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# Modeling climate-mediated shifts: An almost periodic approach to exploitation and interference competition

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## Abstract

*We consider an almost periodic competition model to analyze the simultaneous effects of exploitation and interference competition. Based on the analysis of the model, we prove the existence, uniqueness, and global stability of an almost periodic solution under certain conditions. Through numerical simulations, we illustrate the results obtained analytically. Finally, the results are discussed in detail from a biological perspective.*

**Keywords** . Exploitation competition, interference competition, globally stable attractor, almost periodic functions, competitive systems.

## Resumen

*Consideramos un modelo casi periódico para analizar los efectos simultáneos de la competencia por explotación e interferencia. Basados en el análisis del modelo, probamos la existencia, unicidad y estabilidad global de una solución casi periódica bajo ciertas condiciones. A través de simulaciones numéricas, ilustramos los resultados obtenidos analíticamente. Finalmente, los resultados son discutidos en detalle desde una perspectiva biológica.*

**Palabras clave**. Competencia por explotación, competencia por interferencia, atractor globalmente estable, funciones casi periódicas, sistemas competitivos.

**1. Introduction.** Competition between species is of paramount importance since it plays a fundamental role in supporting ecosystem functions, such as supporting biodiversity and shaping biological communities.

Competition occurs between individuals of the same species (intraspecific competition) and between individuals of different species (interspecific competition). This ecological interaction can be classified into exploitation and interference competition.

Exploitation competition occurs when competitors use and deplete resources, negatively affecting each other. Then it is classified as indirect competition. To exemplify this indirect competition, we mention that exploitative competition has been recorded between two boreal forest

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species that have expanded their range into the Arctic tundra: red foxes (*Vulpes vulpes*) and Arctic foxes (*Vulpes lagopus*) [1, 2, 3, 4].

In contrast, interference competition occurs when competitors prevent others from accessing resources. This is classified as direct competition and, by its nature, can lead to interspecific killing [1, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. In particular, species exhibiting interference competition can display aggressive (such as hoarding or defense) or passive (such as collisions) behavior [15, 16, 17, 18]. Interference competition has been recorded between the collared peccary (*Pecari tajacu*) and the white-tailed deer (*Odocoileus virginianus*) [5, 19, 20].

The species mentioned above have in common that their feeding habits are affected by seasonal changes. For example, when foxes are distributed in North America, they feed year-round primarily on arvicoline rodents, but when their prey becomes scarce in winter, the sea ice provides alternative resources. In an analogous way, the collared peccary and white-tailed deer increase their interference competition when the number of ephemeral ponds decreases during the dry season.

The effects on the population dynamics due to competition have been analyzed using different mathematical tools. Competition between species has been modeled using differential equations with constant rates, density-dependent rates, periodic rates and almost periodic rates [15, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. Since we are interested in analyzing exploitation and interference competition in a scenario where the ecological interaction is affected by climate-mediated shifts, we propose an almost periodic model, as modeling environmental drivers with periodic functions is very restrictive [32]. From the analysis of the model, we prove the existence and uniqueness of an almost periodic global attractor.

This work is organized as follows: in Section 2, we present the almost periodic model. In Section 3, we provide some basic results on competitive systems with almost periodic functions. In Section 4, we establish a theorem concerning the existence, uniqueness and stability of almost periodic solutions of the the model. In Section 5, we present numerical simulations of the solutions of the model, which describe different scenarios associated with the model. Finally, in Section 6, we summarize our results and discuss their biological implications.

**2. The almost periodic competition model.** We assume that the per capita growth rate of species $N_i$, for $i = 1, 2$, is modeled by the expression $r_i(t) - a_i(t)N_i$ when species $N_j$ is absent, for $j = 1, 2$. In this case, $r_i(t)$ describes the intrinsic growth rate at zero density (which can also represent available resources) and $a_i(t)$ models population regulation factors due to intraspecific competition. To model exploitation competition due to the presence of species $N_j$, we use the term $\beta_j(t)\left(1 - e^{-\gamma_j(t)N_j}\right)$, for $i \neq j$. This term describes how the per capita growth rate of species $N_i$ decreases because a portion of the resources, represented by $\beta_j(t)\left(1 - e^{-\gamma_j(t)N_j}\right)$, is depleted by species $N_j$ at rates $\beta_j(t)$ and $\gamma_j(t)$. That is, both species share resources. Finally, interference competition is modeled by the term $\alpha_{ij}(t)N_j$, for $i \neq j$.

Notice that a necessary condition for both populations to increase is that $\beta_i(t) < r_i(t)$. That is, the resources cannot be totally shared, and a portion $(r_i(t) - \beta_j(t))$ must be available exclusively for species $N_i$.

