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COMMUNICATION

Some Results for Generalized Local Homology Modules

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Abstract
We prove some results on the finiteness of co-associated primes of generalized local homology modules inspired by the conjecture of Grothendieck and the question of Huneke. We also show equivalent properties of minimax local homology modules. Here, we get applications for the generalized local homology module, in a general theory of modules.
Keywords . local cohomology modules, linearly compact module, (generalized) local homology, (co-)associated prime

Resumen
Demostramos algunos resultados sobre la finitud de primos coasociados de módulos de homología local generalizados, inspirados en la conjetura de Grothendieck y la cuestión de Huneke. También mostramos propiedades equivalentes de módulos de homología local minimax. Aquí, encontramos aplicaciones para el módulo de homología local generalizado en una teoría general de módulos.
Palabras clave. Módulos de cohomología local, módulo linealmente compacto, homología local (generalizada), primo (co-)asociado.

1. Introduction. Throughout this paper, R is a commutative Noetherian ring with non-zero identity. Moreover, throughout this paper, R has a topological structure.

Local cohomology was introduced by Grothendieck and many people have worked about understanding their structure, (non)-vanishing and finiteness properties. For example, Grothendieck's non-vanishing theorem is one of the important theorems in local cohomology. For more details on local cohomology modules, see [2].

Now, in [6], J. Herzog introduced the definition of generalized local cohomology, which is an extension of the local cohomology of A. Grothendieck (see [2]). Let I be an ideal of a Noetherian commutative ring R and M, N R-modules. In [9], we define the i-th generalized local homology module $\mathrm{H}_i^I(M,N)$ of M, N with respect to I by

$$\mathrm{H}_i^I(M,N) = \varprojlim_t \mathrm{Tor}_i^R(M/I^tM, N)\ .$$

This definition is in some sense dual to J. Herzog's definition of generalized local cohomology modules [6], and in fact a generalization of the usual local homology modules ([[9], [10]]).

The purpose of this paper is to study the finiteness of co-associated primes of generalized local homology modules. We also study some properties of minimax local homology modules. By duality, we get some properties of Herzog's generalized local cohomology modules. The organization of the paper is as follows.

In Section 2, we put some prerequisites.

In Section 3, we presented some results of the theory in question.

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In Section 4, we have the application.

Here, we use the properties of commutative algebra and homological algebra for the development of the results (see [1] and [12]).

2. Prerequisites. We begin by recalling the concept of linearly compact defined by I.G. Macdonald [8]. A Hausdorff linearly topologized R-module M is said to be linearly compact if for any family $\mathfrak{F}$ of closed cosets (i.e., cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathfrak{F}$ have a non-empty intersection. A Hausdorff linearly topologized R-module M is called semi-discrete if every submodule of M is closed.

Thus, a discrete R-module is semi-discrete. It is clear that the Artinian R-modules are linearly compact with the discrete topology. So, the class of semi-discrete linearly compact modules contains all Artinian modules. Moreover, if $(R, \mathfrak{m})$ is a complete ring, then the finitely generated R-modules are also linearly compact and discrete. The following are some basic properties of linearly compact modules.

Lemma 2.1. ([8, 3.5, 3.14, 3.15])

(1) *Let M be a Hausdorff linearly topologized R-module, N a closed submodule of M. Then M is linearly compact if and only if N and M/N are linearly compact.*

(2) *If M is a linearly compact module, then for each positive integer t, I^tM is a closed submodule of M. Moreover, $I(\cap_{t>0} I^tM) = \cap_{t>0} I^tM$.*

Denoting by $\varprojlim_t^i$ the i-th right derived functor of the inverse limit $\varprojlim_t$ we have the following lemma.

Lemma 2.2. (see [7, 7.1]) *Let $\{M_t\}$ be an inverse system of linearly compact modules with continuous homomorphisms. Then $\varprojlim_t^i M_t = 0$, for all $i > 0$. Therefore, if*

$$0 \to \{M_t\} \to \{N_t\} \to \{P_t\} \to 0$$

is a short exact sequence of inverse systems of R-modules provided $\{M_t\}$ is an inverse system of linearly compact modules with continuous homomorphisms, then the sequence of inverse limits

$$0 \to \varprojlim_t M_t \to \varprojlim_t N_t \to \varprojlim_t P_t \to 0$$

is exact.

