Journal homepage http://revistas.unitru.edu.pe/index.php/SSMM

SELECCIONES MATEMÁTICAS
Universidad Nacional de Trujillo
ISSN: 2411-1783 (Online)
2025; Vol.12(2):439-468.

Euler-Rodrigues Rotation for computes the tangent vector and curvature vector of the intersection curve of two surface in 3D Lorentz-Minkowski space

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Received, Set. 31, 2025; *Accepted, Dec. 28, 2025;* *Published, Dec. 27, 2025*

How to cite this article:
Cintra Neto LAR, Aléssio O. *Euler-Rodrigues Rotation for computes the tangent vector and curvature vector of the intersection curve of two surface in 3D Lorentz-Minkowski space*. Selecciones Matemáticas. 2025;12(2):439–468.
https://doi.org/10.17268/sel.mat.2025.02.14

Abstract

We present method computes the tangent and curvature vector of the intersection curve of two surface, parametric/implicit or implicit/implicit, in Lorentz-Minkowski space $\mathbb{E}^3_1$, by applying a Euler-Rodrigues rotation to a vector projected onto the tangent space. The axis of rotation is the normal vector of the surface (the surfaces can be timelike, spacelike or lightlike), therefore three types of rotations, since the normal vectors can be: spacelike, lightlike, or timelike.

Keywords . Euler-Rodrigues formula, tangential intersection, Lorentz-Minkowski space, Surface-Surface intersection.

1. Introduction. Differential geometry of intersection curves for tangential intersections in $\mathbb{E}^3$, the available literature is relatively limited. Ye and Maekawa [1] proposed an algorithm for evaluating higher-order derivatives of the tangential intersection curve of two surfaces, considering all three types of surface-surface intersection problems in $\mathbb{E}^3$. Çalışkan and Düldül [2] computed the unit tangent vector and the geodesic torsion of the tangential intersection curve of two surfaces, also addressing all three intersection types. Nassar et al. [3] investigated the differential geometric properties of the Frenet apparatus (t, n, b, κ, τ) of intersection curves of two implicit surfaces in $\mathbb{E}^3$, treating both transversal and tangential cases using the Implicit Function Theorem.

The Rodrigues rotation formula was introduced into the surface intersection problem by Bahar and Mustafa in [4]. In that paper, the authors proposed new approaches for analyzing both tangential and transversal intersections of two surfaces in Euclidean 3-space using Rodrigues rotation formula. In [5, 6], the authors employed Rodrigues rotation formula to derive the geometric properties of the transversal intersection curve of two regular parametric and implicit surfaces in $\mathbb{E}^3$, including the computation of geodesic curvature and geodesic torsion.

Bahar and Mustafa Düldül [4] introduced two new approaches for analyzing the tangential intersection of two surfaces in Euclidean 3-space, utilizing Rodrigues' rotation formula and Willmore's method. For tangential intersections in $\mathbb{E}^4$, the authors of [7] studied the non-transversal intersection of parametric-parametric-parametric hypersurfaces. In [8], they addressed the non-transversal intersection of implicit-implicit-parametric and implicit-parametric-parametric hypersurfaces, and in [9], the non-transversal intersection of implicit-implicit-implicit hypersurfaces in $\mathbb{E}^4$.

The differential geometry of intersection curves arising from tangential intersections in Lorentz - Minkowski spaces $\mathbb{E}^3_1$ was studied only by Aléssio and Cintra Neto [10], in this work the authors compute the tangent vector of tangential intersection curves of two surfaces parametric in the three-dimensional

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Lorentz- Minkowski space $\mathbb{E}_1^3$, where the combination of the surfaces (spacelike, timelike, or lightlike) for the use of Rodrigues' rotation formula.

Differently from the works [10], in this paper compute the tangent vector and curvature vector of tangential intersection curves of two surfaces (parametric/implicit or implicit/implicit) in the three-dimensional Lorentz-Minkowski space $\mathbb{E}_1^3$, where the combination of the surfaces (spacelike, timelike, or lightlike) for the use of Rodrigues' rotation formula.

The remainder of the paper is organized as follows: Section 2 introduces some notation and definitions, and reviews relevant aspects of differential geometry in the Lorentz- Minkowski 3-space $\mathbb{E}_1^3$. Section 3 presents the Euler-Rodrigues formula adapted to Minkowski 3-space. Section 4 focuses on the differential geometry of the tangential intersection curve of two surfaces in $\mathbb{E}_1^3$, where we propose our method. Some numerical results are provided in Section 5, and concluding remarks are given in Section 6.

2. Preliminaries. In this paper we denote by the Lorentz-Minkowski 3-space $\mathbb{E}_1^3$, the pair $(\mathbb{R}^3, \langle,\rangle_L)$ where $\mathbb{R}^3$ is a three dimensional real vector space equipped with a Lorentz metric (inner product) of signature (2,1). That is, if $\mathbf{v} = (x_1, x_2, x_3)$ and $\mathbf{u} = (y_1, y_2, y_3)$, then

$$\langle \mathbf{v}, \mathbf{u}\rangle_L = x_1y_1 + x_2y_2 - x_3y_3.$$

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{E}_1^3$ are said to be *orthogonal* if $\langle \mathbf{u}, \mathbf{v}\rangle_L = 0$. An arbitrary vector $\mathbf{u}$ in $\mathbb{E}_1^3$ which satisfies $\langle \mathbf{u}, \mathbf{u}\rangle_L = \pm 1$ is called a *unit vector*.

We say that an arbitrary vector $\mathbf{v} \neq 0$ in $\mathbb{E}_1^3$ is called *spacelike*, *timelike* or *lightlike*(*null*), if respectively holds $\langle \mathbf{v}, \mathbf{v}\rangle_L > 0$, $\langle \mathbf{v}, \mathbf{v}\rangle_L < 0$ or $\langle \mathbf{v}, \mathbf{v}\rangle_L = 0$. In particular, the vector $\mathbf{v} = 0$ is *spacelike*. If $\mathbf{v} = (x_1, x_2, x_3) \in \mathbb{E}_1^3$ we define it norm by

$$\|\mathbf{v}\|_L = |\langle \mathbf{v}, \mathbf{v}\rangle_L|^{\frac{1}{2}} = \sqrt[2]{|x_1x_1 + x_2x_2 - x_3x_3|}.$$

The timelike vectors can be separate in two disjoint sets, with describes the next definition.

Definition 2.1. *Let $\mathcal{F}$ be the set of all timelike vectors in $\mathbb{E}_1^3$. If $\mathbf{u}$ is a timelike vector, the timelike cone of $\mathbf{u}$ is the set*

$$\mathcal{C}(\mathbf{u}) = \{\mathbf{v} \in \mathcal{F} | \langle \mathbf{u}, \mathbf{v}\rangle_L < 0\}.$$

Definition 2.2. *[11] Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{E}_1^3$. The Lorentizian vector product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is to the unique vector denoted by $\boldsymbol{u} \times_L \boldsymbol{v}$ that satisfies*

$$\langle \boldsymbol{u} \times_L \boldsymbol{v}, \boldsymbol{w}\rangle_L = det(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \tag{2.1}$$

where $det(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ is the determinant of the matrix obtained by replacing by columns the coordinates of the three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

We also define the vector product of $\mathbf{u}$ and $\mathbf{v}$ (in that order) as the unique vector $\mathbf{u} \times_L \mathbf{v} \in \mathbb{E}_1^3$ such that

$$\mathbf{u} \times_L \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & -\mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), -(u_1v_2 - u_2v_1)),$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical basis of $\mathbb{E}_1^3$ and $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. We have $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical basis of $\mathbb{R}_1^3$, which satisfy $\mathbf{e}_1 \times_L \mathbf{e}_2 = -\mathbf{e}_3$, $\mathbf{e}_2 \times_L \mathbf{e}_3 = \mathbf{e}_1$ and $\mathbf{e}_3 \times_L \mathbf{e}_1 = \mathbf{e}_2$.

Corollary 2.1. *[11] (Lagrange's Identities). Let $\boldsymbol{u}, \boldsymbol{v} \in E_1^3$. Then*

$$\langle \boldsymbol{u} \times_L \boldsymbol{v}, \boldsymbol{u} \times_L \boldsymbol{v}\rangle_L = -det \begin{bmatrix} \langle \boldsymbol{u}, \boldsymbol{u}\rangle_L & \langle \boldsymbol{u}, \boldsymbol{v}\rangle_L \\ \langle \boldsymbol{v}, \boldsymbol{u}\rangle_L & \langle \boldsymbol{v}, \boldsymbol{v}\rangle_L \end{bmatrix}. \tag{2.2}$$

Remark 2.1. *[11] Let us observe that if $\mathbf{u}$ and $\mathbf{v}$ are two non-degenerate vectors, then $B = \{\boldsymbol{u}, \ \boldsymbol{v}, \boldsymbol{u} \times_L \boldsymbol{v}\}$ is basis of $\in \mathbb{E}_1^3$. However, and in contrast to the Euclidean space, the causal character of $\boldsymbol{u}$ and $\boldsymbol{v}$ determines if the basis is or is not positively oriented. Exactly, if $\mathbf{u}$ and $\mathbf{v}$ are spacelike vectors of $\mathbb{E}_1^3$ then $\mathbf{u} \times_L \mathbf{v}$ is a timelike and B is negatively oriented because $det(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u} \times_L \boldsymbol{v}) = \langle \boldsymbol{u} \times_L \boldsymbol{v}, \boldsymbol{u} \times_L \boldsymbol{v}\rangle_L < 0$. Is $\mathbf{u}$ and $\mathbf{v}$ have different causal character, then B is positively oriented.*

2.1. Curves and Surfaces in E_1^3.

Definition 2.3. *A regular curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ can locally be a spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (lightlike).*

Definition 2.4. *For any non-lightlike vector $\mathbf{v} \neq 0$, we set its indicator ϵ_v to be the sign of $\langle \mathbf{v}, \mathbf{v} \rangle_L$, that is, $\epsilon_v = 1$ if $\mathbf{v}$ is spacelike, $\epsilon_v = -1$ if $\mathbf{v}$ is timelike, and $\epsilon_v = 0$ if $\mathbf{v}$ is lightlike.*

The Frenet-Serret trihedron can be found in [10] and [12].

Definition 2.5. *[12] [Regular Parametrized Surface] A smooth map $S : U \subset \mathbb{R}^2 \longrightarrow \mathbb{E}_1^3$ is called a regular parametrized surface if $D_S(u, v)$ has full rank for all $(u, v) \in U$.*

Definition 2.6. *[12] [Regular Implicit Surface] Let $a \in \mathbb{R}$ is a regular value for f, then $f^{-1}(a) = \{(x, y, z) \in D | f(x, y, z) = a\}$ is a regular surface in $\mathbb{E}_1^3$.*

2.1.1. Normal vector of parametric Spacelike, Timelike and Ligthlike Surfaces in $\mathbb{E}_1^3$. The **unit normal** vector field $\mathbf{N}$ of a parametric spacelike (or a timelike) surface $\mathbf{M}$ is given by

$$\mathbf{N} = \frac{\mathbf{S}_u \times_L \mathbf{S}_v}{\|\mathbf{S}_u \times_L \mathbf{S}_v\|_L}. \tag{2.3}$$

The **unit normal** vector field $\mathbf{N}$ of a parametric lightlike surface $\mathbf{M}$ is given by

$$\mathbf{N} = \mathbf{S}_u \times_L \mathbf{S}_v. \tag{2.4}$$

Remark 2.2. *Since $\boldsymbol{S}_u \times_L \boldsymbol{S}_v$ is lightlike the vectors $\boldsymbol{S}_u$ and $\boldsymbol{S}_v$ can be either lightlike or spacelike. If $\langle \boldsymbol{S}_u, \boldsymbol{S}_v \rangle_L = 0$, then one element of the sett $\{\boldsymbol{S}_u, \boldsymbol{S}_v\}$ is lightlike and the other is spacelike.*

2.1.2. Normal vector of implicit Spacelike, Timelike and Ligthlike [12] Surfaces in $\mathbb{E}_1^3$. Consider an arbitrary implicit spacelike (timelike or lightlike) surface

$$S = \{(x, y, z);\ f(x, y, z) = 0\},$$

where $f(x, y, z) : \mathbb{E}_1^3 \longrightarrow \mathbb{R}$ is a differential function. The vector field

$$\nabla_L f = (f_x, f_y, -f_z), \tag{2.5}$$

is orthogonal to any vector of the tangent plane.

*The **unit normal** vector field $\mathbf{N}$ of a parametric spacelike (or a timelike) surface $\mathbf{M}$*
. The **unit normal** is given by

$$\mathbf{N} = \frac{\nabla_L f}{\|\nabla_L f\|_L}.$$

*The **normal** vector field $\mathbf{N}$ of a parametric lightlike surface $\mathbf{M}$*
. The **normal** vector is given by

$$\mathbf{N} = \nabla_L f.$$

2.2. Curves in Surface in E_1^3.

2.2.1. Curves in Parametric Surface in E_1^3. Consider an parametric surface represented by $S : U \subset \mathbb{R}^2 \longrightarrow V \cap S \subset \mathbb{E}_1^3$ and let $\alpha(s)$ a curve in the surface defined by $\alpha(s) = \mathbf{S}(u(s), v(s))$. The $\alpha'(s)$, $\alpha''(s)$ and $\alpha'''(s)$ is

$$\alpha'(s) = \mathbf{S}_u u' + \mathbf{S}_v v', \tag{2.6}$$

$$\alpha''(s) = \mathbf{S}_u u'' + \mathbf{S}_v v'' + \mathbf{S}_{uu}(u')^2 + 2\mathbf{S}_{uv} u'v' + \mathbf{S}_{vv}(v')^2. \tag{2.7}$$

$$\begin{aligned}\alpha'''(s) = \mathbf{S}_u u''' + \mathbf{S}_v v''' + \mathbf{S}_{uuu}(u')^3 + 3\mathbf{S}_{uuv}(u')^2 v' + 3\mathbf{S}_{uvv} u'(v')^2 + \mathbf{S}_{vvv}(v')^3 + \\ 3\left(\mathbf{S}_{uu} u'u'' + \mathbf{S}_{vv} v'v'' + \mathbf{S}_{uv}(u''v' + u'v'')\right).\end{aligned} \tag{2.8}$$

