big)^2 - cf_1^2 - 2bf_1 + g_1^2 + \big(f_2'\big)^2 + cf_2^2 - 2bf_2\big](u_1^0, u_2^0) = 0. \tag{3.19}$$

*Proof:* From Lemma 3.1, we have that

$$\Omega = e^{\varphi}\big(f_1(u_1) + f_2(u_2)\big), \tag{3.20}$$

$$W = -\lambda_2\Omega + g_1(u_1), \tag{3.21}$$

with

$$(\lambda_2 - \lambda_1)e^{\varphi} f_1' - g_1' = 0, \tag{3.22}$$

where $\lambda_1$ and $\lambda_2$ are the principal curvatures of $X$, given by (3.12).

Using the Theorem 2.1, consider $S = (\Omega_1)^2 + (\Omega_2)^2 + W^2$, where $\Omega_i = e^{-\varphi}\Omega_{,i}$. Thus

$$\frac{S}{\Omega} = e^{-2\varphi}\Omega + 2(\varphi' f_1' e^{-\varphi} - \lambda_2 g_1) + \frac{(f_1')^2 + (f_2')^2 + g_1^2}{\Omega}. \tag{3.23}$$

Using (2.16), the associated surface will be isothermic when

$$\Delta\Omega - e^{2\varphi}\big(\lambda_1 + \lambda_2\big)W = \frac{Se^{2\varphi}}{\Omega}.$$

Therefore, we obtain that the functions $f_1$, $f_2$ and $g_1$ satisfy

$$f_2'' + f_1'' + (\lambda_2 - \lambda_1)g_1 e^{\varphi} = \frac{(f_1')^2 + (f_2')^2 + g_1^2}{f_1 + f_2} \tag{3.24}$$

Differentiating (3.24) with respect to $u_1$ and $u_2$ and using (3.17) we obtain

$$\frac{f_2'''}{f_2'} = \frac{-\big(-f_2'' + f_1'' + (\lambda_2 - \lambda_1)e^{\varphi} g_1\big)}{f_1 + f_2} \tag{3.25}$$

$$\frac{\big(f_1'' + (\lambda_2 - \lambda_1)e^{\varphi} g_1\big)'}{f_1'} = \frac{\big(-f_2'' + f_1'' + (\lambda_2 - \lambda_1)e^{\varphi} g_1\big)}{f_1 + f_2}. \tag{3.26}$$

Thus, $\dfrac{\big(-f_2'' + f_1'' + (\lambda_2 - \lambda_1)e^{\varphi} g_1\big)}{f_1 + f_2} = c$, $c \neq 1$ is a constant.

Therefore, we conclude that $f_1$, $f_2$ and $g_1$ satisfy the equations (3.18).

**Lemma 3.3.** *Consider the surface of revolution $X$ parametrized by (3.14) as an isothermic surface. A parametrized surface $\widetilde{X}(u_1, u_2)$ is an isothermic surface locally associated to $X$ by a Ribaucour transformation as in Theorem 2.3 with additional relation given by $T_1 - T_2 = 0$, if and only if, up to a rigid motion of $\mathbb{R}^3$, it is given by (2.3) where $S = \big(\Omega_1\big)^2 + \big(\Omega_2\big)^2 + W^2$, $\Omega_i = e^{-\varphi}\Omega_{,i}$, $\Omega = e^{\varphi}\big(f_1 + f_2\big)$, $W = -\lambda_2\Omega + g_1$, $\lambda_i$, $1 \le i \le 2$ are given by (3.12) and $f_1 = f_1(u_1)$, $f_2 = f_2(u_2)$ and $g_1 = g_1(u_1)$ satisfy*

$$\begin{aligned} &f_2 = A_2\cos(u_2) + B_2\sin(u_2) + b, \\ &(\lambda_2 - \lambda_1)e^{\varphi} f_1' - g_1' = 0, \\ &f_1'' - f_1 + (\lambda_2 - \lambda_1)e^{\varphi} g_1 = b. \end{aligned} \tag{3.27}$$