The almost periodic model with exploitation and interference competition is given by the following coupled system of ordinary differential equations:

$$\begin{aligned} \frac{dN_1}{dt} &= N_1\left[r_1(t) - a_1(t)N_1 - \alpha_{12}(t)N_2 - \beta_2(t)\left(1 - e^{-\gamma_2(t)N_2}\right)\right], \\ \frac{dN_2}{dt} &= N_2\left[r_2(t) - a_2(t)N_2 - \alpha_{21}(t)N_1 - \beta_1(t)\left(1 - e^{-\gamma_1(t)N_1}\right)\right]. \end{aligned} \tag{2.1}$$

Model (2.1) generalizes a model with constant rates proposed by Ayala in [33].

In the following section, we present some basic concepts and properties of almost periodic functions.

**3. Almost periodic functions in competitive systems.** We refer the reader to [34, 35] and [36, 37] for a detailed description about almost periodic functions and competitive systems respectively.

**Definition 3.1.** *We say that a continuous function $\phi \in C^0(\mathbb{R})$ is* almost periodic *if every sequence $\{\phi(t + \alpha_n)\}$ of translations of $\phi$ has a subsequence that converges uniformly in $\mathbb{R}$.*

There are different but equivalent definitions in the literature. We denote by $AP(\mathbb{R})$ this set; it is well known that for every $\phi \in AP(\mathbb{R})$, we can associate its Fourier series:

$$\phi \sim \sum_{n\in\mathbb{N}} c(\phi, \lambda_n)e^{i\lambda_n t},$$

as well as a *mean*

$$\mathcal{M}(\phi) := \lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(t)\,dt,$$

which is a linear bounded functional, $\mathcal{M} : AP(\mathbb{R}) \to \mathbb{R}$. The corresponding Parseval's Theorem for almost periodic functions says

$$\mathcal{M}\left[|\phi|^2\right] = \sum_{n\in\mathbb{N}} |c(\phi,\lambda_n)|^2.$$

Define the *sud-est* or $(IV)$-partial order as follows: For every $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$, we say that $u \leq_{IV} v$, if $u_1 \leq v_1$ and $u_2 \geq v_2$. If $u_1 < v_1$ and $u_2 > v_2$, then we write $u \ll_{IV} v$. This partial order is explicitly defined in [37].

Consider a system

$$\begin{aligned} \dot{x} &= f(t, x(t), y(t)),\\ \dot{y} &= g(t, x(t), y(t)), \end{aligned} \tag{3.1}$$

where $f, g$ are $\mathrm{C}^1$ in an open $D \subset \mathbb{R}^2$ and continuous functions on $t$. Recall that (3.1) is said a *competitive system* in $\mathbb{R} \times D$ if

$$f_y(t, x, y) \leq 0, \text{ and } g_x(t, x, y) \leq 0, \ \forall\, t \in \mathbb{R}, (x, y) \in D. \tag{3.2}$$

We say that a pair of differentiable functions $(a(t), b(t))$ is a $(IV)$-*sub-solution pair* of (3.1) if

$$\begin{aligned} \dot{a} &\leq f(t, a(t), b(t)),\\ \dot{b} &\geq g(t, a(t), b(t)), \text{ for all } t. \end{aligned} \tag{3.3}$$

Analogously a pair of differentiable functions $(A(t), B(t))$ is a $(IV)$-*super-solution pair* if

$$\begin{aligned} \dot{A} &\geq f(t, A(t), B(t)),\\ \dot{B} &\leq g(t, A(t), B(t)), \text{ for all } t. \end{aligned} \tag{3.4}$$

We say that sub- and super-solution pairs are ordered if for all $t$ we have $a(t) < A(t)$ and $b(t) > B(t)$, i.e. if $(a(t), b(t)) \ll_{IV} (A(t), B(t))$.

An important feature for competitive system (3.1) related to almost periodic orbits was established in [38], Theorem 3.2. Explicitly the following result holds.