Therefore, the projection of the vectors $\alpha'(s), \alpha''(s)$ and $\alpha'''(s)$ onto the unit normal vector field ($\mathbf{N}$) of the surface $S(u, v)$ are given respectively by

$$\langle \boldsymbol{\alpha}', \boldsymbol{N} \rangle_L = 0, \tag{2.9}$$

$$\langle \boldsymbol{\alpha}'', \boldsymbol{N} \rangle_L = \langle \mathbf{S}_{uu}, \boldsymbol{N} \rangle_L (u')^2 + 2 \langle \mathbf{S}_{uv}, \boldsymbol{N} \rangle_L u'v' + \langle \mathbf{S}_{vv}, \boldsymbol{N} \rangle_L (v')^2. \tag{2.10}$$

$$\langle \boldsymbol{\alpha}''', \boldsymbol{N} \rangle_L \quad = \quad III + II, \tag{2.11}$$

where

$$III \quad = \quad \langle \mathbf{S}_{uuu}, \boldsymbol{N} \rangle_L (u')^3 + 3 \langle \mathbf{S}_{uuv}, \boldsymbol{N} \rangle_L (u')^2 v' + 3 \langle \mathbf{S}_{uvv}, \boldsymbol{N} \rangle_L u'(v')^2 + \langle \mathbf{S}_{vvv}, \boldsymbol{N} \rangle_L (v')^3,$$

$$II \quad = \quad 3 \langle \mathbf{S}_{uu}, \boldsymbol{N} \rangle_L u'u'' + 3 \langle \mathbf{S}_{vv}, \boldsymbol{N} \rangle_L v'v'' + 3 \langle \mathbf{S}_{uv}, \boldsymbol{N} \rangle_L (u''v' + u'v'').$$

2.2.2. Curves in Implicit Surface in E_1^3. Consider an implicit surface represented by $f : \mathbb{E}_1^3 \longrightarrow \mathbb{E}$ and let $\alpha(s)$ a curve in the surface defined by $\alpha(s) = \{(x(s), y(s), z(s)) | f(\alpha(s)) = 0\}$ in the implicit surface $f(x, y, z) = 0$. Then, we have that:

$$\frac{\partial f}{\partial s} = f_x x' + f_y y' + f_z z' = 0, \tag{2.12}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial s^2} \quad &= \quad f_x x'' + f_y y'' + f_z z'' + \\ & \quad f_{xx}(x')^2 + f_{yy}(y')^2 + f_{zz}(z')^2 + 2(f_{xy}x'y' + f_{yz}y'z' + f_{xz}x'z') = 0. \end{aligned} \tag{2.13}$$

$$\begin{aligned} \frac{\partial^3 f}{\partial s^3} \quad &= \quad f_x x''' + f_y y''' + f_z z''' \\ &+ \quad 3 \{ f_{xx} x' x'' + f_{yy} y' y'' + f_{zz} z' z'' + f_{xy}(x'' y' + x' y'') \\ &+ \quad f_{xz}(x'' z' + x' z'') + f_{yz}(y'' z' + y' z'') + f_{xxy}(x')^2 y' + f_{xxz}(x')^2 z' \\ &+ \quad f_{xyy} x'(y')^2 + f_{xzz} x'(z')^2 + f_{yyz}(y')^2 z' + f_{yzz} y'(z')^2 + f_{xyz} x' y' z' \} \\ &+ \quad f_{xxx}(x')^3 + f_{yyy}(y')^3 + f_{zzz}(z')^3 = 0. \end{aligned} \tag{2.14}$$

Therefore, the projection of the vectors $\alpha'(s), \alpha''(s)$ and $\alpha'''(s)$ onto the unit normal vector field ($\mathbf{N}$) of the surfaces $f(x, y, z) = 0$ are given respectively by

$$\langle \nabla_L f, \alpha'(s) \rangle_L = 0, \tag{2.15}$$

$$\langle \nabla_L f, \alpha''(s) \rangle_L \quad = \quad - \langle (\nabla_L f)', \alpha'(s) \rangle_L , \tag{2.16}$$

$$\langle \nabla_L f, \alpha'''(s) \rangle_L \quad = \quad -2 \langle (\nabla_L f)', \alpha''(s) \rangle_L - \langle (\nabla_L f)'', \alpha'(s) \rangle_L , \tag{2.17}$$

where

$$H_L f = \begin{bmatrix} f_{xx} & f_{xy} & -f_{xz} \\ f_{yx} & f_{yy} & -f_{yz} \\ f_{zx} & f_{zy} & -f_{zz} \end{bmatrix} = \begin{bmatrix} \nabla_L f_x \\ \nabla_L f_y \\ \nabla_L f_z \end{bmatrix}.$$

$$\begin{aligned}(\nabla_L f)' &= \left(\langle \nabla_L f_x, \alpha'(s)\rangle_L, \langle \nabla_L f_y, \alpha'\rangle_L, \langle -\nabla_L f_z, \alpha'(s)\rangle_L\right), \\ (\nabla_L f)' &= \left(\langle H_L f[0,0:3], \alpha'(s)\rangle_L, \langle H_L f[1,0:3], \alpha'\rangle_L, \langle -H_L f[2,0:3], \alpha'(s)\rangle_L\right),\end{aligned} \tag{2.18}$$

$$\begin{aligned}(\nabla_L f)'' &= \diamond_1 + \diamond_2, \\ \diamond_1 &= \left(\langle (\nabla_L f_x)', \alpha'(s)\rangle_L, \langle (\nabla_L f_y)', \alpha'\rangle_L, \langle -(\nabla_L f_z)', \alpha'(s)\rangle_L\right), \\ \diamond_2 &= \left(\langle \nabla_L f_x, \alpha''(s)\rangle_L, \langle \nabla_L f_y, \alpha''\rangle_L, \langle -\nabla_L f_z, \alpha''(s)\rangle_L\right).\end{aligned} \tag{2.19}$$

$$\nabla_L f_j = (f_{jx}, f_{jy}, -f_{jz}), \; j \in \{x, y, z\}. \tag{2.20}$$

$$(\nabla_L f_j)' = \left(\langle \nabla_L f_{jx}, \alpha'(s)\rangle_L, \langle \nabla_L f_{jy}, \alpha'\rangle_L, \langle -\nabla_L f_{jz}, \alpha'(s)\rangle_L\right), \; j \in \{x, y, z\}, \tag{2.21}$$

$$\nabla_L f_{jk} = (f_{jkx}, f_{jky}, -f_{jkz}), \; j, k \in \{x, y, z\}. \tag{2.22}$$

3. Euler-Rodrigues formula in Minkowski 3-space ($\mathbb{E}^3_1$). In this work, the Lorentz metric (inner product) of signature $(+,+,-)$, then the semi-skew-symmetric matrix W obtained from the vector $\mathbf{w} = (w_x, w_y, w_z)$ **axis with unit length** be

$$W = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ w_y & -w_x & 0 \end{bmatrix}.$$

Case 1. *[13, 14] Assume that $\mathbf{w} = (w_x, w_y, w_z)$ is unit spacelike vector, then we get*

$$\mathcal{R}_{sp}(\theta, \mathbf{w}) = I_3 + sinh(\theta)W + (-1 + cosh(\theta))W^2, \tag{3.1}$$

Case 2. *[13, 14] Assume that $\mathbf{w} = (w_x, w_y, w_z)$ is unit timelike axis, then we get*

$$\mathcal{R}_{tm}(\theta, \mathbf{w}) = I_3 + sin(\theta)W + (1 - cos(\theta))W^2, \tag{3.2}$$

Case 3. *[13] If $\mathbf{w} = (w_x, w_y, w_z)$ is lightlike axis, then we get*

$$\mathcal{R}_{lg}(\theta, \mathbf{w}) = I_3 + \theta\, W + \frac{\theta^2}{2}\, W^2. \tag{3.3}$$

Where

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3.1. Operator $\mathcal{D}_L$. In the paper [4], the authors defined the operator $\mathcal{D}(\mathbf{w}) = \mathbf{u} \times_E \mathbf{w}$, where $\times_E$ cross product Euclidean space. In this subsection, we defined $\mathcal{D}_L$. Let $\mathbf{w}$ be a nonzero vector in $\mathbb{E}^3_1$. We define $\mathcal{D}_L$ as

$$\mathcal{D}_L(\mathbf{w}) = \mathbf{u} \times_L \mathbf{w}. \tag{3.4}$$

Where $\times_L$ is cross product Lorentzian spaces.

Proposition 3.1. *If the vector $\mathbf{u}$ is chosen arbitrary such that it is linearly independent with $\mathbf{w}$, then $\mathcal{D}_L(\mathbf{w})$ yields never a zero vector and also we can see that* *i)*

$$\langle \mathcal{D}_L(\mathbf{w}), \mathcal{D}_L(\mathbf{w})\rangle_L = \langle \mathbf{u} \times_L \mathbf{w}, \mathbf{u} \times_L \mathbf{w}\rangle_L = -det \begin{vmatrix} \langle \mathbf{u}, \mathbf{u}\rangle_L & \langle \mu, \mathbf{w}\rangle_L \\ \langle \mathbf{w}, \mathbf{u}\rangle_L & \langle \mathbf{w}, \mathbf{w}\rangle_L \end{vmatrix} = \lambda^2 - \varepsilon_u \varepsilon_w,$$

where $\langle \mu, \mathbf{w}\rangle_L = \lambda$. $\varepsilon_u = \langle \mathbf{u}, \mathbf{u}\rangle_L$ *and* $\varepsilon_w = \langle \mathbf{w}, \mathbf{w}\rangle_L$.

ii) If $\mathbf{u}$ *and* $\mathbf{w}$ *are spacelike vectors, we have* $\mathcal{D}_L(\mathbf{w})$ *is spacelike or* $\mathcal{D}_L(\mathbf{w})$ *is timelike or* $\mathcal{D}_L(\mathbf{w})$ *is lightlike ;*

iii) If $\mathbf{u}$ *and* $\mathbf{w}$ *are timelike vectors, we have* $\mathcal{D}_L(\mathbf{w})$ *is spacelike;*

iv) If $\mathbf{u}$ *and* $\mathbf{w}$ *are lightlike (linearly independent), them* $\mathcal{D}_L(\mathbf{w})$ *is spacelike vector;*

v) If $\mathbf{u}$ *is timelike(spacelike) and* $\mathbf{w}$ *is spacelike (timelike), them* $\mathcal{D}_L(\mathbf{w})$ *is spacelike vector;*

vi) If $\mathbf{u}$ *is lightlike (spacelike) and* $\mathbf{w}$ *is spacelike (lightlike) with* $\langle \mathbf{u}, \mathbf{w}\rangle_L = 0$ *them* $\mathcal{D}_L(\mathbf{w})$ *is lightlike vector;*

vii) If $\mathbf{u}$ *is lightlike (spacelike) and* $\mathbf{w}$ *is spacelike (lightlike) with* $\langle \mathbf{u}, \mathbf{w}\rangle_L \neq 0$ *them* $\mathcal{D}_L(\mathbf{w})$ *is spacelike vector;*

viii) If $\mathbf{u}$ *is lightlike (timelike) and* $\mathbf{w}$ *is timelike (lightlike), them* $\mathcal{D}_L(\mathbf{w})$ *spacelike vector,* $\lambda \neq 0$.

The operator $\mathcal{D}_L$ will play the main role together with the **Euler-Rodrigues formula in Minkowski 3-space** for finding the tangent vector at the tangential intersection point of the intersection curve of two surfaces.

Remark 3.1. *The rotation* $\mathcal{R}_{tm}(\theta, \mathbf{w}) = I_3 + sin(\theta)W + (1 - cos(\theta))W^2$ *of vector* $\frac{\mathcal{D}_L(\mathbf{w})}{\|\mathcal{D}_L(\mathbf{w})\|_L}$ *still keeps it unitary, but the rotation* $\mathcal{R}_{sp}(\theta, \mathbf{w}) = I_3 + sinh(\theta)W + (1 - cosh(\theta))W^2$ *does not maintain its unit length. See the figure below.*

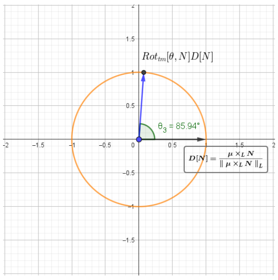

Figure 3.1: $\mathcal{R}_{tm}(\theta, \mathbf{N})\frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}$

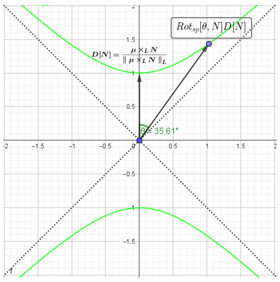

Figure 3.2: $\mathcal{R}_{sp}(\theta, \mathbf{N})\frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}$

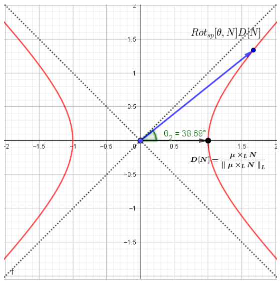

Figure 3.3: $\mathcal{R}_{sp}(\theta, \mathbf{N})\frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}$

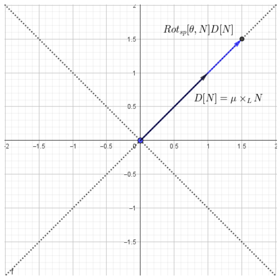

Figure 3.4: $\mathcal{R}_{lg}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N})$

Remark 3.2. *The* $\mathcal{R}_{sp}(\theta, \mathbf{w})\dfrac{\mathcal{D}_L(\mathbf{w})}{\|\mathcal{D}_L(\mathbf{w})\|_L}$ *transform the timelike vectors to timelike vectors, the spacelike vectors to spacelike vectors and the lightlike vectors to lightlike vectors.*

Theorem 3.1. *[10] The* $\mathcal{R}_{lg}(\theta, \mathbf{w})\mathcal{D}_L(\mathbf{w})$ *transform the lightlike vectors to lightlike vectors.*

4. Differential Geometry of Tangential Intersection Curve of Two Surfaces in $\mathbb{E}^3_1$. Now, let us assume the two surfaces S^A and S^B intersect tangentially at a point P_0 where $P_0 = S^A(u_0, v_0)$ *and* $f^B(P_0) = 0$ or $f^A(P_0) = 0$ *and* $f^B(P_0) = 0$ on the intersection curve $\alpha(s)$, i.e., $N^A(P_0) \parallel N^B(P_0)$ at P_0. If the surfaces are spacelike or timelike, by orienting the surfaces properly we can assume that $N^A(P_0) = N^B(P_0) = N$. If the surfaces are lightlike, $N^A(P_0) = \delta N^B(P_0)$, for some real $\delta \in \mathcal{R}$.

4.1. Method Euler-Rodrigues formula Rotation for tangential intersection. The rotation with rotation angle θ around the axis in the direction of $\mathbf{w}$ is $\mathcal{R}_{type}(\theta, \mathbf{w})$, where $type = \{sp, tm, lg\}$, depending on whether $\mathbf{w}$ is spacelike or timelike or lightlike, respectively.

4.1.1. The tangent vector $\alpha'(s)$, of the tangential intersection curve.. The rotation will be given by $\mathcal{R}_{type}(\theta, \mathbf{N})\dfrac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}$ or $\mathcal{R}_{type}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N})$ depending on whether the vector $\mathcal{D}_L(\mathbf{N})$ is timelike, spacelike or lightlike.