*b are real constants. With initial conditions satisfying*

$$\left[(f_1')^2 - f_1^2 - 2bf_1 + g_1^2\right](u_1^0, u_2^0) = c_1. \tag{3.28}$$

*Proof:* Proceeding as in the proof of Lemma 3.2, by using (2.17), we have

$$f_2'' + f_2 = f_1'' - f_1 + (\lambda_2 - \lambda_1)e^{\varphi} g_1. \tag{3.29}$$

Thus, $f_2 = A_1 \cos(u_2) + A_2 \sin(u_2) + b$ and $f_1$ satisfies $f_1'' - f_1 + (\lambda_2 - \lambda_1)e^{\varphi} g_1 = b$. From Lemma 3.1, we have $(\lambda_2 - \lambda_1)e^{\varphi} f_1' - g_1' = 0$. Concluding thus the proof.

Next, we have the final lemma that will provide some expressions that will be useful in the proof of Theorem 3.1.

**Lemma 3.4.** *Consider $\widetilde{X}$ to be the isothermic surface associated to the isothermic surface of revolution by Ribaucour transformations, as given by Lemma 3.2, then the first fundamental form of $\widetilde{X}$, $\widetilde{I} = F^2(du_1^2 + du_2^2)$ and the principal curvatures, are given by*

$$F = \frac{|(c-1)(f_1 + f_2)e^{\varphi}|}{|2b + f_1 + f_2 + c(f_1 - f_2) + 2\varphi' f_1' - 2\lambda_2 g_1 e^{\varphi}|}, \tag{3.30}$$

$$\widetilde{\lambda}_1 = \frac{-WT_1 - \lambda_1 S}{S - \Omega T_2}, \qquad \widetilde{\lambda}_2 = \frac{WT_2 + \lambda_2 S}{S - \Omega T_2}, \tag{3.31}$$

*with*

$$T_2 = 2e^{-\varphi}\left(b - cf_2 + f_1 + f_2 + \varphi' f_1' - \lambda_2 g_1 e^{\varphi}\right), \tag{3.32}$$

*where the functions $\varphi$, $\Omega$, $W$, $\lambda_1$, $\lambda_2$, $f_1$ and $f_2$ are given by Lemma 3.2.*

*Proof:* In fact, from Theorem 2.1, the coefficients of the first fundamental form and the principal curvatures of $\widetilde{X}$ are given by

$$\widetilde{\lambda}_i = \frac{WT_i + \lambda_i S}{S - \Omega T_i}, \qquad \widetilde{g}_{ii} = \left(\frac{S - \Omega T_i}{S}\right)^2 g_{ii} \quad 1 \le i \le 2. \tag{3.33}$$

Since $X$ is a isothermic surface of revolution given by (3.14), then $g_{ii} = e^{2\varphi}$ and from Lemmas 3.1 and 3.2, using (2.18) we have that $T_2$ is given by (3.32) and $S - \Omega T_1 = -(S - \Omega T_2)$, thus we obtain (3.31). Moreover, from (3.23), (3.24), (3.3) and (3.5), we get

$$\frac{S}{\Omega} = e^{-\varphi}\left(2b + f_1 + f_2 + c(f_1 - f_2) + 2\varphi' f_1' - 2\lambda_2 g_1 e^{\varphi}\right).$$

This last equation and (3.32), we obtain

$$S = (f_1 + f_2)\left(2b + f_1 + f_2 + c(f_1 - f_2) + 2\varphi' f_1' - 2\lambda_2 g_1 e^{\varphi}\right), \tag{3.34}$$

$$\frac{S}{\Omega} - T_2 = (c-1)e^{-\varphi}(f_1 + f_2). \tag{3.35}$$

Substituting these two equations in (3.33), we have that the coefficients of the first fundamental of $\widetilde{X}$ is given by (3.30).