**Theorem 3.1.** *Assume that system (3.1) has almost periodic continuous components, $f, g$, with respect to $t$ uniformily respect to $(x, y) \in D$. Assume $f, g$ are of class $C^1$ in $(x, y)$. If it is competitive and has $(IV)$-ordered sub- and super-solution pairs $(a(t), b(t)) \ll_{IV} (A(t), B(t))$. Then the system has an almost periodic solution $(x(t), y(t))$, satisfying*

$$a(t) \leq x(t) \leq A(t),\ b(t) \geq y(t) \geq B(t), \quad \forall t \geq 0, \tag{3.5}$$

*such that for every sub- super-solution pair we have*

$$a(t) \leq x_0 \leq A(t), \qquad B(t) \leq y_0 \leq b(t).$$

*Furthermore, any solution of (3.1), with initial condition $(x(0), y(0))$ satisfying $a(0) < x(0) < A(0)$ and $b(0) > y(0) > B(0)$, converges to the product of the strips*

$$(\check{x}(t), \hat{x}(t)) \times (\hat{y}(t), \check{y}(t)),$$

*where $(\check{x}(t), \check{y}(t)) \leq_{IV} (\hat{x}(t), \hat{y}(t))$ are $(IV)$-minimal, maximal almost periodic solution, respectively.*

**4. Results.** As customary, for an almost periodic function $v : \mathbb{R} \to \mathbb{R}$, we denote

$$v_* := \inf_{t\in\mathbb{R}} v(t) \quad \text{and} \quad v^* := \sup_{t\in\mathbb{R}} v(t).$$

Now we state our main result.

**Theorem 4.1.** *Assume $r_i(t), a_i(t), \alpha_{ij}(t), \beta_i(t) \geq 0$ and $\gamma_i(t) \geq 0$ are continuous almost-periodic functions (not all constant) with $r_{i*} > 0, \alpha_{ij*}$ and $a_{i*} > 0$. Suppose that*

$$r_{i*} - \beta_j^* > \alpha_{ij}^* \frac{r_j^*}{a_{j*}}, \ i \neq j. \tag{4.1}$$

*Then, the following statements are valid.*

*i) There exists at least one almost periodic solution $(N_1, N_2)$ of (2.1) whose components are positive.*

*ii) If*

$$(\alpha_{12} + \beta_2\gamma_2)^*(\alpha_{21} + \beta_1\gamma_1)^* < a_{1*}a_{2*}, \tag{4.2}$$

*there exists a unique almost periodic solution in $\mathbb{R}^2_{\geq 0}$ and any other solution of (2.1) with positive initial conditions converges to this almost periodic solution, when $t \to \infty$.*

*Proof:* A direct account shows us that system (2.1) is competitive in $\mathbb{R}^2_{\geq 0}$. For i) we need to prescribe suitable sub- and super-solution pairs. To develop a sub-solution pair, we consider

$$(a(t), b(t)) = \left(\epsilon, \ \frac{r_2^*}{a_{2*}}\right),$$

these functions satisfy inequalities in (3.3), indeed

$$\begin{aligned}
a'(t) = 0 &\leq \epsilon\left[r_{1*} - a_1^*\epsilon - \alpha_{12}^*\left(\frac{r_2^*}{a_{2*}}\right) - \beta_2^* + \beta_2(t)e^{-\gamma_2(t)\left(\frac{r_2^*}{a_{2*}}\right)}\right] \\
&\leq \epsilon\left[r_1(t) - a_1(t)\epsilon - \alpha_{12}(t)\left(\frac{r_2^*}{a_{2*}}\right) - \beta_2(t)\left(1 - e^{-\gamma_2(t)\left(\frac{r_2^*}{a_{2*}}\right)}\right)\right]. \\
b'(t) = 0 &\geq \left(\frac{r_2^*}{a_{2*}}\right)\left[r_2^* - a_{2*}\left(\frac{r_2^*}{a_{2*}}\right) - \alpha_{21}(t)\epsilon - \beta_1(t)\left(1 - e^{-\gamma_1(t)\epsilon}\right)\right] \\
&\geq \left(\frac{r_2^*}{a_{2*}}\right)\left[r_2(t) - a_2(t)\left(\frac{r_2^*}{a_{2*}}\right) - \alpha_{21}(t)\epsilon - \beta_1(t)\left(1 - e^{-\gamma_1(t)\epsilon}\right)\right].
\end{aligned}$$

Using (4.1), we have that $\left(\epsilon, \ \frac{r_2^*}{a_{2*}}\right)$ is a sub-solution pair for $\epsilon > 0$ small enough.