If $\mathbf{N}$ is spacelike(lightlike), the $\mathcal{R}_{sp}(\theta, \mathbf{N})\dfrac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}$($\mathcal{R}_{lg}(\theta, \mathbf{N})\dfrac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}$) may not preserve the length of vector $\dfrac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}$.

Parametric-implicit surfaces Spacelike or Timelike.

The **vector tangent** of the intersection two surfaces: parametric-implicit is

$$\alpha'(s) \quad = \quad S_u^A u'(s) + S_v^A v'(s) = (x'(s), y'(s), z'(s)) \quad . \tag{4.1}$$

The equation (4.1) consists of five variables $u', v', x'(s), y'(s)$ and $z'(s)$.

The projection of the vector $\alpha''(s)$ onto $\mathbf{N}^A$ and $\mathbf{N}^B$,

$$\left\langle \mathbf{N}^A, \alpha''(s) \right\rangle_L = \left\langle \mathbf{N}^B, \alpha''(s) \right\rangle_L \quad , \tag{4.2}$$

by using (2.10) and (2.16) produces the equation.

$$e^A (u')^2 + 2 f^A u'v' + g^A (v')^2 = -\frac{\left\langle \left(\nabla_L f^B\right)', \alpha'(s) \right\rangle_L}{\|\nabla_L f^B\|_L}. \tag{4.3}$$

The vector tangent (4.1) can be write in terms of rotation angle θ

$$\alpha'(\theta) \quad = \quad S_u^A u'(\theta) + S_v^A v'(\theta) = (x'(\theta), y'(\theta), z'(\theta)) = \mathcal{R}_{type}(\theta, \mathbf{N}) \frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}, \tag{4.4}$$

or

$$\alpha'(\theta) \quad = \quad S_u^A u'(\theta) + S_v^A v'(\theta) = (x'(\theta), y'(\theta), z'(\theta)) = \mathcal{R}_{type}(\theta, \mathbf{N}) \mathcal{D}_L(\mathbf{N}) \quad , \tag{4.5}$$

the u' and v' values can be obtained in terms of the rotation angle θ.

$$\begin{aligned} u'(\theta) &= \frac{\left\langle \alpha'(\theta) \times_L S_v^A, \mathbf{N} \right\rangle_L}{\left\langle S_u^A \times_L S_v^A, \mathbf{N} \right\rangle_L}, \\ v'(\theta) &= \frac{\left\langle \alpha'(\theta) \times_L S_u^A, \mathbf{N} \right\rangle_L}{\left\langle S_v^A \times_L S_u^A, \mathbf{N} \right\rangle_L}, \end{aligned} \tag{4.6}$$

and the x', y' and z' values can be obtained in terms of the rotation angle θ.

$$\alpha'(\theta) = (x'(\theta), y'(\theta), z'(\theta)).$$

Substituting these solutions $u'(s) = u'(\theta),\ v'(s) = v'(\theta),\ x'(s) = x'(\theta),\ y'(s) = y'(\theta),\ z'(s) = z'(\theta)$ in the equation (4.3), we have the trigonometric equation

$$e^A (u'(\theta))^2 + 2 f^A u'(\theta) v'(\theta) + g^A (v'(\theta))^2 = -\frac{\left\langle \left(\nabla_L f^B\right)', \alpha'(\theta) \right\rangle_L}{\|\nabla_L f^B\|_L}, \tag{4.7}$$

where

$$\left(\nabla_L f^B\right)' \quad = \quad \left(\left\langle \nabla_L f_x^B, \alpha'(\theta) \right\rangle_L, \left\langle \nabla_L f_y^B, \alpha'(\theta) \right\rangle_L, \left\langle -\nabla_L f_z^B, \alpha'(\theta) \right\rangle_L \right) \tag{4.8}$$

Solving the equation (4.7), we can be find the rotation angle θ. Substituting the results into (4.5) yields the tangent vector.

Parametric-implicit surfaces Lighlike.

If the surfaces are lightlike, $N^A(P_0) = \delta N^B(P_0)$, for some real $\delta \in \mathcal{R}$. The vector tangent of the intersection two surfaces: parametric-implicit is

The **vector tangent** of the intersection two surfaces: parametric-implicit is

$$\alpha'(s) = S_u^A u'(s) + S_v^A v'(s) = (x'(s), y'(s), z'(s)) \tag{4.9}$$

The equation (4.9) consists of five variables $u', v', x'(s), y'(s)$ and $z'(s)$.

The projection of the vector $\alpha''(s)$ onto $\mathbf{N}^A$ and $\mathbf{N}^B$,

$$\left\langle \mathbf{N}^A, \alpha''(s) \right\rangle_L = \delta \left\langle \mathbf{N}^B, \alpha''(s) \right\rangle_L \ , \tag{4.10}$$

by using (2.10) and (2.16) produces the equation.

$$e^A(u')^2 + 2f^A u'v' + g^A(v')^2 = -\delta \left\langle \left(\nabla_L f^B\right)', \alpha'(s) \right\rangle_L . \tag{4.11}$$

The vector tangent (4.9) can be write in terms of rotation angle θ

$$\alpha'(\theta) = S_u^A u'(\theta) + S_v^A v'(\theta) = (x'(\theta), y'(\theta), z'(\theta)) = \mathcal{R}_{lg}(\theta, \mathbf{N}) \frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L} \tag{4.12}$$

or

$$\alpha'(\theta) = S_u^A u'(\theta) + S_v^A v'(\theta) = (x'(\theta), y'(\theta), z'(\theta)) = \mathcal{R}_{lg}(\theta, \mathbf{N}) \mathcal{D}_L(\mathbf{N}) \ , \tag{4.13}$$

the u' and v' values can be obtained in terms of the rotation angle θ.

$$\begin{aligned} u'(\theta) &= \frac{\left\langle \alpha'(\theta) \times_E S_v^A, \mathbf{N} \right\rangle_L}{\left\langle S_u^A \times_E S_v^A, \mathbf{N} \right\rangle_L}, \\ v'(\theta) &= \frac{\left\langle \alpha'(\theta) \times_E S_u^A, \mathbf{N} \right\rangle_L}{\left\langle S_v^A \times_E S_u^A, \mathbf{N} \right\rangle_L}. \end{aligned} \tag{4.14}$$

Remark 4.1. *we use* $\left\langle S_u^A \times_E S_v^A, \mathbf{N} \right\rangle_L$ *instead of* $\left\langle S_u^A \times_L S_v^A, \mathbf{N} \right\rangle_L$ *would have a value of zero.*
and the x', y' and z' values can be obtained in terms of the rotation angle θ.

$$\alpha'(\theta) = (x'(\theta), y'(\theta), z'(\theta)).$$

Substituting these solutions $u'(s) = u'(\theta),\ v'(s) = v'(\theta),\ x'(s) = x'(\theta),\ y'(s) = y'(\theta),\ z'(s) = z'(\theta)$ in the equation (4.11), we have the trigonometric equation

$$e^A(u'(\theta))^2 + 2f^A u'(\theta)v'(\theta) + g^A(v'(\theta))^2 = -\left\langle \left(\nabla_L f^B\right)', \alpha'(\theta) \right\rangle_L , \tag{4.15}$$

where

$$\left(\nabla_L f^B\right)' = \left(\left\langle \nabla_L f_x^B, \alpha'(\theta) \right\rangle_L, \left\langle \nabla_L f_y^B, \alpha'(\theta) \right\rangle_L, \left\langle -\nabla_L f_z^B, \alpha'(\theta) \right\rangle_L \right). \tag{4.16}$$

Solving the equation (4.15), we can be find the rotation angle θ. Substituting the results into (4.13) yields the tangent vector.

Implicit-implicit surfaces Spacelike or Timelike.
The **vector tangent** is

$$\alpha'(s) = (x'(s), y'(s), z'(s)) = \mathcal{R}_{type}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}), \tag{4.17}$$

or

$$\alpha'(s) = (x'(s), y'(s), z'(s)) = \mathcal{R}_{type}(\theta, \mathbf{N})\frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}. \tag{4.18}$$

The projection of the vector α'' onto $\mathbf{N}^A$ and $\mathbf{N}^B$,

$$\left\langle \mathbf{N}^A, \alpha'' \right\rangle_L = \left\langle \mathbf{N}^B, \alpha'' \right\rangle_L \ , \tag{4.19}$$

by using (2.16) produces the equation

$$\left\langle (\nabla_L f^A)' - \frac{\|\nabla_L f^A\|_L}{\|\nabla_L f^B\|_L}(\nabla_L f^B)', \alpha'(s) \right\rangle_L = 0 \ , \tag{4.20}$$

where

$$(\nabla_L f)' \quad = \quad \left(\langle \nabla_L f_x, \alpha'(s)\rangle_L, \langle \nabla_L f_y, \alpha'(s)\rangle_L, \langle -\nabla_L f_z, \alpha'(s)\rangle_L\right). \tag{4.21}$$

The x', y' and z' values can be obtained in terms of the rotation angle θ.

$$\alpha'(\theta) = (x'(\theta), y'(\theta), z'(\theta)) = \mathcal{R}_{type}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}),$$

or

$$\alpha'(\theta) = (x'(\theta), y'(\theta), z'(\theta)) = \mathcal{R}_{type}(\theta, \mathbf{N})\frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}. \tag{4.22}$$

Substituting these solutions $x'(\theta)$, $y'(\theta)$, $z'(\theta)$ in the equation (4.20), we have the trigonometric equation

$$\left\langle (\nabla_L f^A)' - \frac{\|\nabla_L f^A\|_L}{\|\nabla_L f^B\|_L}(\nabla_L f^B)', \alpha'(\theta) \right\rangle_L = 0, \tag{4.23}$$

$$(\nabla_L f^B)' \quad = \quad \left(\left\langle \nabla_L f^i_x, \alpha'(\theta)\right\rangle_L, \left\langle \nabla_L f^i_y, \alpha'(\theta)\right\rangle_L, \left\langle -\nabla_L f^i_z, \alpha'(\theta)\right\rangle_L\right). \tag{4.24}$$

$i \in \{A, B\}$.

Solving the equation (4.23), we can be find the rotation angle θ. Substituting the results into (4.18) yields the tangent vector.

Implicit-implicit surfaces Lightlike.
The **vector tangent** is

$$\alpha'(s) = (x'(s), y'(s), z'(s)) = \mathcal{R}_{lg}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}), \tag{4.25}$$

or

$$\alpha'(s) = (x'(s), y'(s), z'(s)) = \mathcal{R}_{lg}(\theta, \mathbf{N})\frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}. \tag{4.26}$$

The projection of the vector α'' onto $\mathbf{N}^A = \nabla_L f^A$ and $\delta\mathbf{N}^B = \delta\nabla_L f^B$ produces the equation.

$$\left\langle \mathbf{N}^A, \alpha'' \right\rangle_L = \left\langle \delta\mathbf{N}^B, \alpha'' \right\rangle_L, \tag{4.27}$$

by using (2.16) produces the equation

$$\left\langle (\nabla_L f^A)' - \delta(\nabla_L f^B)', \alpha'(s) \right\rangle_L = 0, \tag{4.28}$$

where

$$(\nabla_L f)' = \left(\langle \nabla_L f_x, \alpha'(s) \rangle_L , \langle \nabla_L f_y, \alpha'(s) \rangle_L , \langle -\nabla_L f_z, \alpha'(s) \rangle_L \right). \tag{4.29}$$

The x', y' *and* z' values can be obtained in terms of the rotation angle θ.

$$\alpha'(\theta) = (x'(\theta), y'(\theta), z'(\theta)) = \mathcal{R}_{lg}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N})$$

or

$$\alpha'(\theta) = (x'(\theta), y'(\theta), z'(\theta)) = \mathcal{R}_{lg}(\theta, \mathbf{N})\frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}. \tag{4.30}$$

Substituting these solutions $x'(\theta),\ y'(\theta),\ z'(\theta)$ in the equation (4.20), we have the trigonometric equation

$$\left\langle (\nabla_L f^A)' - \delta(\nabla_L f^B)', \alpha'(\theta) \right\rangle_L = 0, \tag{4.31}$$

$$\left(\nabla_L f^i\right)' = \left(\left\langle \nabla_L f^i_x, \alpha'(\theta) \right\rangle_L , \left\langle \nabla_L f^i_y, \alpha'(\theta) \right\rangle_L , \left\langle -\nabla_L f^i_z, \alpha'(\theta) \right\rangle_L \right). \tag{4.32}$$

$i \in \{A, B\}$.

Solving the equation (4.31), we can be find the rotation angle θ. Substituting the results into (4.26) yields the tangent vector.

4.1.2. Solution of the Equations (4.7,4.15,4.23,4.31).

Theorem 4.1. *Let S^A and S^B be spacelike surfaces that intersect tangentially at a point $P_0 = S^A(u_0, v_0) = S^B(p_0, q_0)$, i.e., $N^A(u_0, v_0) \parallel N^B(p_0, q_0)$ at P_0. Since the surfaces are spacelike, the normal vector $\mathbf{N}$ is timelike. Therefore, the corresponding rotation is of timelike type: $\mathcal{R}_{tm}(\theta, \mathbf{N})$. As a result, the transformation $\mathcal{R}_{tm}(\theta, \mathbf{N})\mathcal{D}_L(N)$ is similar the Rodrigues rotation formula was introduced into the surface intersection problem by Bahar and Mustafa in [4].*

Theorem 4.2. *Let S^A and S^B be timelike surfaces that intersect tangentially at a point $P_0 = S^A(u_0, v_0) = S^B(p_0, q_0)$, i.e., $N^A(u_0, v_0) \parallel N^B(p_0, q_0)$ at P_0. Since the surfaces are timelike, the normal vector $\mathbf{N}$ is spacelike. Therefore, the corresponding rotation is of spacelike type: $\mathcal{R}_{sp}(\theta, \mathbf{N})$. As a result, the transformation $\mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(N)$ maps lightlike vectors to lightlike vectors. Proof:* See theorem 1 of the [14] article. □

Theorem 4.3. *[10] Let S^A and S^B be lighlike surfaces that intersect tangentially at a point $P_0 = S^A(u_0, v_0) = S^B(p_0, q_0)$, i.e., $N^A(u_0, v_0) \parallel N^B(p_0, q_0)$ at P_0. Since the surfaces are lighlike, the normal vector $\mathbf{N}$ is lighlike. Therefore, the corresponding rotation is of lighlike type: $\mathcal{R}_{lg}(\theta, \mathbf{N})$. As a result, the transformation $\mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(N)$ maps lightlike vectors to lightlike vectors.*

Remark 4.2.

To analyze the solutions of the trigonometric equations (4.7,4.15,4.23,4.31) in the variable θ, we need to separate into three cases: When $\mathbf{N}$ is timelike or is lightlike or is spacelike.