Proof of Theorem 3.1. Consider $\widetilde{X}$ to be the isothermic surface associated to the isothermic surface of revolution by Ribaucour transformations, as given by Lemma 3.2. By Theorem 2.3, assuming that $T_1 + T_2 = \dfrac{2S}{\Omega}$, we have that the functions

$$\phi_1 = \frac{\epsilon\sqrt{2}e^{\varphi}\left(2W(S - \Omega T_2) + S\Omega(\lambda_2 - \lambda_1)\right)}{2\Omega S}, \tag{3.36}$$

$$\Phi_1 = \frac{-\epsilon\sqrt{2}e^{\varphi}\left(2W + \Omega(\lambda_2 + \lambda_1)\right)}{2\Omega}, \tag{3.37}$$

are solutions of the Calapso equation, where $W$ and $\Omega$ are given by Lemma 3.1. Using (3.34) and (3.35), we obtain that the functions (3.2) and (3.1) are solutions of the Calapso equation, where by Lemma 3.2, $f_1 = f_1(u_1)$, $f_2 = f_2(u_2)$, and $g_1 = g_1(u_1)$ satisfy (3.3)-(3.5).

Consider $\widetilde{X}$ to be the isothermic surface associated to the isothermic surface of revolution by Ribaucour transformations, as given by Lemma 3.3. By Theorem 2.3, assuming that $T_1 - T_2 = 0$, we have that the function

$$\phi_2 = \frac{\epsilon\sqrt{2}e^{\varphi}\big(-2WT_1 - S(\lambda_1+\lambda_2)\big)}{2S},$$

is solution of the Calapso equation. Proceeding in a similar manner to what was proven in the previous case, using the Lemma 3.3, we obtain that

$$\phi_2 = \frac{2\epsilon\sqrt{2}\big(\lambda_2 e^{\varphi}(f_1+f_2) - g_1\big)\big(b + f_1 - g_1\lambda_2 e^{\varphi}\big)}{c_2 + 2(f_1+f_2)\big(b+f_1-g_1\lambda_2 e^{\varphi} + \varphi' f_1'\big)} - \frac{\epsilon\sqrt{2}e^{\varphi}\big(\lambda_2+\lambda_1\big)}{2}. \tag{3.38}$$

Where $f_2 = A_1\cos(u_2) + A_2\sin(u_2) + b$, $f_1 = f_1(u_1)$, and $g_1 = g_1(u_1)$ satisfy (3.27), with initial conditions, satisfying

$$\big[(f_1')^2 - f_1^2 - 2bf_1 + g_1^2\big](u_1^0, u_2^0) = c_1.$$

□

**Remark 3.1.** *For each solution of the Calapso equation given by Theorem 3.1, we can obtain solutions of the Zoomeron and Davey–Stewartson III equations.*

1. *The transformation $u_2 \to iu_2$ transforms the Calapso equation into the Zoomeron equation*

$$\left(\frac{\omega_{,u_1u_2}}{w}\right)_{,u_1u_1} - \left(\frac{\omega_{,u_1u_2}}{w}\right)_{,u_2u_2} + \left(\omega^2\right)_{,u_1u_2} = 0.$$

*Therefore, if $\phi(u_1,u_2)$ is a solution of the Calapso equation, then $\Phi(u_1,u_2) = \phi(u_1, iu_2)$ is a solution of the Zoomeron equation.*

2. *For each solution $\Phi(u_1,u_2)$ of the Zoomeron equation, there corresponds a two-parameter family of solutions $u(u_1,u_2,t)$ given by*

$$u = e^{i(\nu u_1 + \mu u_2 + \mu\nu t)}\phi(u_1+\mu t, u_2+\nu t), \qquad \rho = \frac{\big|u\big|_{,u_1u_2}}{|u|},$$

*which solve the Davey–Stewartson III equation*

$$iu_{,t} = u_{,u_1u_2} - \rho u, \qquad \rho_{,u_1u_1} - \rho_{,u_2u_2} + \left(\big|u\big|^2\right)_{,u_1u_2} = 0.$$

**4. Examples.** In this section, we will apply Theorem 3.1 to obtain explicit solutions of the Calapso equation and, consequently, of the Zoomeron and Davey–Stewartson III equations.