For a super-solution pair; we take

$$(A(t), B(t)) = \left(\frac{r_1^*}{a_{1*}}, \epsilon\right),$$

these functions verify:

$$\begin{aligned}
A'(t) = 0 &\geq \left(\frac{r_1^*}{a_{1*}}\right)\left[r_1^* - a_{1*}\left(\frac{r_1^*}{a_{1*}}\right) - \alpha_{12}(t)\epsilon - \beta_2(t)\left(1 - e^{-\gamma_2(t)\epsilon}\right)\right] \\
&\geq \left(\frac{r_1^*}{a_{1*}}\right)\left[r_1(t) - a_1(t)\left(\frac{r_1^*}{a_{1*}}\right) - \alpha_{12}(t)\epsilon - \beta_2(t)\left(1 - e^{-\gamma_2(t)\epsilon}\right)\right]. \\
B'(t) = 0 &\leq \epsilon\left[r_{2*} - a_2^*\epsilon - \alpha_{21}^*\left(\frac{r_1^*}{a_{1*}}\right) - \beta_1^* + \beta_1(t)e^{-\gamma_1(t)\left(\frac{r_1^*}{a_{1*}}\right)}\right] \\
&\leq \epsilon\left[r_2(t) - a_2(t)\epsilon - \alpha_{21}(t)\left(\frac{r_1^*}{a_{1*}}\right) - \beta_1(t)\left(1 - e^{-\gamma_1(t)\left(\frac{r_1^*}{a_{1*}}\right)}\right)\right].
\end{aligned}$$

Thus by (4.1) and $\epsilon > 0$ small enough, they constitute a super-solution pair.

Therefore, by Theorem 3.1 there exists at least one almost periodic solution for system (2.1). This concludes the part i).

ii) For uniqueness, we consider a maximal pair $(\hat{N}_1, \hat{N}_2)$ and minimal pair $(\check{N}_1, \check{N}_2)$ of almost periodic solutions. We just need to prove that $\hat{N}_1(t) = \check{N}_1(t)$ and $\hat{N}_2(t) = \check{N}_2(t)$, for which we will use the following assertion.

**Claim 1.** *Let $\phi, \psi$ be almost periodic functions such that*

$$\phi(t) \geq \psi(t) \geq 0, \qquad \mathcal{M}[\phi] = \mathcal{M}[\psi].$$

*Then $\phi(t) = \psi(t)$ for every $t \in \mathbb{R}$.*

Since the mean $\mathcal{M}\left[(\ln \hat{N}_i)'\right] = \mathcal{M}\left[(\ln \check{N}_i)'\right] = 0$, then

$$\mathcal{M}[r_i - \beta_j] + \mathcal{M}[\beta_j \left(e^{-\gamma_j \hat{N}_j}\right)] = \mathcal{M}\left[a_i \hat{N}_i\right] + \mathcal{M}\left[\alpha_{ij} \hat{N}_j\right],$$
$$\mathcal{M}[r_i - \beta_j] + \mathcal{M}[\beta_j \left(e^{-\gamma_j \check{N}_j}\right)] = \mathcal{M}\left[a_i \check{N}_i\right] + \mathcal{M}\left[\alpha_{ij} \check{N}_j\right], \ i \neq j.$$

Thus

$$\mathcal{M}\left[a_i \left(\hat{N}_i - \check{N}_i\right)\right] + \mathcal{M}\left[\alpha_{ij} \left(\hat{N}_j - \check{N}_j\right)\right] = \mathcal{M}\left[\beta_j \left(e^{-\gamma_j \hat{N}_j} - e^{-\gamma_j \check{N}_j}\right)\right]. \tag{4.3}$$

The mean value theorem yields

$$e^{-\gamma_j \hat{N}_j} - e^{-\gamma_j \check{N}_j} = -\gamma_j e^{-\gamma_j \tilde{N}_j} \left(\hat{N}_j - \check{N}_j\right), \text{ for some, } \tilde{N}_j. \tag{4.4}$$

Therefore, from the equations (4.3) and (4.4) we have

$$\mathcal{M}\left[a_1 \left(\hat{N}_1 - \check{N}_1\right)\right] + \mathcal{M}\left[\alpha_{12} \left(\hat{N}_2 - \check{N}_2\right)\right] = \mathcal{M}\left[-\beta_2 \gamma_2 e^{-\gamma_2 \tilde{N}_2} \left(\hat{N}_2 - \check{N}_2\right)\right]. \tag{4.5}$$

Since $\check{N}_1 \leq \hat{N}_1$, $\hat{N}_2 \leq \check{N}_2$ and $e^{-\gamma_j \tilde{N}_j} \leq 1$, rewriting (4.5), we get

$$\mathcal{M}\left[a_1 \left(\hat{N}_1 - \check{N}_1\right)\right] \leq \mathcal{M}\left[(\alpha_{12} + \beta_2 \gamma_2) \left(\check{N}_2 - \hat{N}_2\right)\right]. \tag{4.6}$$

In the same way, we have

$$\mathcal{M}\left[a_2 \left(\check{N}_2 - \hat{N}_2\right)\right] \leq \mathcal{M}\left[(\alpha_{21} + \beta_1 \gamma_1) \left(\hat{N}_1 - \check{N}_1\right)\right]. \tag{4.7}$$