- *If $\mathbf{N}$ is **timelike**, we have the following cases depending upon the number of solutions:*
  - *(a) If equation has no solution, then P is the isolated contact point.*
  - *(b) If equation has one simple solution, then we have one intersection curve passing through P.*
  - *(c) If equation has several simple solutions, then P is a branch point, i.e. we have another branch passing through P.*
  - *(d) If equation vanishes, then surfaces have at least second order contact at P.*
- *If $\mathbf{N}$ is **spacelike** in $p \in M$, T_pM is a timelike plane, then T_pM contains **two** linearly independent lightlike vectors, timelike and spacelike vector, therefore the vector tangent can be spacelike, timelike or lightlike. Since the rotation $\mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N})$ transform the timelike vectors to timelike vectors, the spacelike vectors to spacelike vectors and the lightlike vectors to lightlike vectors, we*

must choose four vector $\mu_{\mathbf{i}}$ $i \in \{1,2,3,4\}$ for $\mathcal{D}_L(N) = \mu_{\mathbf{i}} \times_L \mathbf{N}$. We can choose the vector $\mu_{\mathbf{1}}$ such that $\mathcal{D}_L(N) = \mu_{\mathbf{1}} \times_L \mathbf{N}$ be lightlike and $\mu_{\mathbf{2}}$ such that $\mathcal{D}_L(N) = \mu_{\mathbf{2}} \times_L \mathbf{N}$ lightlike and $\mu_{\mathbf{3}}$ such that $\mathcal{D}_L(N) = \mu_{\mathbf{3}} \times_L \mathbf{N}$ spacelike and $\mu_{\mathbf{4}}$ such that $\mathcal{D}_L(N) = \mu_{\mathbf{4}} \times_L \mathbf{N}$ timelike.

$$eq1 = \left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L, \ for\ \mu_{\mathbf{1}},$$

$$eq2 = \left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L, \ for\ \mu_{\mathbf{2}},$$

$$eq3 = \left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L, \ for\ \mu_{\mathbf{3}},$$

$$eq4 = \left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L, \ for\ \mu_{\mathbf{4}}.$$

eq1	eq2	eq3	eq4	case	solution
$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	a	If equation has no solution, then P is the isolated contact point.
$\equiv 0$	$\neq 0$	$\neq 0$	$\neq 0$	b	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_{\mathbf{1}} \times_L \mathbf{N}$ for any θ.
$\neq 0$	$\equiv 0$	$\neq 0$	$\neq 0$	b	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_{\mathbf{2}} \times_L \mathbf{N}$ for any θ.
$\neq 0$	$\neq 0$	$= 0$	$\neq 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_{\mathbf{3}} \times_L \mathbf{N},\ if\theta_i$ is solutions to the $eq3 = 0$.
$\neq 0$	$\neq 0$	$\neq 0$	$= 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_{\mathbf{4}} \times_L \mathbf{N}$, if θ_i is solutions to the $eq4 = 0$.
$\neq 0$	$\equiv 0$	$\neq 0$	$= 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_{\mathbf{2}} \times_L \mathbf{N}$ for any θ. $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_{\mathbf{4}} \times_L \mathbf{N}$, if θ_i is solutions to the $eq4 = 0$.
$\neq 0$	$\equiv 0$	$= 0$	$\neq 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_{\mathbf{2}} \times_L \mathbf{N}$, for any θ. $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_{\mathbf{3}} \times_L \mathbf{N}$, if θ_i is solutions to the $eq3 = 0$.
$\equiv 0$	$\equiv 0$	$\equiv 0$	$\neq 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_{\mathbf{1}} \times_L \mathbf{N};\ \alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_{\mathbf{2}} \times_L \mathbf{N},\ \forall\,\theta.$
$\equiv 0$	$\equiv 0$	$\neq 0$	$\equiv 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_{\mathbf{1}} \times_L \mathbf{N};\ \alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_{\mathbf{2}} \times_L \mathbf{N},\ \forall\,\theta.$
$\equiv 0$	$\equiv 0$	$\neq 0$	$\neq 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_{\mathbf{1}} \times_L \mathbf{N};\ \alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_{\mathbf{2}} \times_L \mathbf{N},\ \forall\,\theta.$
$\equiv 0$	$\neq 0$	$\neq 0$	$= 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_{\mathbf{1}} \times_L \mathbf{N}$,f or any θ $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_{\mathbf{4}} \times_L \mathbf{N}$, if θ_i is solutions to the $eq4 = 0$.
$\equiv 0$	$\neq 0$	$= 0$	$\neq 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_{\mathbf{1}} \times_L \mathbf{N}$ for any θ. $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_{\mathbf{3}} \times_L \mathbf{N}$, if θ_i is solutions to the $eq3 = 0$.
$\neq 0$	$\neq 0$	$= 0$	$= 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_{\mathbf{3}} \times_L \mathbf{N},\ if\ \theta_i$ is solutions to the $eq3 = 0$. $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_{\mathbf{4}} \times_L \mathbf{N},\ if\ \theta_i$ is solutions to the $eq4 = 0$.
$\equiv 0$	$\neq 0$	$= 0$	$= 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_{\mathbf{1}} \times_L \mathbf{N}$, for any θ. $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_{\mathbf{3}} \times_L \mathbf{N},\ if\ \theta_i$, is solutions to the $eq3 = 0$. $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_{\mathbf{4}} \times_L \mathbf{N},\ if\ \theta_i$ is solutions to the $eq4 = 0$.
$\neq 0$	$\equiv 0$	$= 0$	$= 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_{\mathbf{2}} \times_L \mathbf{N}$ for any θ. $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_{\mathbf{3}} \times_L \mathbf{N},\ if\theta_i$ is solutions to the $eq3 = 0$ $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_{\mathbf{4}} \times_L \mathbf{N},\ if\theta_i$ is solutions to the $eq4 = 0$
$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	d	have at least second order contact at P

Table 4.1: Solutions

Where the cases are:

(a) If equation has no solution, then P is the isolated contact point.
(b) If equation has one simple solution, then we have one intersection curve passing through P.
(c) If equation has several simple solutions, then P is a branch point, i.e. we have another branch passing through P.
(d) If equation vanishes, then surfaces have at least second order contact at P.

- *If* $\mathbf{N}$ *is* ***lightlike*** *in* $p \in M$, T_pM *is a lightlike plane, then the* T_pM *contains only one vector lightlike vector and spacelike vector, but not a timelike one, therefore vector tangent can be spacelike or lightlike.*
  Since $\mathcal{D}_L(\mathbf{N})$ *can be spacelike or lightlike, the rotation* $\mathcal{R}_{lg}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N})$ *transform the* $\mathcal{D}_L(\mathbf{N})$ *spacelike vector to spacelike or lighlike vectors and* $\mathcal{D}_L(\mathbf{N})$ *lightlike vectors to lightlike vectors, we must choose* $\mu_\mathbf{1}$ *such that* $\mathcal{D}_L(N) = \mu_\mathbf{1} \times_L \mathbf{N}$ *be lightlike and* $\mu_\mathbf{2}$ *such that* $\mathcal{D}_L(N) = \mu_\mathbf{2} \times_L \mathbf{N}$ *spacelike. For the choice of* μ*, see Proposition (3.1).*

4.1.3. The curvature vector $\alpha''(s)$**, of the tangential intersection curve..** The rotation will be given by $\mathcal{R}_{type}(\phi, \alpha'(s))\dfrac{\mathcal{D}_L(\alpha'(s))}{\|\mathcal{D}_L(\alpha'(s))\|_L}$ or $\mathcal{R}_{type}(\phi, \alpha'(\mathbf{s}))\mathcal{D}_L(\alpha'(s))$, depending on whether the vector $\mathcal{D}_L(\alpha'(s))$ is timelike, spacelike or lightlike.

If $\alpha'(s)$ is spacelike or lightlike, the $\mathcal{R}_{sp}(\phi, \alpha'(\mathbf{s}))\dfrac{\mathcal{D}_L(\alpha'(s))}{\|\mathcal{D}_L(\alpha'(s))\|_L}$ or $\mathcal{R}_{lg}(\phi, \alpha'(s))\mathcal{D}_L(\alpha'(s))$ may not preserve the length of vector $\dfrac{\mathcal{D}_L(\alpha'(s))}{\|\mathcal{D}_L(\alpha'(s))\|_L}$ or $\mathcal{D}_L(\alpha'(s))$, respectively.

Parametric-implicit surfaces Spacelike or Timelike .

The **curvature vector** of the intersection two surfaces: parametric-implicit is

$$\begin{aligned} \alpha''(s) &= \mathbf{S}_u u'' + \mathbf{S}_v v'' + \mathbf{S}_{uu}(u')^2 + 2\mathbf{S}_{uv}u'v' + \mathbf{S}_{vv}(v')^2, \\ &= (x''(s), y''(s), z''(s))\,. \end{aligned} \tag{4.33}$$

The equation (4.33) consists of five variables $u'', v'', x''(s), y''(s)$ *and* $z''(s)$. These quantities can be expressed in terms of $\kappa\sin(\phi)$ and $\kappa\cos(\phi)$.

The first equation is obtained for projection of the vector α''' onto $\mathbf{N}^A$ and $\mathbf{N}^B$,

$$\left\langle \mathbf{N}^A, \alpha'''(s) \right\rangle_L = \left\langle \mathbf{N}^B, \alpha'''(s) \right\rangle_L, \tag{4.34}$$

by using (2.11) and (2.17), produces the equation.

$$III(u', v') + II(u', v', u'', v'') = \frac{-3\left\langle (\nabla_L f^B)', \alpha''(s) \right\rangle_L - \left\langle \diamond_1^B, \alpha'(s) \right\rangle_L}{\|\nabla_L f^B\|_L}\;. \tag{4.35}$$

The curvature vector (4.33) can be write in terms of rotation angle ϕ

$$\alpha''(\phi) = \kappa\mathcal{R}_{type}(\theta, \alpha'(s))\frac{\mathcal{D}_L(\alpha'(s))}{\|\mathcal{D}_L(\alpha'(s))\|_L}, \tag{4.36}$$

if $\alpha''(\phi)$ is timelike, spacelike, and

$$\alpha''(\phi) = \mathcal{R}_{type}(\theta, \alpha'(s))\mathcal{D}_L(\alpha'(s))\;\;, \tag{4.37}$$

if $\alpha''(\phi)$ is lightlike.

The u'' *and* v'' values can be obtained in terms of the $k\cos(\phi)$ and $k\sin(\phi)$

$$\begin{aligned} u''(\phi) &= \frac{\left\langle \alpha''(\phi) \times_L S_v^A, \mathbf{N}^A \right\rangle_L}{\langle S_u^A \times_L S_v^A, \mathbf{N}^A \rangle_L}, \\ v''(\phi) &= \frac{\left\langle \alpha''(\phi) \times_L S_u^A, \mathbf{N}^A \right\rangle_L}{\langle S_v^A \times_L S_u^A, \mathbf{N}^A \rangle_L}. \end{aligned} \tag{4.38}$$

The x'', y'' and z'' values can be obtained in terms of the $k\cos(\phi)$ and $k\sin(\phi)$

$$\alpha''(\phi) = (x''(\phi), y''(\phi), z''(\phi)) .$$

The second equation is obtained for equation

$$\left\langle \nabla_L f^B, \alpha''(\phi) \right\rangle_L = -\left\langle (\nabla_L f^B)', \alpha'(s_0) \right\rangle_L . \tag{4.39}$$

We obtain the linear system in the variables $k\cos(\phi)$ and $k\sin(\phi)$ from the equations (4.36,4.39).

$$\begin{cases} II(u'(s_0), v'(s_0), u''(\phi), v''(\phi)) + \dfrac{3\left\langle (\nabla_L f^B)', \alpha''(\phi) \right\rangle_L}{\|\nabla_L f^B\|_L} = -III(u'(s_0), v'(s_0)) - \dfrac{\left\langle \diamond_1^B, \alpha'(s_0) \right\rangle_L}{\|\nabla_L f^B\|_L} \\ \left\langle \nabla_L f^B, \alpha''(\phi) \right\rangle_L = -\left\langle (\nabla_L f^B)', \alpha'(s_0) \right\rangle_L . \end{cases} \tag{4.40}$$

where

$$(\nabla_L f)' = \left(\langle \nabla_L f_x, \alpha'(s_0) \rangle_L , \langle \nabla_L f_y, \alpha'(s_0) \rangle_L , \langle -\nabla_L f_z, \alpha'(s_0) \rangle_L \right) , \tag{4.41}$$

$$\diamond_1 = \left(\langle (\nabla_L f_x)', \alpha'(s_0) \rangle_L , \langle (\nabla_L f_y)', \alpha'(s_0) \rangle_L , \langle -(\nabla_L f_z)', \alpha'(s_0) \rangle_L \right) . \tag{4.42}$$

Solving the system (4.40) gives us the $\kappa\cos(\phi), \kappa\sin(\phi)$. Substituting the results into (4.37 or 4.38) yields the curvature vector. From this, one can derive the curvature, the normal vector, and the binormal vector.

Parametric-implicit surfaces Lighlike.

The **curvature vector** of the intersection two surfaces: parametric-implicit is

$$\begin{aligned} \alpha''(s) &= \mathbf{S}_u u'' + \mathbf{S}_v v'' + \mathbf{S}_{uu}(u')^2 + 2\mathbf{S}_{uv} u'v' + \mathbf{S}_{vv}(v')^2, \\ &= (x''(s), y''(s), z''(s)) . \end{aligned} \tag{4.43}$$

The equation (4.43) consists of five variables $u'', v'', x''(s), y''(s)$ and $z''(s)$. These quantities can be expressed in terms of $\kappa\sin(\phi)$ and $\kappa\cos(\phi)$.