**Example 4.1.** *If $e^{\varphi} = 1$ e $B = u_1$. Then the functions*

$$\phi = -\frac{\epsilon\sqrt{2}}{2} + \epsilon\sqrt{2}\,(c-1)\,\frac{g}{2b_1 + (c-1)(f-g)}, \tag{4.1}$$

$$\Phi = -\epsilon\sqrt{2}\,\frac{f}{f+g} + \frac{\epsilon\sqrt{2}}{2}, \tag{4.2}$$

*are solutions to the Calapso equation with $c \neq 1$ and $f = f(u_1)$ and $g = g(u_2)$ functions given by*

$$f(u_1) = \begin{cases} \dfrac{b_1}{1-c} + A_1\cos\big(\sqrt{1-c}\,u_1\big) + B_1\sin\big(\sqrt{1-c}\,u_1\big), & c<1,\\[2mm] -\dfrac{b_1}{c-1} + A_1\cosh\big(\sqrt{c-1}\,u_1\big) + B_1\sinh\big(\sqrt{c-1}\,u_1\big), & c>1, \end{cases} \tag{4.3}$$

$$g(u_2) = \begin{cases} \dfrac{b_1}{c} + A_2\cos\big(\sqrt{c}\,u_2\big) + B_2\sin\big(\sqrt{c}\,u_2\big), & c>0,\ c\neq 1,\\[2mm] \dfrac{b_1}{2}u_2^2 + A_2u_2 + B_2, & c=0,\\[2mm] \dfrac{b_1}{c} + A_2\cosh\big(\sqrt{-c}\,u_2\big) + B_2\sinh\big(\sqrt{-c}\,u_2\big), & c<0. \end{cases} \tag{4.4}$$

*with the constants $A_1$, $B_1$, $c$, and $b_1$ satisfying*

$$\begin{cases} (c-1)(B_1^2-A_1^2)+\dfrac{b_1^2}{c-1} + c(A_2^2+B_2^2)-\dfrac{b_1^2}{c}=0, & c>1,\\ (1-c)(A_1^2+B_1^2)-\dfrac{b_1^2}{1-c} + c(A_2^2+B_2^2)-\dfrac{b_1^2}{c}=0, & 0<c<1,\\ A_1^2+B_1^2+A_2^2-b_1^2-2b_1B_2=0, & c=0,\\ (1-c)(A_1^2+B_1^2)-\dfrac{b_1^2}{1-c} + c(A_2^2-B_2^2)-\dfrac{b_1^2}{c}=0, & c<0. \end{cases} \tag{4.5}$$

*Moreover, if $c=1$, the function*

$$\phi_2=\frac{-2\epsilon\sqrt{2}\,(b+c_3)\,g(u_2)}{c_2+2(b+c_3)\big(f(u_1)+g(u_2)\big)}+\frac{\epsilon\sqrt{2}}{2}. \tag{4.6}$$

*is also solution to the Calapso equation, where*

$$f(u_1) = \frac{b+c_3}{2}u_1^2+A_2u_1+B_2, \tag{4.7}$$
$$g(u_2) = A_1\cos u_2+B_1\sin u_2+b, \tag{4.8}$$

*with the constants satisfying $c_2=A_1^2+A_2^2+B_1^2+c_3^2-b^2-2(b+c_3)B_2$.*

*In fact, the equation (3.4) becomes $-f_1'-g_1'=0$, since $\lambda_1=0$ and $\lambda_2=-1$. Thus $g_1=-f_1+c_3$ and from (3.5) we get*

$$f_1''-(-1+c)f_1-c_3=b.$$

*Adding term by term $cc_3$ to this last equation and also to (3.3), we obtain*

$$f_1''-(-1+c)f_1-c_3+cc_3=b+cc_3, \tag{4.9}$$
$$f_2''+cf_2+cc_3=b+cc_3. \tag{4.10}$$