From the previous relations we obtain

$$\begin{aligned} a_{2*} \mathcal{M}\left[\check{N}_2 - \hat{N}_2\right] &\leq (\alpha_{21} + \beta_1 \gamma_1)^* \mathcal{M}\left[\hat{N}_1 - \check{N}_1\right] \\ &\leq \frac{(\alpha_{21} + \beta_1 \gamma_1)^* (\alpha_{12} + \beta_2 \gamma_2)^*}{a_{1*}} \mathcal{M}\left[\check{N}_2 - \hat{N}_2\right]. \end{aligned}$$

If $\mathcal{M}\left[\check{N}_2 - \hat{N}_2\right] > 0$, the above inequality contradicts the condition in ii). Therefore $\mathcal{M}\left[\check{N}_2\right] = \mathcal{M}\left[\hat{N}_2\right]$, whence $\check{N}_2 = \hat{N}_2$ by Claim 1. This in turn implies that $\hat{N}_1 = \check{N}_1$.

Now we prove Claim 1.

*Proof:* Since $\phi(t), \psi(t)$ are almost periodic, then they are bounded. Hence,

$$\begin{aligned} 0 \leq \mathcal{M}\left[\phi^2 - \psi^2\right] &\leq \mathcal{M}\left[(\phi - \psi)(\phi + \psi)\right] \\ &\leq (2 \sup\{\phi(t)\}) \cdot \mathcal{M}[\phi - \psi] = 0. \end{aligned}$$

Therefore, $\mathcal{M}\left[\phi^2\right] = \mathcal{M}\left[\psi^2\right]$. Thus,

$$0 \leq \mathcal{M}\left[(\phi - \psi)^2\right] \leq 2\left(\mathcal{M}\left[\phi^2\right] - \mathcal{M}[\phi\psi]\right) \leq 2\left(\mathcal{M}\left[\phi^2\right] - \mathcal{M}\left[\psi^2\right]\right) = 0.$$

If we apply Parseval's Theorem on the sum of the squares of the Fourier coefficients of $\phi - \psi$ we get $\phi \equiv \psi$. □

Now we can conclude the proof of Theorem 4.1.

First note that the rectangle $\Omega_\epsilon := \{(N_1, N_2) \in \mathbb{R}^2 \ : \ \epsilon \leq N_i \leq \frac{r_i^*}{a_{i*}}\}$ is a global attractor of system (2.1) in $\Gamma_\epsilon := \{(N_1, N_2) \in \mathbb{R}^2 \ : \ \epsilon \leq N_i\}$. Indeed, if $\frac{r_1^*}{a_{1*}} \leq N_1$, then observing the first component of the vector field associated to (2.1), we get

$$\begin{aligned} \frac{dN_1}{dt} &= N_1\left[r_1(t) - a_1(t)N_1 - \alpha_{12}(t)N_2 - \beta_2(t)(1 - e^{-\gamma_2(t)N_2})\right] \\ &\leq N_1\left[r_1^* - a_{1*}\left(\frac{r_1^*}{a_{1*}}\right) - \alpha_{12}(t)N_2 - \beta_2(t)(1 - e^{-\gamma_2(t)N_2})\right] \leq 0. \end{aligned}$$

That is, the component $N_1$ decreases. Similarly, if $\frac{r_2^*}{a_{2*}} \leq N_2$, thus the orbit of any initial condition in $\Gamma_\epsilon$ outside of $\Omega_\epsilon$ will eventually enter to $\Omega_\epsilon$.

We have a single almost periodic orbit that is an attractor in $\Omega_\epsilon$, which in turn attracts every orbit in $\Omega_\epsilon$. Taking $\epsilon > 0$ arbitrarily small then, the almost periodic orbit is attractor at the entire set $\mathbb{R}^2_{>0}$. This concludes ii) and therefore end as the proof of Theorem 4.1. □

**5. Numerical simulations.** In this section, numerical simulations of the solutions of the model are presented. Through the use of different values of the parameters, we show that model (2.1) describes different ecological scenarios: coexistence and exclusion of species.