Lemma 2.3. ([9, 2.7]) *If M is a finitely generated R-module and $\{N_s\}$ is an inverse system of linearly compact R-modules with continuous homomorphisms, then for all $i \geq 0$, $\{\operatorname{Tor}_i^R(M, N_s)\}$ forms an inverse system of linearly compact modules with continuous homomorphisms. Moreover, we have*

$$\operatorname{Tor}_i^R(M, \varprojlim_s N_s) \cong \varprojlim_s \operatorname{Tor}_i^R(M, N_s).$$

Let I be an ideal of R. The I-adic completion $\Lambda_I(M)$, of a R-module M, is defined by

$$\Lambda_I(M) = \varprojlim_t M/I^tM.$$

In [3], the i-th local homology module $\mathrm{H}_i^I(M)$ of an R-module M with respect to I is defined by

$$\mathrm{H}_i^I(M) \cong \varprojlim_t \operatorname{Tor}_i^R(R/I^t, M).$$

It is clear that $\mathrm{H}_0^I(M) \cong \Lambda_I(M)$.

Lemma 2.4. ([9, 3.3, 3.10, 4.1]) *Let M be a linearly compact R-module. Then,*

(1) *$\mathrm{H}_i^I(M)$ is also a linearly compact R-module for all $i \geq 0$.*

(2) *$\mathrm{H}_i^I(\cap_t I^tM) \cong 0$, if $i = 0$.*

(3) *$\mathrm{H}_i^I(\cap_t I^tM) \cong \mathrm{H}_i^I(M)$, if $i > 0$.*

(4) *Assume in addition that M is a semi-discrete linearly compact R-module. Then, $\mathrm{H}_0^I(M) = 0$ if and only if $xM = M$, for some $x \in I$.*

Let M and N be R-modules. In [10], the i-th generalized local homology module $\mathrm{H}_i^I(M, N)$ of M, N with respect to I is defined by

$$\mathrm{H}_i^I(M, N) = \varprojlim_t \operatorname{Tor}_i^R(M/I^tM, N).$$

When $i = 0$, $\mathrm{H}_0^I(M, N) \cong \Lambda_I(M, N)$, in which

$$\Lambda_I(M, N) = \varprojlim_t (R/I^t \otimes_R M \otimes_R N).$$

In particular, $\mathrm{H}_i^I(R, N) = \mathrm{H}_i^I(N)$.

Lemma 2.5. ([14, 2.3(i)]) *If M is a finitely generated R-module and N is a linearly compact R-module, then for all $i \geq 0$, $\mathrm{H}_i^I(M, N)$ is a linearly compact R-module.*

Lemma 2.6. ([14, 3.4]) *Let M be a finitely generated module, and N a linearly compact R-module. If N is complete in the I-adic topology (i.e., $\Lambda_I(N) \cong N$), then for all $i \geq 0$, there is an isomorphism*

$$\mathrm{Tor}_i^R(M, N) \cong \mathrm{H}_i^I(M, N).$$

The co-support $\mathrm{Cosupp}_R(M)$ of an R-module M is the set of primes $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M with $\mathrm{Ann}_R(L) \subseteq \mathfrak{p}$ (see [13, 2.1]).

Note that a module is cocyclic if it is a submodule of $E(R/\mathfrak{m})$. If

$$0 \to N \to M \to K \to 0$$

is an exact sequence of R-modules, then

$$\mathrm{Cosupp}_R(M) = \mathrm{Cosupp}_R(N) \bigcup \mathrm{Cosupp}_R(K) \quad (\text{see } [13, 2.7]).$$

Lemma 2.7. ([4, 3.14]) *Let M be a finitely generated R-module and N a linearly compact R-module. Then*

$$\mathrm{Cosupp}_R(\mathrm{H}_i^I(M, N)) \subseteq \mathrm{Supp}_R(M) \bigcap \mathrm{Cosupp}_R(N) \bigcap V(I),$$

for all $i \geq 0$.

A prime ideal $\mathfrak{p}$ is called co-associated to a non-zero R-module M if there is an artinian homomorphic image T of M with $\mathfrak{p} = \mathrm{Ann}_R(T)$ (see [13]).