*Defining $f=f_1-c_3$, $g=f_2+c_3$ and when $c\neq 1$ by defining $b_1=b+cc_1$, we have*

$$f''-(c-1)f=b_1, \tag{4.11}$$
$$g''+cg=b_1. \tag{4.12}$$

*Therefore, by Theorem 3.1, if $c\neq 1$ we obtain (4.1) and (4.2), and if $c=1$ we obtain (4.6), as solutions of the Calapso equation. Moreover, Theorem 3.1 implies that the constants satisfy (4.5) if $c\neq 1$, and $c_2=A_1^2+A_2^2+B_1^2+c_3^2-b^2-2c_3B_2-2bB_2$ if $c=1$.*

In the next application of Theorem 3.1, we take $\varphi=\frac{\sqrt{2}\,u_1}{2}$ and $B=e^{\varphi}$. Whose proof is an immediate consequence of Theorem 3.1 with arguments similar to those used in Example 4.1.

**Example 4.2.** *Consider $\varphi=\frac{\sqrt{2}\,u_1}{2}$ and $B=e^{\varphi}$. Then the functions*

$$\phi=-\frac{\epsilon}{2}-\frac{\epsilon(1-c)g}{2b_1+cf+(1-c)g+\sqrt{2}\,f'}, \tag{4.13}$$

$$\Phi=-\frac{\epsilon f}{f+g}+\frac{\epsilon}{2}. \tag{4.14}$$

*are solutions to the Calapso equation with $c\neq 1$ and $f=f(u_1)$ and $g=g(u_2)$ functions given by*

$$f(u_1)=\begin{cases} A_1\cos\big(\sqrt{c-\frac12}\,u_1\big)+B_1\sin\big(\sqrt{c-\frac12}\,u_1\big)+\dfrac{b_1}{c-\frac12}, & c>1,\\ A_1\cos\big(\sqrt{c-\frac12}\,u_1\big)+B_1\sin\big(\sqrt{c-\frac12}\,u_1\big)+\dfrac{b_1}{c-\frac12}, & \frac12<c<1,\\ \dfrac{b_1}{2}\,u_1^2+A_1u_1+B_1, & c=\frac12,\\ A_1\cosh\big(\sqrt{\frac12-c}\,u_1\big)+B_1\sinh\big(\sqrt{\frac12-c}\,u_1\big)+\dfrac{b_1}{c-\frac12}, & c<\frac12. \end{cases} \tag{4.15}$$

$$g(u_2)=\begin{cases} A_2\cos\big(\sqrt{c}\,u_2\big)+B_2\sin\big(\sqrt{c}\,u_2\big)+\dfrac{b_1}{c}, & c>0,\ c\neq 1,\\ \dfrac{b_1}{2}\,u_2^2+A_2u_2+B_2, & c=0,\\ A_2\cosh\big(\sqrt{-c}\,u_2\big)+B_2\sinh\big(\sqrt{-c}\,u_2\big)+\dfrac{b_1}{c}, & c<0.\end{cases} \tag{4.16}$$

*with the constants $A_1$, $B_1$, $c$, and $b_1$ satisfying*

$$\begin{cases} \left(c-\frac{1}{2}\right)(B_1^2-A_1^2)+\dfrac{b_1^2}{c-\frac{1}{2}}+c(A_2^2+B_2^2)-\dfrac{b_1^2}{c}=0, & c>\frac{1}{2},\\ A_1^2-2b_1B_1+\frac{1}{2}(A_2^2+B_2^2)-2b_1^2=0, & c=\frac{1}{2},\\ \left(\frac{1}{2}-c\right)(A_1^2+B_1^2)-\dfrac{b_1^2}{\frac{1}{2}-c}+c(A_2^2+B_2^2)-\dfrac{b_1^2}{c}=0, & 0<c<\frac{1}{2},\\ \frac{1}{2}(A_1^2+B_1^2)-2b_1^2+A_2^2-2b_1B_2=0, & c=0,\\ \left(\frac{1}{2}-c\right)(A_1^2+B_1^2)-\dfrac{b_1^2}{\frac{1}{2}-c}+c(A_2^2-B_2^2)-\dfrac{b_1^2}{c}=0, & c<0.\end{cases}$$