The first equation is obtained for projection of the vector α''' onto $\mathbf{N}^A$ and $\mathbf{N}^B$,

$$\left\langle \mathbf{N}^A, \alpha'''(s) \right\rangle_L = \left\langle \mathbf{N}^B, \alpha'''(s) \right\rangle_L , \tag{4.44}$$

by using (2.11) and (2.17), produces the equation.

$$III(u', v') + II(u', v', u'', v'') = -3\left\langle (\nabla_L f^B)', \alpha''(s) \right\rangle_L - \left\langle \diamond_1^B, \alpha'(s) \right\rangle_L . \tag{4.45}$$

The curvature vector (4.43) can be write in terms of rotation angle ϕ

$$\alpha''(\phi) = \kappa \mathcal{R}_{type}(\phi, \alpha'(s)) \frac{\mathcal{D}_L(\alpha'(s))}{\|\mathcal{D}_L(\alpha'(s))\|_L}, \tag{4.46}$$

if $\alpha''(\phi)$ is timelike, spacelike, and

$$\alpha''(\phi) = \mathcal{R}_{type}(\phi, \alpha'(s)) \mathcal{D}_L(\alpha'(s)), \tag{4.47}$$

if $\alpha''(\phi)$ is lightlike.
The u'' and v'' values can be obtained in terms of the $k\cos(\phi)$ and $k\sin(\phi)$

$$\begin{aligned} u''(\phi) &= \frac{\left\langle \alpha''(\phi) \times_E S_v^A, \mathbf{N}^A \right\rangle_E}{\langle S_u^A \times_L S_v^A, \mathbf{N}^A \rangle_L}, \\ v''(\phi) &= \frac{\left\langle \alpha''(\phi) \times_E S_u^A, \mathbf{N}^A \right\rangle_E}{\langle S_v^A \times_L S_u^A, \mathbf{N}^A \rangle_L}. \end{aligned} \tag{4.48}$$

The x'', y'' and z'' values can be obtained in terms of the $k\cos(\phi)$ and $k\sin(\phi)$

$$\alpha''(\phi) = (x''(\phi), y''(\phi), z''(\phi)).$$

The second equation is obtained for equation

$$\left\langle \nabla_L f^B, \alpha''(\phi) \right\rangle_L = -\left\langle (\nabla_L f^B)', \alpha'(s_0) \right\rangle_L. \tag{4.49}$$

We obtain the linear system in the variables $k\cos(\phi)$ and $k\sin(\phi)$ from the equations (4.45,4.49).

$$\begin{cases} II(u'(s_0), v'(s_0), u''(\phi), v''(\phi)) + 3\left\langle (\nabla_L f^B)', \alpha''(\phi) \right\rangle_L = -III(u'(s_0), v'(s_0)) - \left\langle \diamond_1^B, \alpha'(s_0) \right\rangle_L \\ \left\langle \nabla_L f^B, \alpha''(\phi) \right\rangle_L = -\left\langle (\nabla_L f^B)', \alpha'(s_0) \right\rangle_L. \end{cases} \tag{4.50}$$

where

$$(\nabla_L f)' = \left(\langle \nabla_L f_x, \alpha'(s_0) \rangle_L, \langle \nabla_L f_y, \alpha'(s_0) \rangle_L, \langle -\nabla_L f_z, \alpha'(s_0) \rangle_L \right), \tag{4.51}$$

$$\diamond_1 = \left(\langle (\nabla_L f_x)', \alpha'(s_0) \rangle_L, \langle (\nabla_L f_y)', \alpha'(s_0) \rangle_L, \langle -(\nabla_L f_z)', \alpha'(s_0) \rangle_L \right).$$

Solving the system gives us the $\kappa\cos(\phi), \kappa\sin(\phi)$. Substituting the results into (4.46 or 4.47) yields the curvature vector. From this, one can derive the curvature, the normal vector, and the binormal vector.

Implicit-implicit surfaces Spacelike or Timelike.

The **curvature vector** is

$$\alpha''(s) = (x''(s), y''(s), z''(s)). \tag{4.52}$$

The equation (4.52) consists of five variables $x''(s), y''(s)$ and $z''(s)$. These quantities can be expressed in terms of $\kappa\sin(\phi)$ and $\kappa\cos(\phi)$.
The first equation is obtained for projection of the vector α''' onto $\mathbf{N}^A$ and $\mathbf{N}^B$,

$$\left\langle \mathbf{N}^A, \alpha'''(s) \right\rangle_L = \left\langle \mathbf{N}^B, \alpha'''(s) \right\rangle_L, \tag{4.53}$$

by using (2.17), produces the equation.

$$\frac{-3\left\langle (\nabla_L f^A)', \alpha''(s) \right\rangle_L - \left\langle \diamond_1^A, \alpha'(s) \right\rangle_L}{\|\nabla_L f^A\|_L} = \frac{-3\left\langle (\nabla_L f^B)', \alpha''(s) \right\rangle_L - \left\langle \diamond_1^B, \alpha'(s) \right\rangle_L}{\|\nabla_L f^B\|_L}. \tag{4.54}$$

The curvature vector (4.52) can be write in terms of rotation angle ϕ

$$\alpha''(\phi) = \kappa \mathcal{R}_{type}(\phi, \alpha'(s)) \frac{\mathcal{D}_L(\alpha'(s))}{\|\mathcal{D}_L(\alpha'(s))\|_L}, \tag{4.55}$$

if $\alpha''(\phi)$ is timelike, spacelike, and

$$\alpha''(\phi) = \mathcal{R}_{type}(\phi, \alpha'(s))\mathcal{D}_L(\alpha'(s)) \ , \tag{4.56}$$

if $\alpha''(\phi)$ is lightlike.
The second equation is obtained for equation

$$\langle \nabla_L f, \alpha''(\phi)\rangle_L = -\langle(\nabla_L f)', \alpha'(s_0)\rangle_L .$$

We obtain the linear system in the variables $k\cos(\phi)$ and $k\sin(\phi)$ from the equation (4.54).

$$\begin{cases} \left\langle \frac{-3(\nabla_L f^A)'}{\|\nabla_L f^A\|_L} + \frac{3(\nabla_L f^B)'}{\|\nabla_L f^B\|_L}, \alpha''(\phi)\right\rangle_L = \left\langle \frac{\diamond_1^A}{\|\nabla_L f^A\|_L} - \frac{\diamond_1^B}{\|\nabla_L f^B\|_L}, \alpha'(s_0)\right\rangle_L \\ \langle \nabla_L f, \alpha''(\phi)\rangle_L = -\langle(\nabla_L f)', \alpha'(s_0)\rangle_L . \end{cases} \tag{4.57}$$

where

$$(\nabla_L f)' = \left(\langle \nabla_L f_x, \alpha'(s_0)\rangle_L , \langle \nabla_L f_y, \alpha'(s_0)\rangle_L , \langle -\nabla_L f_z, \alpha'(s_0)\rangle_L\right), \tag{4.58}$$

$$\diamond_1 = \left(\langle(\nabla_L f_x)', \alpha'(s_0)\rangle_L , \langle(\nabla_L f_y)', \alpha'(s_0)\rangle_L , \langle -(\nabla_L f_z)', \alpha'(s_0)\rangle_L\right). \tag{4.59}$$

Solving the system (4.57) gives us the $\kappa\cos(\phi), \kappa\sin(\phi)$. Substituting the results into (4.55 or 4.56) yields the curvature vector. From this, one can derive the curvature, the normal vector, and the binormal vector.

Implicit-implicit surfaces Lightlike.
The **curvature vector** is

$$\alpha''(s) = (x''(s), y''(s), z''(s)) . \tag{4.60}$$

The equation (4.60) consists of five variables $x''(s), y''(s)$ *and* $z''(s)$. These quantities can be expressed in terms of $\kappa\sin(\phi)$ and $\kappa\cos(\phi)$.
The first equation is obtained for projection of the vector α''' onto $\mathbf{N}^A$ and $\mathbf{N}^B$,

$$\left\langle \mathbf{N}^A, \alpha'''(s)\right\rangle_L = \left\langle \mathbf{N}^B, \alpha'''(s)\right\rangle_L \ , \tag{4.61}$$

by using (2.17), produces the equation.

$$-3\left\langle(\nabla_L f^A)', \alpha''(s)\right\rangle_L - \left\langle \diamond_1^A, \alpha'(s)\right\rangle_L = -3\left\langle(\nabla_L f^B)', \alpha''(s)\right\rangle_L - \left\langle \diamond_1^B, \alpha'(s)\right\rangle_L \ . \tag{4.62}$$

The curvature vector (4.60) can be write in terms of rotation angle ϕ

$$\alpha''(\phi) = \kappa\mathcal{R}_{type}(\phi, \alpha'(s))\frac{\mathcal{D}_L(\alpha'(s))}{\|\mathcal{D}_L(\alpha'(s))\|_L}, \tag{4.63}$$

if $\alpha''(\phi)$ is timelike, spacelike, and

$$\alpha''(\phi) = \mathcal{R}_{type}(\phi, \alpha'(s))\mathcal{D}_L(\alpha'(s)) \ , \tag{4.64}$$

if $\alpha''(\phi)$ is lightlike.
The second equation is obtained for equation

$$\langle \nabla_L f, \alpha''(\phi)\rangle_L = -\langle(\nabla_L f)', \alpha'(s_0)\rangle_L .$$

We obtain the linear system in the variables $k\cos(\phi)$ and $k\sin(\phi)$ from the equation (4.62).

$$\begin{cases} \left\langle -3(\nabla_L f^A)' + 3(\nabla_L f^B)', \alpha''(\phi)\right\rangle_L & = \left\langle \diamond_1^A - \diamond_1^B, \alpha'(s_0)\right\rangle_L \\ \langle \nabla_L f, \alpha''(\phi)\rangle_L & = -\langle (\nabla_L f)', \alpha'(s_0)\rangle_L , \end{cases} \tag{4.65}$$

where

$$(\nabla_L f)' = \left(\langle \nabla_L f_x, \alpha'(s_0)\rangle_L , \langle \nabla_L f_y, \alpha'(s_0)\rangle_L , \langle -\nabla_L f_z, \alpha'(s_0)\rangle_L\right), \tag{4.66}$$

$$\diamond_1 = \left(\langle (\nabla_L f_x)', \alpha'(s_0)\rangle_L , \langle (\nabla_L f_y)', \alpha'(s_0)\rangle_L , \langle -(\nabla_L f_z)', \alpha'(s_0)\rangle_L\right). \tag{4.67}$$

Solving the system (4.65) gives us the $\kappa\cos(\phi), \kappa\sin(\phi)$. Substituting the results into (4.63 or 4.64) yields the curvature vector. From this, one can derive the curvature, the normal vector, and the binormal vector.

4.1.4. Solution of the Systems (4.40,4.50,4.57,4.65). The linear systems in terms of the $(\kappa\cos(\phi), \kappa\sin(\phi))$ if $\alpha''(s)$ is timelike or spacelike and in the terms of the $(\cos(\phi), \sin(\phi))$ if $\alpha''(s)$ is lightlike.

To analyze the solutions of the systems (4.40,4.50,4.57,4.65) in the variable ϕ, we need to separate into three cases: When α'' is timelike or is lightlike or is spacelike.

- If $\alpha'(s)$ is **timelike**, then $\alpha''(s)$ is **spacelike**, and the system has a unique simple solution.
- If $\alpha'(s)$ is **spacelike**, then $\alpha''(s)$ may be **timelike**, **spacelike**, or **lightlike**. In this case, the system have solution depending on the choice of μ_i. If we choose $\mu_\mathbf{1}$ such that $\mathcal{D}_L(\alpha'(s)) = \mu_\mathbf{1} \times_L \alpha'(s)$ is lightlike and the system has no solution, then we choose $\mu_\mathbf{2}$ such that $\mathcal{D}_L(\alpha'(s)) = \mu_\mathbf{2} \times_L \alpha'(s)$ is spacelike. If this case still has no solution, we then choose $\mu_\mathbf{3}$ such that $\mathcal{D}_L(\alpha'(s)) = \mu_\mathbf{3} \times_L \alpha'(s)$ is timelike.
- If $\alpha'(s)$ is **lightlike**, then $\alpha''(s)$ may be **spacelike** or **lightlike**. In this case, the system has a unique simple solution depending on the choice of μ_i. If we choose $\mu_\mathbf{1}$ such that $\mathcal{D}_L(\alpha'(s)) = \mu_\mathbf{1} \times_L \alpha'(s)$ is lightlike and the system has no solution, then we choose $\mu_\mathbf{2}$ such that $\mathcal{D}_L(\alpha'(s)) = \mu_\mathbf{2} \times_L \alpha'(s)$ is spacelike.

5. Examples.

In this section, we present some examples that illustrate our new methods.

5.1. Example of implicit-implicit surface intersection.

Example 5.1. *Let S^A and S^B be the implicit surfaces given by*

$$f^A(x, y, z) = 2x^2 + 2y - 2z^3 - 1 = 0, \quad f^B(x, y, z) = 8x + 4y^2 - 4z^3 - 5 = 0.$$

Then

$$\nabla_L f^A = (4x, 2, 6z^2), \ \ \nabla_L f^B = (8, 8y, 12z^2),$$

we have $N^A = N^B = N$ at $P = (1, \frac{1}{2}, 1)$, i.e. S^A and S^B intersect tangentially at P. We have $N^A = \frac{(4,2,6)}{\|(4,1,6)\|_L} = N^B = \frac{(8,4,12)}{\|(8,4,12)\|_L} = N = (1, 0.5, 1.5)$.

*a) The **tangent vector** $\alpha'(s) = (x'(s), y'(s), z'(s))$.*

Since $N = (1, 0.5, 1.5)$ is timelike, then can be any $\mu_\mathbf{1}$, let $\mu_\mathbf{1} = (0, 0, -1)$. Then $\mathcal{D}_L(\mathbf{N}) = (0.5, -1.0, 0)$

$$\begin{aligned} \mathcal{R}_{tm}\frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L} &= (1.34164078649987\sin(\theta) + 0.447213595499958\cos(\theta), \\ &\quad 0.670820393249937\sin(\theta) - 0.894427190999916\cos(\theta), \\ &\quad 1.11803398874989\sin(\theta)) \\ \alpha'(\theta) &= \mathcal{R}_{tm}\frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}, \end{aligned}$$

and from (4.23) we obtain

$$\begin{aligned} \frac{\langle(\nabla_L f^A)', \alpha'(\theta)\rangle_L}{\|\nabla_L f^A\|_L} - \frac{\langle(\nabla_L f^B)', \alpha'(\theta)\rangle_L}{\|\nabla_L f^B\|_L} &= 0. \\ -1.2 * sin(2 * theta) + 0.975 * cos(2 * theta) - 0.375 &= 0. \end{aligned} \tag{5.1}$$

Thus, we have two solutions for $\theta : \{\theta_1 = -1.10714871779409, \theta_2 = 0.218668945873942\}$
For $\theta_1 = -1.10714871779409$*, we have*

$$\alpha'(\theta_1) = (-1.0, -1.0, -1.0), \tag{5.2}$$

$$\mathbf{t}(t_0) = \frac{\alpha'(\theta_1)}{\|\alpha'(\theta_1)\|_L} = (-1, -1, -1). \tag{5.3}$$

For $\theta_2 = 0.218668945873942$*, we have*

$$\alpha'(\theta_2) = (0.727606875108999, -0.727606875108999, 0.242535625036333), \tag{5.4}$$

$$\mathbf{t}(t_0) = \frac{\alpha'(\theta_2)}{\|\alpha'(\theta_2)\|_L} = (0.727606875108999, -0.727606875108999, 0.242535625036333). \tag{5.5}$$

*b) The **curvature vector*** $\alpha''(s) = (x''(s), y''(s), z''(s))$.