To illustrate a coexistence scenario, we use the following almost periodic functions:

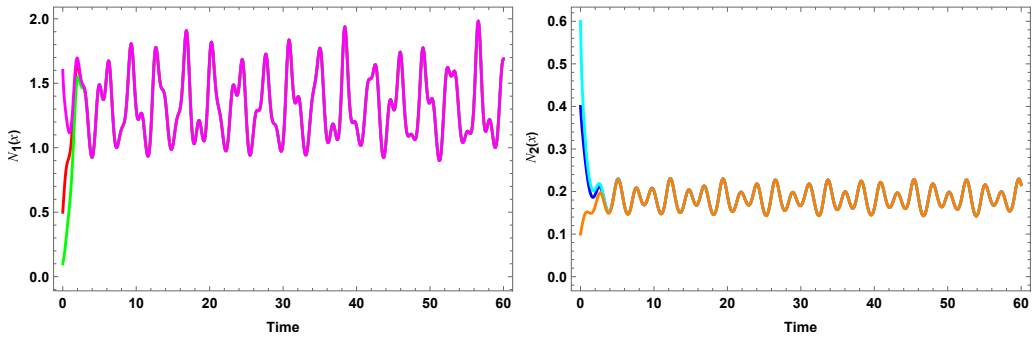
$$\begin{aligned} r_1(t) &= 2.5\left(1.2 + 0.1\sin\left(\sqrt{10}t\right) + 0.2\cos\left(\sqrt{17}t\right)\right), \\ r_2(t) &= 1.6\left(1.1 + 0.1\sin\left(\sqrt{3}t\right) + 0.3\cos\left(\sqrt{7}t\right)\right), \\ a_1(t) &= 1.2\left(1.8 + 0.03\sin\left(\sqrt{2}t\right) + 0.5\cos\left(\sqrt{3}t\right)\right), \\ a_2(t) &= 1.92\left(3.27 + 0.2\sin\left(\sqrt{7}t\right) + 0.1\cos\left(\sqrt{3}t\right)\right), \\ \alpha_{12}(t) &= 0.08\left(1.5 + 0.3\sin\left(\sqrt{5}t\right) + 0.2\cos\left(\sqrt{2}t\right)\right), \\ \alpha_{21}(t) &= 0.02\left(1.4 + 0.1\sin\left(\sqrt{5}t\right) + 0.4\cos\left(\sqrt{7}t\right)\right), \\ \beta_1(t) &= 0.5\left(1.3 + 0.02\sin\left(\sqrt{3}t\right) + 0.04\cos\left(\sqrt{7}t\right)\right), \\ \beta_2(t) &= 0.8\left(1.1 + 0.08\sin\left(\sqrt{10}t\right) + 0.09\cos\left(\sqrt{17}t\right)\right), \\ \gamma_1(t) &= 1.1\left(1.2 + 0.02\sin\left(\sqrt{11}t\right) + 0.4\cos\left(\sqrt{3}t\right)\right). \\ \gamma_2(t) &= 1.0\left(1.6 + 0.2\sin\left(\sqrt{11}t\right) + 0.1\cos\left(\sqrt{3}t\right)\right). \end{aligned} \tag{5.1}$$

We calculate the infimum and supremum values required to apply Theorem 4.1, which are presented in Table 5.1.

| | | | | |
|---|---|---|---|---|
| $r_{1*} = 2.2500$ | $r_{2*} = 1.1545$ | $a_{1*} = 1.5240$ | $\alpha_{12}^* = 0.15944$ | $\beta_1^* = 1.0138$ |
| $r_1^* = 3.7418$ | $r_2^* = 2.3985$ | $a_{2*} = 5.7026$ | $\alpha_{21}^* = 0.037980$ | $\beta_2^* = 0.67992$ |

Table 5.1: Infimum and supremum values of the functions used in the coexistence scenario.

Using the values from Table 5.1, we find that $r_{1*} - \beta_2^* = 1.57008$, $r_{2*} - \beta_1^* = 0.1407$, $\alpha_{12}^* \frac{r_2^*}{a_{2*}} = 0.06706008487$, and $\alpha_{21}^* \frac{r_1^*}{a_{1*}} = 0.09325037008$. Furthermore, it follows that $(\alpha_{12} + \beta_2\gamma_2)^*(\alpha_{21} + \beta_1\gamma_1)^* = (1.4265)\,(1.8307) = 2.61149355$ and $a_{1*}a_{2*} = (1.5240)\,(5.7026) = 8.69076240$. Consequently, conditions (4.1) and (4.2) of Theorem 4.1 are satisfied. Therefore, a unique coexistence scenario exists, which is a global attractor. In Figure 5.1, we show that for different initial conditions, the solutions of the model converge to a unique almost periodic solution.