*Moreover, if $c=1$, the function*

$$\phi_2 \;=\; -\frac{\epsilon(g+c_3)(2b+f+c_3)}{c_2+(f+g)\big(2b+f+c_3+\sqrt{2}\,f'\big)}+\frac{\epsilon}{2}. \tag{4.17}$$

*is also solution to the Calapso equation, where*

$$f(u_1) \;=\; A_2\cosh\left(\frac{u_1}{\sqrt{2}}\right)+B_2\sinh\left(\frac{u_1}{\sqrt{2}}\right)-2b-c_3, \tag{4.18}$$

$$g(u_2) \;=\; A_1\cos\,u_2+B_1\sin\,u_2+b, \tag{4.19}$$

*with the constants satisfying* $c_2=\dfrac{2A_1^2-A_2^2+2B_1^2+B_2^2-6b^2-4bc_3}{2}$.

In the next example, we consider $\varphi=\ln(\cosh u_1)$ and $B=u_1$, and as before, we obtain

**Example 4.3.** *Consider $\varphi=\ln(\cosh u_1)$ and $B=u_1$. Then the functions*

$$\phi \;=\; \epsilon\sqrt{2}+\frac{\epsilon\sqrt{2}(1-c)\big((f_1+f_2)-g_1\big)}{2b+f_1+f_2+c(f_1-f_2)+2\tanh(u_1)f_1'-2g_1}, \tag{4.20}$$

$$\Phi \;=\; \frac{\epsilon\sqrt{2}g_1}{f_1+f_2}-\epsilon\sqrt{2}. \tag{4.21}$$

*are solutions to the Calapso equation with $c\neq 1$ and $f_1=f_1(u_1)$, $g_1=g_1(u_1)$ and $f_2=f_2(u_2)$ functions given by*

$$f_1''-cf_1+\frac{2}{\cosh\,u_1}g_1=b, \tag{4.22}$$

$$g_1'=\frac{2}{\cosh\,u_1}f_1'. \tag{4.23}$$

$$f_2(u_2)=\begin{cases} A_2\cos\big(\sqrt{c}\,u_2\big)+B_2\sin\big(\sqrt{c}\,u_2\big)+\dfrac{b_1}{c}, & c>0,\ c\neq 1,\\ \dfrac{b_1}{2}\,u_2^2+A_2u_2+B_2, & c=0,\\ A_2\cosh\big(\sqrt{-c}\,u_2\big)+B_2\sinh\big(\sqrt{-c}\,u_2\big)+\dfrac{b_1}{c}, & c<0.\end{cases} \tag{4.24}$$

*with initial conditions satisfying*

$$\big[(f_1')^2-cf_1^2-2bf_1+g_1^2+(f_2')^2+cf_2^2-2bf_2\big](u_1^0,u_2^0)=0. \tag{4.25}$$

*Moreover, if $c = 1$, the function*

$$\phi_2 \quad = \quad \frac{2\epsilon\sqrt{2}\big((f_1+f_2)-g_1\big)\big(b+f_1-g_1\big)}{c_2+2(f_1+f_2)\big(b+f_1-g_1+\tanh(u_1)f_1'\big)}. \tag{4.26}$$

*is also solution to the Calapso equation, where $f_1$ and $g_1$ are given by (4.22) with $c = 1$ and $f_2$ is given by (4.24) with the constants satisfying $c_2 = A_2^2 + B_2^2 - b^2 + c_1$, where $c_1$ is given by (3.8).*

**Remark 4.1.** *The solutions given in these three examples correspond to solutions of the Calapso equation obtained from isothermic surfaces locally associated, respectively, with the cylinder, cone, and catenoid through Ribaucour transformations.*

**5. Conclusions.** From the results obtained in this work, we observe the following: By applying Ribaucour transformations to isothermic surfaces of revolution, we obtain explicit families of solutions of the Calapso equation depending on functions of one variable.

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