Since $\alpha' = (-1.0, -1.0, -1.0)$ *is spacelike, then* $\alpha''(s)$ *may be **timelike**, **spacelike**, or **lightlike**. In this case, the system has a unique simple solution depending on the choice of* μ_i*. If we choose* $\mu_{\mathbf{1}}$ *such that* $\mathcal{D}_L(\alpha') = \mu_{\mathbf{1}} \times_L \alpha'$ *is lightlike and the system has no solution, then we choose* $\mu_{\mathbf{2}}$ *such that* $\mathcal{D}_L(\alpha') = \mu_{\mathbf{2}} \times_L \alpha'$ *is spacelike. If this case still has no solution, we then choose* $\mu_{\mathbf{3}}$ *such that* $\mathcal{D}_L(\alpha') = \mu_{\mathbf{3}} \times_L \alpha'$ *is timelike.*

- *If we choose* $\mu_{\mathbf{1}} = (-1, 0, 0)$ *such that* $\mathcal{D}_L(\alpha') = \mu_{\mathbf{1}} \times_L \alpha' = (0, -1, -1)$ *is lightlike*

The curvature vector $alpha''(\phi) = kappa * \mathcal{R}_{sp}\mathcal{D}_L(\alpha'(s))$

$$\begin{aligned} \mathcal{R}_{sp} &= I3 + sp.sinh(phi) * S + (-1 + sp.cosh(phi)) * S^2, \\ \alpha''(\phi) &= kappa * \mathcal{R}_{sp}\mathcal{D}_L(\alpha'(s)), \\ \alpha''(\phi) &= (0.0, kappa * (-1.0 * sinh(phi) - 1.0 * cosh(phi)), kappa * (-1.0 * sinh(phi) - 1.0 * cosh(phi)))). \end{aligned}$$

We obtain the linear system in the variables $k\cos(\phi)$ *and* $k\sin(\phi)$ *from the equation (4.57).*

$$\begin{cases} 3.0 * kappa * cosh(phi) + 3.0 * kappa * sinh(phi) = 0.0, \\ 8.0 * kappa * cosh(phi) + 8.0 * kappa * sinh(phi) = 16.0. \end{cases} \tag{5.6}$$

The system (5.6) has no solution.

- *If we choose* $\mu_{\mathbf{2}} = (0, 0, 1$ *such that* $\mathcal{D}_L(\alpha') = \mu_{\mathbf{2}} \times_L \alpha' = (1, -1, 0)$ *is spacelike*

$$\begin{aligned} \mathcal{R}_{sp} &= I3 + sp.sinh(phi) * S + (-1 + sp.cosh(phi)) * S^2, \\ \alpha''(\phi) &= kappa * \mathcal{R}_{sp}\frac{\mathcal{D}_L(\alpha'(s))}{\|\mathcal{D}_L(\alpha'(s))\|_L}, \\ &= (-0.707106781186548 * kappa * sinh(phi) + 0.707106781186548 * kappa * cosh(phi), \\ &\quad -0.707106781186548 * kappa * sinh(phi) - 0.707106781186547 * kappa * cosh(phi), \\ &\quad -1.4142135623731 * kappa * sinh(phi)) \end{aligned}$$

We obtain the linear system in the variables $k\cos(\phi)$ *and* $k\sin(\phi)$ *from the equation (4.57).*

$$\begin{cases} 0.0 * kappa * sinh(phi) + 4.24264068711928 * kappa * cosh(phi) = 0.0, \\ 8.48528137423862 * kappa * sinh(phi) + 2.82842712474619 * kappa * cosh(phi) = 16.0. \end{cases} \tag{5.7}$$

The system (5.7) have solution $kappa * sinh(phi) = 1.8856180831641158$ *and* $kappa * cosh(phi) = 0.0$.

Substituting the results into

$$\begin{aligned}\alpha''(s) &= (0.707106781186548 * kappa * sinh(phi) - 0.707106781186548 * kappa * cosh(phi), \\ &\quad -0.707106781186548 * kappa * sinh(phi) - 0.707106781186547 * kappa * cosh(phi), \\ &\quad -1.4142135623731 * kappa * sinh(phi))\end{aligned}$$

we have

$$\alpha''(s) = (-1.33333333333333, -1.33333333333333, -2.66666666666666).$$

From this, one can derive the curvature, the normal vector and the binormal vector.

$$\kappa = \langle \alpha''(s), \alpha''(s) \rangle_L = 1.88561808316413,$$

$$\mathbf{n} = \frac{\alpha''(s)}{\kappa} = (-0.707106781186544, -0.707106781186544, -1.41421356237309)$$

$$\mathbf{b} = \alpha'(s) \times_L \mathbf{n} = (-0.707106781186548, 0.707106781186547, 0).$$

5.2. Example of parametric-implicit surface intersection.

Example 5.2. *Let us consider the surface* S^A *and* S^B *by the parametric equations*

$$S^A(u,v) = (0.6 * cos(v) * cos(u), 0.8 * cos(v) * sin(u), sin(v)) . \quad f^B(x,y,z) = \frac{x^2}{0.45^2} + \frac{y^2}{0.8^2} + \frac{z^2}{1.25^2} - 1.$$

Since the unit normal vectors of these surfaces at the intersection point $P = (0, 0.8, 0)$, *i.e,* $S^A(\frac{\pi}{2}, 0) = (0, 0.8, 0)$ *and* $f^B(0, 0.8, 0) = 0$ *are* $N^A \parallel N^B$ $\big(N^A = (0,-1,0), N^B = (0,1,0)\big)$, *these surfaces intersect tangentially at* P. *The vectors* $S_u^A(\frac{\pi}{2}, 0) = (-0.6, 0, 0)$, $S_v^A(\frac{\pi}{2}, 0) = (0, 0, 1)$, $\nabla_L f^B(0, 0.8, 0) = (0, 2.5, 0)$, *produce* $\mathbf{N}^A = \frac{S_u^A \times_L S_v^A}{\|S_u^A \times_L S_v^A\|_L} = (0, -1, 0)$ *and* $\mathbf{N}^B = \frac{\nabla_L f^B(0,0.8,0)}{\|\nabla_L f^B(0,0.8,0)\|_L} = (0, 1, 0)$.

The vector normal $\mathbf{N}^A = -\mathbf{N}^B$ *are spacelikes. Since surfaces is timelike, by orienting the surfaces properly we can assume that* $N^A(P_0) = N^B(P_0) = N = (0, 1, 0)$. *Let us now apply our second method to find the tangential direction. We must test the four equations:*

*a) The **tangent vector*** $\alpha'(s) = (x'(s), y'(s), z'(s))$.

Since $N = (0, 1, 0)$ *is spacelike, we must test the four equations:*

- $eq1 = \langle \mathbf{N}^A, \alpha''(\theta) \rangle_L - \langle \mathbf{N}^B, \alpha''(\theta) \rangle_L$, *for* $\mu_\mathbf{1}$

  Let $\mu_\mathbf{1} = (1, 0, 1)$ *be lightlike, we get* $\mathcal{D}_L = \mu_\mathbf{i} \times_L \mathbf{N} = (-1, 0, -1)$ *is lightlike. Then, from (3.1) we way write*

$$\alpha'(s) = \mathcal{R}_{sp}(\theta, \mathbf{N}) \mathcal{D}_L(\mathbf{N}),$$

$$\alpha'(s) = \begin{bmatrix} cosh(\theta) & 0 & sinh(\theta) \\ 0 & 1 & 0 \\ sinh(\theta) & 0 & cosh(\theta) \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix},$$

$$\alpha'(s) = \mathcal{R}_{sp}\mathcal{D}_L(\mathbf{N}) = (-1.0 * sinh(\theta) - 1.0 * cosh(\theta), 0, -1.0 * sinh(\theta) - 1.0 * cosh(\theta)) .$$

From the equation (4.6) we have

$$\begin{aligned} u' &= 1.66666666666667 * sinh(\theta) + 1.66666666666667 * cosh(\theta), \\ v' &= -1.0 * sinh(\theta) - 1.0 * cosh(\theta), \end{aligned}$$

The coefficients of the second fundamental form are

$$e^A = -0.8,\ f^A = 0,\ g^A = -0.8,$$

and

$$H_L f^B = \begin{bmatrix} 9.87654320987654 & 0 & 0 \\ 0 & 3.125 & 0 \\ 0 & 0 & -1.28 \end{bmatrix}.$$

If we substitute these results into (4.5), we have

$$\begin{aligned} e^A(u'(\theta))^2 + 2f^A u'(\theta)v'(\theta) + g^A(v'(\theta))^2 + \frac{\langle(\nabla_L f^B)', \alpha'(\theta)\rangle_L}{\|\nabla_L f^B\|_L} &= 0, \\ 1.44039506172839 * exp(2 * theta) &= 0. \end{aligned}$$

We have

$$eq1 \neq 0.$$

any θ.

- $eq2 = \langle \mathbf{N}^A, \alpha''(\theta)\rangle_L - \langle \mathbf{N}^B, \alpha''(\theta)\rangle_L,\ for\ \mu_\mathbf{2}$
We need to choose $\mu_\mathbf{2}$ such that $\mathcal{D}_L(N) = \mu_\mathbf{2} \times_L \mathbf{N}$ is lightlike, but linearly independent with $\mathcal{D}_L(N) = \mu_\mathbf{1} \times_L \mathbf{N}$.
Choosing $\mu_\mathbf{2} = (-1, 0, 1)$ such that $\mathcal{D}_L = \mu_\mathbf{2} \times_L \mathbf{N} = (-1, 0, 1)$ is lightlike. Then, from (3.1) we way write,

$$\alpha'(s) = \mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}),$$

$$\alpha'(s) = \begin{bmatrix} cosh(\theta) & 0 & sinh(\theta) \\ 0 & 1 & 0 \\ sinh(\theta) & 0 & cosh(\theta) \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

$$\alpha'(s) = \mathcal{R}_{sp}\mathcal{D}_L(\mathbf{N}) = (1.0 * sinh(\theta) - 1.0 * cosh(\theta), 0, -1.0 * sinh(\theta) + 1.0 * cosh(\theta)).$$

From the equation (4.6) we have

$$\begin{aligned} u' &= -1.66666666666667 * sinh(\theta) + 1.66666666666667 * cosh(\theta), \\ v' &= -1.0 * sinh(\theta) + 1.0 * cosh(\theta). \end{aligned}$$

The coefficients of the second fundamental form are

$$e^A = -0.8,\ f^A = 0,\ g^A = -0.8.$$

and

$$H_L f^B = \begin{bmatrix} 9.87654320987654 & 0 & 0 \\ 0 & 3.125 & 0 \\ 0 & 0 & -1.28 \end{bmatrix}.$$

$$\begin{aligned}(\nabla_L f)' &= \left(\langle \nabla_L f_x, \alpha'(s)\rangle_L, \langle \nabla_L f_y, \alpha'\rangle_L, \langle -\nabla_L f_z, \alpha'(s)\rangle_L\right),\\ (\nabla_L f^B)' &= \left(\left\langle H_L f^B[0,0:3], \alpha'(s)\right\rangle_L, \left\langle H_L f^B[1,0:3], \alpha'\right\rangle_L, \left\langle -H_L f^B[2,0:3], \alpha'(s)\right\rangle_L\right).\end{aligned}$$

If we substitute these results into (4.5), we have

$$\begin{aligned}e^A(u'(\theta))^2 + 2f^A u'(\theta)v'(\theta) + g^A(v'(\theta))^2 + \frac{\left\langle (\nabla_L f^B)', \alpha'(\theta)\right\rangle_L}{\|\nabla_L f^B\|_L} &= 0,\\ 1.44039506172839 * exp(2 * theta) &= 0.\end{aligned}$$

We have

$$eq2 \neq 0,$$

any θ.

- $eq3 = \left\langle \mathbf{N}^A, \alpha''(\theta)\right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta)\right\rangle_L,\ for\ \mu_{\mathbf{3}}$.
  Choosing $\mu_{\mathbf{3}} = (0,0,1)$ *such that* $\mathcal{D}_L = \mu_{\mathbf{3}} \times_L \mathbf{N} = (-1,0,0)$ *is spacelike. Then, from (3.1) we way write,*

$$\alpha'(s) = \mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}),$$

$$\alpha'(s) = \begin{bmatrix} cosh(\theta) & 0 & sinh(\theta)\\ 0 & 1 & 0\\ sinh(\theta) & 0 & cosh(\theta)\end{bmatrix}\begin{bmatrix}-1\\0\\0\end{bmatrix},$$

$$\alpha'(s) = \mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}) = (-1.0 * cosh(\theta), 0, -1.0 * sinh(\theta)).$$

From the equation (4.6) we have

$$u' = \cosh(\theta),$$
$$v' = -\sinh(\theta).$$

The coefficients of the second fundamental form are

$$e^A = -0.8,\ f^A = 0,\ g^A = -0.8,$$

and

$$H_L f^B = \begin{bmatrix} 9.87654320987654 & 0 & 0\\ 0 & 3.125 & 0\\ 0 & 0 & -1.28\end{bmatrix}.$$

$$\begin{aligned}(\nabla_L f)' &= \left(\langle \nabla_L f_x, \alpha'(s)\rangle_L, \langle \nabla_L f_y, \alpha'\rangle_L, \langle -\nabla_L f_z, \alpha'(s)\rangle_L\right),\\ (\nabla_L f^B)' &= \left(\left\langle H_L f^B[0,0:3], \alpha'(s)\right\rangle_L, \left\langle H_L f^B[1,0:3], \alpha'\right\rangle_L, \left\langle -H_L f^B[2,0:3], \alpha'(s)\right\rangle_L\right).\end{aligned}$$

If we substitute these results into (4.5), we have

$$\begin{aligned}e^A(u'(\theta))^2 + 2f^A u'(\theta)v'(\theta) + g^A(v'(\theta))^2 + \frac{\left\langle (\nabla_L f^B)', \alpha'(\theta)\right\rangle_L}{\|\nabla_L f^B\|_L} &= 0,\\ 0.720197530864197 * cosh(2 * theta) + 1.0081975308642 &= 0.\end{aligned}$$

We have

$$eq3 \neq 0,$$

any $\theta \in \mathbb{R}$.