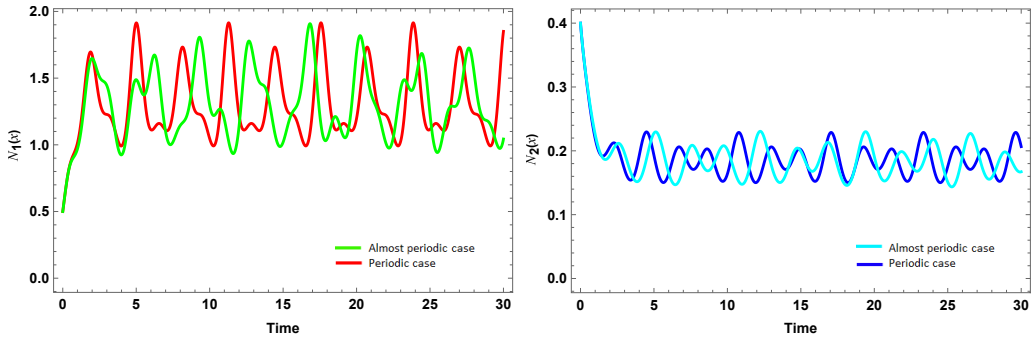



(a) (b)

Figure 5.1: Model (2.1) admits a unique global attractor when the conditions in Theorem 4.1 are satisfied. To illustrate this scenario, initial conditions $(0.5, 0.4)$, $(0.1, 0.6)$, and $(1.6, 0.1)$ are used in the numerical simulations.

Results concerning the almost periodic model given in (2.1) can also be applied to the periodic case. In Figure 5.2, we show that almost periodic and periodic solutions may initially coincide; however, as time passes, the solutions diverge over various time intervals. To show this scenario, we use the following periodic functions and the almost periodic functions given in (5.1).

$$\begin{aligned}
r_1(t) &= 2.5\,(1.2 + 0.1\sin(3t) + 0.2\cos(4t))\,,\\
r_2(t) &= 1.6\,(1.1 + 0.1\sin(2t) + 0.3\cos(3t))\,,\\
a_1(t) &= 1.2\,(1.8 + 0.03\sin(2t) + 0.5\cos(2t))\,,\\
a_2(t) &= 1.92\,(3.27 + 0.2\sin(3t) + 0.1\cos(2t))\,,\\
\alpha_{12}(t) &= 0.08\,(1.5 + 0.3\sin(2t) + 0.2\cos(2t))\,,\\
\alpha_{21}(t) &= 0.02\,(1.4 + 0.1\sin(2t) + 0.4\cos(3t))\,,\\
\beta_1(t) &= 0.5\,(1.3 + 0.02\sin(2t) + 0.04\cos(3t))\,,\\
\beta_2(t) &= 0.8\,(1.1 + 0.08\sin(3t) + 0.09\cos(4t))\,,\\
\gamma_1(t) &= 1.1\,(1.2 + 0.02\sin(3t) + 0.4\cos(2t))\,.\\
\gamma_2(t) &= 1.0\,(1.6 + 0.2\sin(4t) + 0.1\cos(2t))\,.
\end{aligned} \tag{5.2}$$




(a) (b)

Figure 5.2: Comparison between periodic and almost periodic solutions of the model. The numerical simulations are performed using the initial condition $(0.5, 0.4)$.

Coexistence scenarios are relevant from a biodiversity perspective; however, when the conditions of the theorem are not satisfied, different exclusion scenarios can occur. To show these cases, we multiply the almost periodic function $\alpha_{12}(t)$ by 100 such that condition (4.1) for $i = 1$ and $j = 2$ is violated while it remains satisfied for $i = 2$ and $j = 2$. Then, the population of species $N_2$ persists while species $N_1$ goes extinct; see Figure 5.3 case (a). In contrast, when $\alpha_{21}(t)$ is increased by a factor of 100 such that condition (4.1), for $i = 2$ and $j = 1$, is not satisfied, but holds for $i = 1$ and $j = 2$, species $N_1$ persists while species $N_2$ is excluded; see Figure 5.3

case (b). Finally, when conditions (4.1) are not satisfied for either species, the surviving species depends on the initial conditions; see Figure 5.3 cases (c) and (d).