The set of coassociated primes of M is denoted by $\mathrm{Coass}_R(M)$.

It follows that $\mathrm{Coass}_R(M) \subseteq \mathrm{Cosupp}_R(M)$. If

$$0 \to N \to M \to K \to 0$$

is an exact sequence of R-modules, then

$$\mathrm{Coass}_R(K) \subseteq \mathrm{Coass}_R(M) \subseteq \mathrm{Coass}_R(N) \bigcup \mathrm{Coass}_R(K).$$

If M is a semidiscrete linearly compact R-module, then the set $\mathrm{Coass}_R(M)$ is finite (see [9, 2.9]).

We finish the section with the following result.

Lemma 2.8. ([13, 1.21]) *Let M be a finitely generated R-module and N an R-module. Then,*

$$\mathrm{Coass}_R(M \otimes_R N) = \mathrm{Supp}_R(M) \bigcap \mathrm{Coass}_R(N).$$

3. The results. We first recall the concept of minimax module introduced by H. Zoschinger [15]. An R-module M is called a minimax module if there is a finitely generated submodule N of M such that the quotient module M/N is Artinian. Thus, the class of minimax modules includes all finitely generated and all Artinian modules. Moreover, it also includes all semi-discrete linearly compact modules.

Definition 3.1. An R-module M is said to be I-coartinian if $\mathrm{Cosupp}_R(M) \subseteq V(I)$ and $\mathrm{Tor}_i^R(R/I, M)$ is an Artinian R-module for each i (see [11]).

We now have the following proposition.

Proposition 3.1. ([5, 3.5]) *Let M be a minimax linearly compact R-module with $\mathrm{Cosupp}_R(M) \subseteq V(I)$. Then, the R-module M is I-coartinian if and only if M/IM is Artinian.*

Lemma 3.1. ([5, 3.6]) *Let M be an I-coartinian minimax linearly compact R-module. If N is a closed submodule of M, then N and M/N are I-coartinian minimax linearly compact R-modules.*

We now have the following result.

Theorem 3.1. *Let M be a finitely generated R-module and N a semi-discrete linearly compact R-module such that*

$$N/(\cap_{t>0} I^t N)$$

is an Artinian R-module. Let s be a non-negative integer. If $\mathrm{H}_i^I(M, N)$ is minimax for all $i < s$, then $\mathrm{H}_i^I(M, N)$ is I-coartinian for all $i < s$, and

$$\mathrm{H}_s^I(M, N)/I\mathrm{H}_s^I(M, N)$$

is Artinian.

In particular, $\operatorname{Coass}_R(\mathrm{H}^I_s(M,N))$ *is a finite set.*

Proof: The proof is by induction on s. When $s=0$, it is trivial that $\mathrm{H}^I_i(M,N)$ is I-coartinian for all $i<0$. By Lemma 2.2, the short exact sequence of inverse systems of linearly compact modules

$$0 \to \{I^tN\} \to \{N\} \to \{N/I^tN\} \to 0$$

induces a short exact sequence

$$0 \to \bigcap_{t>0} I^tN \to N \to \Lambda_I(N) \to 0 \ .$$

Set $K=\bigcap_{t>0} I^tN$, and then $\Lambda_I(N) \cong N/K$, which is an Artinian R-module. From Lemma 2.3, we have that

$$\Lambda_I(M,N) = \varprojlim_t (R/I^t \otimes_R M \otimes_R N)$$

and so

$$\Lambda_I(M,N) \cong \varprojlim_t (M \otimes_R N/I^tN) \cong M \otimes_R \Lambda_I(N) \ .$$

It follows that $\Lambda_I(M,N)$ is Artinian and then $\Lambda_I(M,N)/I\Lambda_I(M,N)$ is also Artinian.

Let $s>0$. From the inductive hypothesis, we have that $\mathrm{H}^I_i(M,N)$ is I-coartinian for all $i<s-1$, and $\mathrm{H}^I_{s-1}(M,N)/I\mathrm{H}^I_{s-1}(M,N)$ is Artinian. Moreover,

$$\operatorname{Cosupp}_R(\mathrm{H}^I_{s-1}(M,N)) \subseteq V(I) \ ,$$

by Lemma 2.7.