- $eq4 = \langle \mathbf{N}^A, \alpha''(\theta) \rangle_L - \langle \mathbf{N}^B, \alpha''(\theta) \rangle_L, \ for \ \mu_\mathbf{4}$

Choosing $\mu_\mathbf{4} = (1,0,0)$ *such that* $\mathcal{D}_L = \mu_\mathbf{4} \times_L \mathbf{N} = (0,0,-1)$ *is timelike. Then, from (3.1) we way write*

$$\alpha'(s) = \mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}),$$

$$\alpha'(s) = \begin{bmatrix} cosh(\theta) & 0 & sinh(\theta) \\ 0 & 1 & 0 \\ sinh(\theta) & 0 & cosh(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix},$$

$$\alpha'(s) = \mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}) = (-1.0 * sinh(\theta), 0, -1.0 * cosh(\theta)).$$

From the equation (4.6) we have

$$u' = 1.66666666666667 \sinh(\theta),$$
$$v' = -1.0 \cosh(\theta).$$

The coefficients of the second fundamental form are

$$e^A = -0.8, \ f^A = 0, \ g^A = -0.8,$$

and

$$H_L f^B = \begin{bmatrix} 9.87654320987654 & 0 & 0 \\ 0 & 3.125 & 0 \\ 0 & 0 & -1.28 \end{bmatrix}.$$

$$\begin{aligned} (\nabla_L f)' &= \left(\langle \nabla_L f_x, \alpha'(s) \rangle_L, \langle \nabla_L f_y, \alpha' \rangle_L, \langle -\nabla_L f_z, \alpha'(s) \rangle_L\right), \\ (\nabla_L f^B)' &= \left(\langle H_L f^B[0,0:3], \alpha'(s) \rangle_L, \langle H_L f^B[1,0:3], \alpha' \rangle_L, \langle -H_L f^B[2,0:3], \alpha'(s) \rangle_L\right). \end{aligned}$$

If we substitute these results into (4.5), we have

$$\begin{aligned} e^A (u'(\theta))^2 + 2 f^A u'(\theta) v'(\theta) + g^A (v'(\theta))^2 + \frac{\langle (\nabla_L f^B)', \alpha'(\theta) \rangle_L}{\|\nabla_L f^B\|_L} &= 0, \\ 0.720197530864197 * cosh(2 * theta) - 1.0081975308642 &= 0. \end{aligned}$$

We have

$$eq4 = 0,$$

whit solutions $[\theta_1 = -0.433451372988399, \theta_2 = 0.433451372988399]$.
As $\langle \mathbf{N}^A, \alpha''(\theta) \rangle_L - \langle \mathbf{N}^B, \alpha''(\theta) \rangle_L \neq 0$ *for* $\mu_\mathbf{i}, \ i \in \{1,2,3\}$ *and* $\langle \mathbf{N}^A, \alpha''(\theta) \rangle_L - \langle \mathbf{N}^B, \alpha''(\theta) \rangle_L = 0$ *for* $\mu_\mathbf{i}, \ i \in \{4\}$*, we have* $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N}) \ \mu_\mathbf{4} \times_L \mathbf{N}$*, if* θ_i *is solutions to the* $eq4 = 0$.

$\neq 0$	$\neq 0$	$\neq 0$	$= 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N}) \ \mu_\mathbf{4} \times_L \mathbf{N}$, if θ_i is solutions to the $eq4 = 0$.

For $\theta_1 = -0.433451372988399$,
$\alpha'(\theta_1) = (0.447152261947521, 0, -1.09542007712329)$,
and $\theta_2 = 0.433451372988399$.
$\alpha'(\theta_2) = (-0.447152261947521, 0, -1.09542007712329)$.
b) The ***curvature vector*** *is* $\alpha''(s) = (x''(s), y''(s), z''(s))$.

Since $\alpha' = (0.447152261947521, 0, -1.09542007712329)$ *is* ***timelike****, then* $\alpha''(s)$ *is* ***spacelike****, and the system has a unique simple solution. We can be any* $\mu_{\mathbf{i}}$ $i \in \{1,2,3,4\}$. *Let* $\mu_{\mathbf{1}} = (0,0,1)$. *Then* $\mathcal{D}_L(\mathbf{N}) = (0,1,0)$

$$\begin{aligned}
\mathcal{R}_{tm} &= I3 + sp.sin(phi) * S + (1 - sp.cos(phi)) * S^2, \\
\alpha''(\phi) &= kappa * \mathcal{R}_{tm} \frac{\mathcal{D}_L(\alpha'(s))}{\|\mathcal{D}_L(\alpha'(s))\|_L}, \\
\alpha''(\phi) &= (1.09542007712329 * kappa * sin(phi), 1.0 * kappa * cos(phi), -0.447152261947521 * kappa * sin(phi)).
\end{aligned}$$

We obtain the linear system in the variables $k\cos(\phi)$ *and* $k\sin(\phi)$ *from the equations (4.40).*

$$\begin{cases} 2.11660104885168 * kappa * sin(phi) + 0.0 * kappa * cosh(phi) &= \quad 0.0, \\ 0.0 * kappa * sinh(phi) + 2.5 * kappa * cos(phi) &= -3.51069665386727. \end{cases} \tag{5.8}$$

Solving the system (5.8) gives us the $\kappa\sin(\phi) = 0.0$, $\kappa\cos(\phi) = -1.404278661546908$.
Substituting the results into

$$\alpha''(\phi) = (1.09542007712329{*}kappa{*}sin(phi), 1.0{*}kappa{*}cos(phi), -0.447152261947521{*}kappa{*}sin(phi)),$$

we have

$$\alpha''(s) = (0.0, -1.40427866154691, 0.0).$$

From this, one can derive the curvature, the normal vector and the binormal vector.

$$\kappa = \langle \alpha''(s), \alpha''(s) \rangle_L = 1.40427866154691,$$

$$\mathbf{n} = \frac{\alpha''(s)}{\kappa} = (0.0, -1.0, 0.0),$$

$$\mathbf{b} = \alpha'(s) \times_L \mathbf{n} = (1.09542007712329, 0.0, -0.447152261947521).$$

5.3. Example of parametric-implicit surface intersection.

Example 5.3.
Let us consider the surface S^A *and* S^B *by the parametric equations*

$$S^A(u,v) = \left(u, v^4, v\right). \quad f^B(x,y,z) = y.$$

Since the unit normal vectors of these surfaces at the intersection point $P = (1,0,0)$*, i.e,* $S^A(1,0) = (1,0,0)$ *and* $f^B(1,0,0) = 0$ *are* $N^A \parallel N^B$ $\left(N^A = (0,-1,0), N^B = (0,1,0)\right)$*, these surfaces intersect tangentially at P. The vectors* $S^A_u(1,0) = (1,0,0)$, $S^A_v(1,0) = (0,0,1)$, $\nabla_L f^B(1,0,0) = (0,1,0)$*, produce* $\mathbf{N}^A = \frac{S^A_u \times_L S^A_v}{\|S^A_u \times_L S^A_v\|_L} = (0,-1,0)$ *and* $\mathbf{N}^B = \frac{\nabla_L f^B(1,0,0)}{\|\nabla_L f^B(1,0,0)\|_L} = (0,1,0)$.

The vector normal $\mathbf{N}^A = -\mathbf{N}^B$ *are spacelikes. Since surfaces is timelike, by orienting the surfaces properly we can assume that* $N^A(P_0) = N^B(P_0) = N = (0,-1,0)$*. Let us now apply our second method to find the tangential direction. We must test the four equations:*

- $eq1 = \left\langle \mathbf{N}^A, \alpha''(\theta)\right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta)\right\rangle_L,\ for\ \mu_{\mathbf{1}}$
  Let $\mu_{\mathbf{1}} = (1,0,1)$ *be lightlike, we get* $\mathcal{D}_L = \mu_{\mathbf{i}} \times_L \mathbf{N} = (1,0,1)$ *is lightlike. Then, from (3.1) we way write*

$$\alpha'(s) = \mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}),$$

$$\alpha'(s) = \begin{bmatrix} cosh(\theta) & 0 & -sinh(\theta) \\ 0 & 1 & 0 \\ -sinh(\theta) & 0 & cosh(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$\alpha'(s) = \mathcal{R}_{sp}\mathcal{D}_L(\mathbf{N}) = (-1.0 * sinh(\theta) + 1.0 * cosh(\theta), 0, -1.0 * sinh(\theta) + 1.0 * cosh(\theta)).$$

From the equation (4.6) we have

$$u' = \cosh(\theta) - \sinh(\theta),$$
$$v' = -\sinh(\theta) + \cosh(\theta).$$

The coefficients of the second fundamental form are

$$e^A = 0.0,\ f^A = 0,\ g^A = 0.0,$$

and

$$H_L f^B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{aligned} (\nabla_L f)' &= \left(\langle \nabla_L f_x, \alpha'(s)\rangle_L, \langle \nabla_L f_y, \alpha'\rangle_L, \langle -\nabla_L f_z, \alpha'(s)\rangle_L\right), \\ (\nabla_L f^B)' &= \left(\left\langle H_L f^B[0,0:3], \alpha'(s)\right\rangle_L, \left\langle H_L f^B[1,0:3], \alpha'\right\rangle_L, \left\langle -H_L f^B[2,0:3], \alpha'(s)\right\rangle_L\right). \end{aligned}$$

If we substitute these results into

$$\begin{aligned} e^A(u'(\theta))^2 + 2f^A u'(\theta)v'(\theta) + g^A(v'(\theta))^2 + \frac{\left\langle (\nabla_L f^B)', \alpha'(\theta)\right\rangle_L}{\|\nabla_L f^B\|_L} &= 0, \\ 0 &= 0. \end{aligned}$$

We have

$$eq1 \equiv 0.$$

- $eq2 = \left\langle \mathbf{N}^A, \alpha''(\theta)\right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta)\right\rangle_L,\ for\ \mu_2$.
  We need to choose μ_2 *such that* $\mathcal{D}_L(N) = \mu_2 \times_L \mathbf{N}$ *is lightlike, but linearly independent with* $\mathcal{D}_L(N) = \mu_1 \times_L \mathbf{N}$.
  Choosing $\mu_2 = (-1, 0, 1)$ *such that* $\mathcal{D}_L = \mu_2 \times_L \mathbf{N} = (1, 0, -1)$ *is lightlike. Then, from (3.1) we way write,*

$$\alpha'(s) = \mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}),$$

$$\alpha'(s) = \begin{bmatrix} cosh(\theta) & 0 & -sinh(\theta) \\ 0 & 1 & 0 \\ -sinh(\theta) & 0 & cosh(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

$$\alpha'(s) = \mathcal{R}_{sp}\mathcal{D}_L(\mathbf{N}) = (1.0 * sinh(\theta) + 1.0 * cosh(\theta), 0, -1.0 * sinh(\theta) - 1.0 * cosh(\theta)).$$

From the equation (4.6) we have

$$u' = \cosh(\theta) + \sinh(\theta),$$
$$v' = -\sinh(\theta) - \cosh(\theta).$$

The coefficients of the second fundamental form are

$$e^A = 0.0,\ f^A = 0,\ g^A = 0.0,$$

and

$$H_L f^B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{aligned} (\nabla_L f)' &= \left(\langle \nabla_L f_x, \alpha'(s)\rangle_L, \langle \nabla_L f_y, \alpha'\rangle_L, \langle -\nabla_L f_z, \alpha'(s)\rangle_L\right), \\ (\nabla_L f^B)' &= \left(\left\langle H_L f^B[0,0:3], \alpha'(s)\right\rangle_L, \left\langle H_L f^B[1,0:3], \alpha'\right\rangle_L, \left\langle -H_L f^B[2,0:3], \alpha'(s)\right\rangle_L\right). \end{aligned}$$

If we substitute these results into

$$\begin{aligned} e^A(u'(\theta))^2 + 2f^A u'(\theta)v'(\theta) + g^A(v'(\theta))^2 + \frac{\left\langle (\nabla_L f^B)', \alpha'(\theta)\right\rangle_L}{\|\nabla_L f^B\|_L} &= 0, \\ 0 &= 0. \end{aligned}$$

We have

$$eq2 \equiv 0.$$

- $eq3 = \left\langle \mathbf{N}^A, \alpha''(\theta)\right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta)\right\rangle_L,\ for\ \mu_{\mathbf{3}}$
  Choosing $\mu_{\mathbf{3}} = (0,0,1)$ *such that* $\mathcal{D}_L = \mu_{\mathbf{3}} \times_L \mathbf{N} = (1,0,0)$ *is spacelike. Then, from (3.1) we way write,*

$$\alpha'(s) = \mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}),$$

$$\alpha'(s) = \begin{bmatrix} cosh(\theta) & 0 & -sinh(\theta) \\ 0 & 1 & 0 \\ -sinh(\theta) & 0 & cosh(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\alpha'(s) = \mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}) = (1.0 * cosh(\theta), 0, -1.0 * sinh(\theta)).$$

From the equation (4.6) we have

$$\begin{aligned} u' &= \cosh(\theta), \\ v' &= -\sinh(\theta). \end{aligned}$$

The coefficients of the second fundamental form are

$$e^A = 0.0,\ f^A = 0,\ g^A = 0.0,$$

and

$$H_L f^B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{aligned} (\nabla_L f)' &= \left(\langle \nabla_L f_x, \alpha'(s)\rangle_L, \langle \nabla_L f_y, \alpha'\rangle_L, \langle -\nabla_L f_z, \alpha'(s)\rangle_L\right), \\ (\nabla_L f^B)' &= \left(\left\langle H_L f^B[0,0:3], \alpha'(s)\right\rangle_L, \left\langle H_L f^B[1,0:3], \alpha'\right\rangle_L, \left\langle -H_L f^B[2,0:3], \alpha'(s)\right\rangle_L\right). \end{aligned}$$

If we substitute these results into

$$\begin{array}{rcl} e^A(u'(\theta))^2 + 2f^A u'(\theta)v'(\theta) + g^A(v'(\theta))^2 + \dfrac{\left\langle (\nabla_L f^B)', \alpha'(\theta)\right\rangle_L}{\|\nabla_L f^B\|_L} & = & 0, \\ 0 & = & 0. \end{array}$$

We have

$$eq3 \equiv 0.$$

- $eq4 = \left\langle \mathbf{N}^A, \alpha''(\theta)\right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta)\right\rangle_L,\ for\ \mu_\mathbf{4}$

Choosing $\mu_\mathbf{4} = (1, 0, 0)$ *such that* $\mathcal{D}_L = \mu_\mathbf{4} \times_L \mathbf{N} = (0, 0, 1)$ *is timelike. Then, from (3.1) we way write*

$$\alpha'(s) = \mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}),$$

$$\alpha'(s) = \begin{bmatrix} cosh(\theta) & 0 & -sinh(\theta) \\ 0 & 1 & 0 \\ -sinh(\theta) & 0 & cosh(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\alpha'(s) = \mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}) = (-1.0 * sinh(\theta), 0, 1.0 * cosh(\theta)).$$

From the equation (4.6) we have

$$\begin{array}{rcl} u' & = & -\sinh(\theta), \\ v' & = & \cosh(\theta). \end{array}$$

The coefficients of the second fundamental form are

$$e^A = 0.0,\ f^A = 0,\ g^A = 0.0,$$

and

$$H_L f^B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{array}{rcl} (\nabla_L f)' & = & \left(\langle \nabla_L f_x, \alpha'(s)\rangle_L, \langle \nabla_L f_y, \alpha'\rangle_L, \langle -\nabla_L f_z, \alpha'(s)\rangle_L\right), \\ (\nabla_L f^B)' & = & \left(\left\langle H_L f^B[0, 0:3], \alpha'(s)\right\rangle_L, \left\langle H_L f^B[1, 0:3], \alpha'\right\rangle_L, \left\langle -H_L f^B[2, 0:3], \alpha'(s)\right\rangle_L\right). \end{array}$$

If we substitute these results into

$$\begin{array}{rcl} e^A(u'(\theta))^2 + 2f^A u'(\theta)v'(\theta) + g^A(v'(\theta))^2 + \dfrac{\left\langle (\nabla_L f^B)', \alpha'(\theta)\right\rangle_L}{\|\nabla_L f^B\|_L} & = & 0, \\ 0 & = & 0. \end{array}$$

We have

$$eq4 \equiv 0.$$

As $\left\langle \mathbf{N}^A, \alpha''(\theta)\right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta)\right\rangle_L \equiv 0$ *for* $\mu_\mathbf{i},\ i \in \{1, 2, 3, 4\}$, *then surfaces have at least second order contact at P.*

eq1	eq2	eq3	eq4	case	solution
$eq1 \equiv 0$	$eq2 \equiv 0$	$eq3 \equiv 0$	$eq4 \equiv 0$	(d)	have at least second order contact at P.