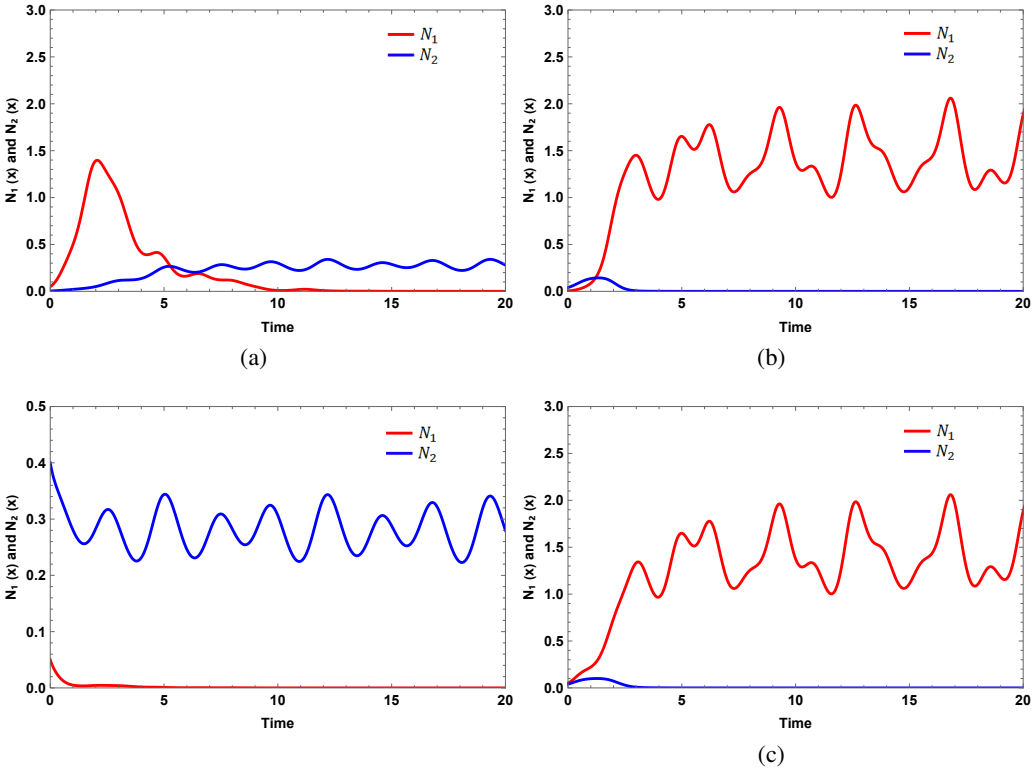



Figure 5.3: Different exclusion scenarios occurs when condition (4.1) is not satisfied for $i, j = 1, 2$. Cases (a) and (b) illustrate competitive exclusion where the outcome is independent of the initial conditions. In contrast, cases (c) and (d) illustrate exclusion scenarios where the outcome depends on the initial conditions, the initial conditions $(0.05, 0.004)$ and $(0.005, 0.04)$ are used, respectively.

**6. Conclusions.** We analyze an almost periodic model which describes competition between species. The proposed model includes terms that describe exploitation and interference competition. In the model, all rates are modeled by nonnegative almost periodic functions. We prove the existence, uniqueness, and stability of an almost periodic solution. For this purpose, we used concepts from competitive systems with almost periodic functions.

Model (2.1) admits at least one almost periodic solution $(N_1, N_2)$ with positive coordinates when the condition

$$r_{i*} - \beta_j^* > \alpha_{ij}^* \frac{r_j^*}{a_{j*}}, \ i \neq j$$

is satisfied for $i, j = 1, 2$.

This condition shows that a big enough portion of exclusive resources for both species favored its coexistence. In this context, a portion of shared resources sufficient enough avoids persistence of both species. In the same direction, the coexistence of species is favored when the interspecific interference rates are small enough; however, this coexistence is avoided when the intraspecific interference rates are small enough.

The competition model presents a unique almost periodic stable solution when the condition

$$(\alpha_{12} + \beta_2\gamma_2)^*(\alpha_{21} + \beta_1\gamma_1)^* < a_{1*}a_{2*}$$

is satisfied. According to this inequality, sufficiently large intraspecific interference rates and low interspecific interference rates promote the uniqueness of the coexistence scenario. This is in line with that fact that individuals shared the same niche show a competence stronger than individuals sharing resources. Uniqueness is also favored for larger depletion rates. This unique coexistence scenario is shown in Figure 5.1.

Analogous results of the existence, uniqueness and stability of solutions of the model are presented in the periodic case. However, we shown in Figure 5.2 that modeling seasonal phenomena

with periodic rates when actually they must be modeled by almost periodic rates can lead to underestimation or overestimation of the population densities, which lead to misleading estimation of the densities populations.

Finally, we show that scenarios of exclusion of species occur when the condition (4.1) does not hold. In this case, two scenarios occur. In a first case, the species whose ecological and demographic rates satisfy condition (4.1) is established while its competitors are excluded. In a second case, when there are no species satisfying condition (4.1), priority effects are presented, where the long-term outcome depends on the initial conditions, as shown in Figure 5.3. From an analysis of the conditions in Theorem 4.1, we can conclude that these scenarios are favored by higher interspecific interference rates and smaller depletion rates.

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