We conclude from Lemma 3.1, that $\mathrm{H}^I_{s-1}(M,N)$ is I-coartinian. Now, the short exact sequence of linearly compact R-modules

$$0 \to K \to N \to N/K \to 0 \ ,$$

gives rise to a long exact sequence of generalized local homology modules

$$\ldots \to \mathrm{H}^I_s(M,K) \to \mathrm{H}^I_s(M,N) \xrightarrow{\alpha} \mathrm{H}^I_s(M,N/K) \to \ldots \quad . \qquad (*)$$

It induces an exact sequence,

$$\mathrm{H}^I_s(M,K) \to \mathrm{H}^I_s(M,N) \to \operatorname{Im}(\alpha) \to 0 \ .$$

Then we have the following exact sequence

$$\mathrm{H}^I_s(M,K)/I\mathrm{H}^I_s(M,K) \to \mathrm{H}^I_s(M,N)/I\mathrm{H}^I_s(M,N) \to \operatorname{Im}(\alpha)/I\operatorname{Im}(\alpha) \to 0 \ .$$

Since N/K is complete in I-adic topology, there is an isomorphism

$$\operatorname{Tor}^R_i(M,N/K) \cong \mathrm{H}^I_i(M,N/K) \ ,$$

for all $i \geq 0$, by Lemma 2.6. So, $\mathrm{H}^I_s(M,N/K)$ is an Artinian R-module and then $\operatorname{Im}(\alpha)/I\operatorname{Im}(\alpha)$, is also an Artinian R-module.

In order to prove the artinianness, of $\mathrm{H}^I_s(M,N)/I\mathrm{H}^I_s(M,N)$ we only need to prove that $\mathrm{H}^I_s(M,K)/I\mathrm{H}^I_s(M,K)$ is an Artinian R-module.

By Lemma 2.4, (3), and, (4), there is an element $x \in I$ such that $xK=K$. Now the short exact sequence

$$0 \to (0:_K x) \to K \xrightarrow{.x} K \to 0 \ ,$$

gives rise to a long exact sequence

$$\ldots \to \mathrm{H}^I_i(M,K) \xrightarrow{.x} \mathrm{H}^I_i(M,K) \to \mathrm{H}^I_{i-1}(M,(0:_K x)) \to \ldots \quad .$$

Thus, we have an induced short exact sequence

$$0 \to \mathrm{H}^I_i(M,K)/x\mathrm{H}^I_i(M,K) \to \mathrm{H}^I_{i-1}(M,(0:_K x)) \to \left(0:_{\mathrm{H}^I_{i-1}(M,K)} x\right) \to 0.$$

According to the above argument, $\mathrm{H}_i^I(M, N/K)$ is an Artinian R-module for all $i \geq 0$, so $\mathrm{H}_i^I(M, N/K)$ is minimax. As $\mathrm{H}_i^I(M, N)$ is minimax for all $i < s$, it follows from the exact sequence $(*)$ that $\mathrm{H}_i^I(M, K)$ is minimax for all $i < s$.

Then, $\mathrm{H}_i^I(M, K)/x\mathrm{H}_i^I(M, K)$ and $\left(0 :_{\mathrm{H}_{i-1}^I(M,K)} x\right)$ are minimax for all $i < s$.

Hence, $\mathrm{H}_{i-1}^I(M, (0 :_K x))$ is also minimax for all $i < s$. It should be noted by [16, $1b(0)$] that $(0 :_K x)$ is an Artinian R-module. From the inductive hypothesis, $\mathrm{H}_{i-1}^I(M, (0 :_K x))$ is I-coartinian for all $i < s$, and $\mathrm{H}_{s-1}^I(M, (0 :_K x))/I\mathrm{H}_{s-1}^I(M, (0 :_K x))$ is Artinian.

We now have that

$$\mathrm{Cosupp}_R(\mathrm{H}_{s-1}^I(M, (0 :_K x))) \subseteq V(I) \ ,$$

by Lemma 2.7, and, $\mathrm{H}_{s-1}^I(M, (0 :_K x))$ is a minimax linearly compact module.