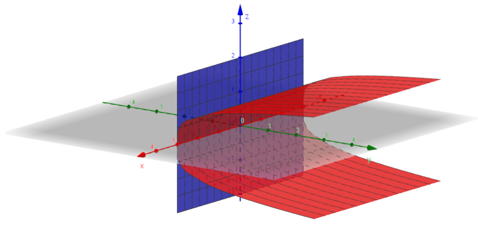

Figure 5.1: Seconde Order Contact

5.4. Example of parametric-implicit surface intersection.

Example 5.4.
Let us consider the surface S^A and S^B by the parametric equations

$$\begin{aligned} S^A(u,v) &= \left(\frac{u^2-1}{2} - v\left(u - \frac{(u^2+1)*v}{2\sqrt{u^2+v^2}}\right), u + v\left(\frac{(u^2+1)*u}{2\sqrt{u^2+v^2}} + \frac{u^2-1}{2}\right),\right. \\ &\left. \frac{u^2+1}{2} + v\left(\frac{-v*(u^2+1)}{2*\sqrt{u^2+v^2}} + u\right)\right), \\ S^B &= f^B(x,y,z) = -x^2 - y^2 + z^2. \end{aligned}$$

Since the unit normal vectors of these surfaces at the intersection point $P = S^A(1,0) = (0,1,1)$ and $f^B(0,1,1) = 0$ are $N^A = N^B = N$, these surfaces intersect tangentially at $P = (0,1,1)$. The vectors $S_u^A(1,0) = (1,1,1)$, $S_v^A(1,0) = (-1,1,1)$, $\nabla_L f^B(P) = (0,-2,-2)$, produce $\mathbf{N}^A = S_u^A \times_L S_v^A = (0,-2,-2)$ and $\mathbf{N}^B = \nabla_L f^B(P) = (0,-2,-2)$. The vector normal $\mathbf{N}^A = \lambda \mathbf{N}^B = \mathbf{N} = (0,-2,-2)$ are lightlikes and $\lambda = 1$. Let us now apply our second method to find the tangential direction.
Since $\mathbf{N}(P) = (0,-2,-2)$ is lightlike, We must test the two equations:

- $eq1 = \left\langle \mathbf{N}^A, \alpha''(\theta)\right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta)\right\rangle_L$, *for* $\mu_\mathbf{1}$
  Let $\mu_\mathbf{1} = (1,0,0)$ be lightlike, we get $\mathcal{D}_L = \mu_\mathbf{i} \times_L \mathbf{N} = (0,2,2)$ is lightlike. Then, from (3.3) we way write

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u' + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} v' = \begin{bmatrix} 1 & 2\theta & -2\theta \\ -2\theta & 1-2\theta^2 & 2\theta^2 \\ -2\theta & 2\theta^2 & 2\theta^2+1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \alpha'(t_0).$$

$$\alpha'(t_0) = \mathcal{R}_{lg}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}) = (0,2,2).$$

From (4.14), we have

$$\begin{aligned} u'(\theta) &= 1, \\ v'(\theta) &= 1, \end{aligned}$$

and

$$H_L f = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

$$
\begin{aligned}
(\nabla_L f)' &= \left(\langle \nabla_L f_x, \alpha'(s)\rangle_L, \langle \nabla_L f_y, \alpha'\rangle_L, \langle -\nabla_L f_z, \alpha'(s)\rangle_L\right), \\
(\nabla_L f^B)' &= \left(\left\langle H_L f^B[0,0:3], \alpha'(s)\right\rangle_L, \left\langle H_L f^B[1,0:3], \alpha'\right\rangle_L, \left\langle -H_L f^B[2,0:3], \alpha'(s)\right\rangle_L\right), \\
(\nabla_L f^B)' &= \left(\langle(-2,0,0),(0,2,2)\rangle_L, \langle(0,-2,0),(0,2,2)\rangle_L, \langle(0,0,2),(0,2,2)\rangle_L\right), \\
(\nabla_L f^B)' &= (0,-4,-4).
\end{aligned}
$$

If we substitute these results into

$$\bar{e}^A(u'(\theta))^2 + 2\bar{f}^A u'(\theta)v'(\theta) + \bar{g}^A(v'(\theta))^2 + \lambda\left\langle(\nabla_L f^B)', \alpha'(\theta)\right\rangle_L = 0, \tag{5.9}$$

$\lambda = 1$ *and from the coefficients of the second fundamental form are we have* $\bar{e}^A = 2,\ \bar{f}^A = -2,\ \bar{g}^A = -4.$

$$
\begin{aligned}
2.1 + 2.(-2).1.1 - 4.1 + 1\langle(0,-4,-4),(0,2,2)\rangle_L &= 0, \\
-6 &= 0.
\end{aligned} \tag{5.10}
$$

We have

$$eq1 = -6 \neq 0.$$

- $eq2 = \left\langle \mathbf{N}^A, \alpha''(\theta)\right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta)\right\rangle_L,\ for\ \mu_\mathbf{2}$
We need to choose $\mu_\mathbf{2}$ *such that* $\mathcal{D}_L(N) = \mu_\mathbf{2} \times_L \mathbf{N}$ *is spacelike.*
Choosing $\mu_\mathbf{2} = (0,0,1)$ *such that* $\mathcal{D}_L = \mu_\mathbf{2} \times_L \mathbf{N} = (2,0,0)$ *is spacelike. Then, from (3.3) we way write,*

$$\alpha'(t_0) = \mathcal{R}_{lg}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}) = (2, -4\theta, -4\theta).$$

From (4.14), we have

$$
\begin{aligned}
u'(\theta) &= 1 - 2\theta, \\
v'(\theta) &= 2\theta - 1,
\end{aligned}
$$

and

$$H_L f = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

$$
\begin{aligned}
(\nabla_L f)' &= \left(\langle \nabla_L f_x, \alpha'(s)\rangle_L, \langle \nabla_L f_y, \alpha'\rangle_L, \langle -\nabla_L f_z, \alpha'(s)\rangle_L\right), \\
(\nabla_L f^B)' &= \left(\left\langle H_L f^B[0,0:3], \alpha'(s)\right\rangle_L, \left\langle H_L f^B[1,0:3], \alpha'\right\rangle_L, \left\langle -H_L f^B[2,0:3], \alpha'(s)\right\rangle_L\right), \\
(\nabla_L f^B)' &= \left(\langle(-2,0,0),(2,-4\theta,-4\theta)\rangle_L, \langle(0,-2,0),(2,-4\theta,-4\theta)\rangle_L, \langle(0,0,2),(2,-4\theta,-4\theta)\rangle_L\right), \\
(\nabla_L f^B)' &= (-4, 8\theta, 8\theta).
\end{aligned}
$$

If we substitute these results into

$$\bar{e}^A(u'(\theta))^2 + 2\bar{f}^A u'(\theta)v'(\theta) + \bar{g}^A(v'(\theta))^2 + \lambda\left\langle(\nabla_L f^B)', \alpha'(\theta)\right\rangle_L = 0, \tag{5.11}$$

where $\lambda = 1$ *and from the coefficients of the second fundamental form are we have* $\bar{e}^A = 2,\ \bar{f}^A = -2,\ \bar{g}^A = -4.$

$$
\begin{aligned}
2.(1-2\theta)^2 + 2.(-2).(1-2\theta)(-2\theta-1) - 4.(-2\theta-1)^2 + 1\langle(-4,8\theta,8\theta),(2,-4\theta,-4\theta)\rangle_L &= 0, \\
-24\theta^2 - 24\theta - 6 &= 0.
\end{aligned}
$$

We have

$$eq2 = -24\theta^2 - 24\theta - 6.0 = 0.$$

Where $\theta_0 = -0.5$ *is the solution to the equation* $eq2 = 0$.
The vector tangent is

$$\alpha'(t_0) = \mathcal{R}_{lg}(\theta_0, \mathbf{N})\mathcal{D}_L(\mathbf{N}) = (2, -4\theta_0, -4\theta_0) = (2, 2, 2).$$

As $\left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L \neq 0$ *for* $\mu_\mathbf{1}$ *and* $\left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L = 0$ *for* $\mu_\mathbf{2}$.
We have $\theta = -0.5$ *is solution to the equation* $eq2 = 0$, *then the equation has one simple solution, then we have one intersection curve passing through P.*

eq1	eq2	case	solutions
$eq1 \neq 0$	$eq2 = 0$	(b)	$\alpha'(t_0) = \mathcal{R}_{lg}(\theta_0, \mathbf{N})\ \mu_\mathbf{2} \times_L \mathbf{N}$ if $\theta_0 = -0.5$.

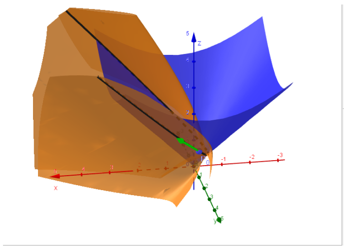

Figure 5.2: $S^A \cap S^B$

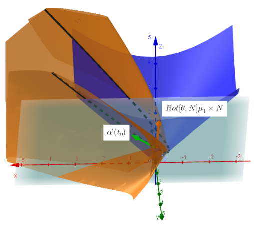

Figure 5.3: $S^A \cap S^B$

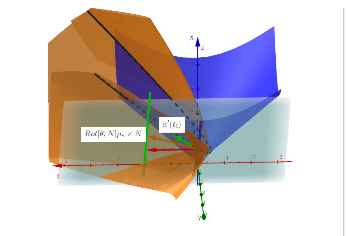

Figure 5.4: $S^A \cap S^B$

5.5. Example of parametric-parametric surface intersection.

Example 5.5.
Let us consider the surface S^A *and* S^B *by the parametric equations*

$$S^A(u, v) = (u, sin(v), 2 + cos(v)),$$

$$f^B(x, y, z) = -x^2 - z^2 + 1.$$

Since the unit normal vectors of these surfaces at the intersection point $P = S^A(0, \pi) = (0, 0, 1)$ *and* $f^B(0, 0, 1) = 0$ *are* $N^A = N^B = N$, *these surfaces intersect tangentially at* $P = (0, 0, 1)$. *The vectors* $S_u^A(0, \pi) = (1, 0, 0)$, $S_v^A(0, \pi) = (0, -1, 0)$, $\nabla_L f^B(0, 1, 1) = (0, 0, 2)$, *produce* $\mathbf{N}^A = \frac{S_u^A \times_L S_v^A}{\|S_u^A \times_L S_v^A\|} = (0, 0, 1)$ *and* $\mathbf{N}^B = \frac{\nabla_L f^B(0,1,1)}{\|\nabla_L f^B(0,1,1)\|} = (0, 0, 1)$.

The vector normal $\mathbf{N}^A = \mathbf{N}^B = \mathbf{N} = (0, 0, 1)$ *are timelike.*

Let $\mu_\mathbf{1} = (1, 0, 0)$, *we get* $\mathcal{D}_L = \mu_\mathbf{1} \times_L \mathbf{N} = (0, -1, 0)$. *Then, from (3.2) we way write*

$$\alpha'(s) = \mathcal{R}_{tm}(\theta, \mathbf{N}) \frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L},$$

$$\alpha'(s) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix},$$

$$\alpha'(s) = \mathcal{R}_{tm}(\theta, \mathbf{N}) \frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L} = (sin(\theta), -cos(\theta), 0).$$

From the equation (4.6) we have

$$u'(\theta) = \sin(\theta),$$
$$v'(\theta) = \cos(\theta).$$

The coefficients of the second fundamental form are

$$e^A = 0.0,\ f^A = 0,\ g^A = -1.0,$$

and

$$H_L f^B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

$$\begin{aligned} (\nabla_L f)' &= \left(\langle \nabla_L f_x, \alpha'(s)\rangle_L, \langle \nabla_L f_y, \alpha'\rangle_L, \langle -\nabla_L f_z, \alpha'(s)\rangle_L\right), \\ (\nabla_L f^B)' &= \left(\left\langle H_L f^B[0,0:3], \alpha'(s)\right\rangle_L, \left\langle H_L f^B[1,0:3], \alpha'\right\rangle_L, \left\langle -H_L f^B[2,0:3], \alpha'(s)\right\rangle_L\right). \end{aligned}$$

If we substitute these results into

$$\begin{aligned} e^A(u'(\theta))^2 + 2f^A u'(\theta)v'(\theta) + g^A(v'(\theta))^2 + \frac{\left\langle (\nabla_L f^B)', \alpha'(\theta)\right\rangle_L}{\|\nabla_L f^B\|_L} &= 0, \\ 0.5 * cos(2 * theta) - 1.5 &= 0, \\ cos(2 * theta) &= \tfrac{1.5}{0.5}, \\ cos(2 * theta) &= 3. \end{aligned}$$

The equation has no solution, then P is the isolated contact point.

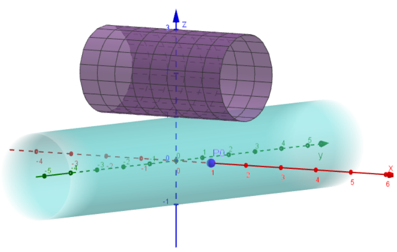

Figure 5.5: $S^A \cap S^B$

6. Conclusions. The application of the Euler–Rodrigues rotation formula in Minkowski 3-space is more intricate than the classical Rodrigues rotation formula in Euclidean 3-space. In the case of a tangential intersection between two timelike surfaces, the tangent vector is computed by applying the rotation to all three types of vectors: spacelike, timelike, and lightlike. For tangential intersections of two lightlike surfaces, the rotation involves spacelike and lightlike vectors. In the case of two spacelike surfaces, the computation of the tangent vector is analogous to that using Rodrigues rotation formula in Euclidean 3-space.

As future work, we intend to extend the method to transversal intersection curves in Lorentz–Minkowski space, computing the tangent vector by rotating a single vector. The generalization of the Euler–Rodrigues rotation method to broader settings in Lorentz–Minkowski space $\mathbb{E}^4_1$ and Euclidean space $\mathbb{E}^4$ remains an open direction for further research.

Authorcontributions. Cintra Neto and Aléssio contributed equally to all aspects of the work, including the conception, design, data collection, analysis, and preparation of the manuscript. Both authors reviewed and approved the final version of the manuscript.

Conflicts of interest. The authors declare that they have no conflict of interest.

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