Therefore, $\mathrm{H}_{s-1}^I(M, (0 :_K x))$ is I-coartinian by Proposition 3.1, and then $\left(0 :_{\mathrm{H}_{s-1}^I(M,K)} x\right)$ is I-coartinian by Lemma 3.1. Now, the last exact sequence induces an exact sequence

$$\mathrm{Tor}_1^R\left(R/I, \left(0 :_{\mathrm{H}_{s-1}^I(M,K)} x\right)\right) \to \mathrm{H}_s^I(M, K)/I\mathrm{H}_s^I(M, K) \to \mathrm{H}_{s-1}^I(M, (0 :_K x))/I\mathrm{H}_{s-1}^I(M, (0 :_K x)) \ .$$

As $\mathrm{Tor}_1^R\left(R/I, \left(0 :_{\mathrm{H}_{s-1}^I(M,K)} x\right)\right)$ is Artinian, we have that the R-module $\mathrm{H}_s^I(M, K)/I\mathrm{H}_s^I(M, K)$ is also Artinian. Thus, $\mathrm{H}_s^I(M, N)/I\mathrm{H}_s^I(M, N)$ is artinian and then

$$\mathrm{Coass}_R(\mathrm{H}_s^I(M, N)/I\mathrm{H}_s^I(M, N))$$

is a finite set. We also have by Lemma 2.7, that

$$\mathrm{Coass}_R(\mathrm{H}_s^I(M, N)) \subseteq \mathrm{Cosupp}_R(\mathrm{H}_s^I(M, N)) \subseteq V(I) \ .$$

Therefore, we have

$$\mathrm{Coass}_R(\mathrm{H}_s^I(M, N)/I\mathrm{H}_s^I(M, N)) = \mathrm{Coass}_R(\mathrm{H}_s^I(M, N)) \bigcap V(I) = \mathrm{Coass}_R(\mathrm{H}_s^I(M, N)) \ ,$$

by Lemma 2.8. The proof is complete. □

4. Application. The following theorem gives us a more general result.

Theorem 4.1. *Let M be a finitely generated R-module and N a semi-discrete linearly compact R-module such that $N/(\cap_{t>0} I^t N)$ is an Artinian R-module. Let s be a non-negative integer. If $\mathrm{H}_i^I(M, N)$ is minimax for all $i < s$, and G is a closed submodule of $\mathrm{H}_s^I(M, N)$ such that $\mathrm{H}_s^I(M, N)/G$ is minimax, then G/IG is Artinian. In particular, we have that $\mathrm{Coass}_R(G)$ is a finite set.*

Proof: It should be noted by Lemma 2.5, that $\mathrm{H}_s^I(M, N)$ is linearly compact. Then $\mathrm{H}_s^I(M, N)/G$ is also linearly compact by Lemma 2.1, (1).

Set $L = \mathrm{H}_s^I(M, N)/G$, the short exact sequence

$$0 \to G \to \mathrm{H}_s^I(M, N) \to L \to 0,$$

induces an exact sequence

$$\mathrm{Tor}_1^R(R/I, L) \to G/IG \to \mathrm{H}_s^I(M, N)/I\mathrm{H}_s^I(M, N) \to L/IL \to 0 \ .$$

It follows from Theorem 3.1, that $\mathrm{H}_s^I(M, N)/I\mathrm{H}_s^I(M, N)$ is Artinian, so is L/IL.

On the other hand, Lemma 2.7, gives

$$\mathrm{Cosupp}_R(L) \subseteq \mathrm{Cosupp}_R(\mathrm{H}_s^I(M, N)) \subseteq V(I) \ .$$

Then L is I-coartinian, by Proposition 3.1.

So, $\mathrm{Tor}_1^R(R/I, L)$ is Artinian and then we have that G/IG is Artinian. Finally, $\mathrm{Coass}_R(G)$ is a finite set.

This finishes the proof. □

Remark 4.1. We have Theorem 4.1 which provides the finiteness of a set of co-associated primes.

5. Conclusions. In this article, we can relate the theory of commutative algebra to the theory of local cohomology modules. With the results of the article, we show the importance of local cohomology theory as a study tool within commutative algebra theory, for example in results on the finiteness of co-associated primes of generalized local homology modules. Here, by making this relationship, we get applications for the generalized local homology module, in a general theory of modules.

Author contributions. The author wrote the article alone.

Conflicts of interest. The author declare no conflict of interest.

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