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On a special class of hypersurfaces in $\mathbb{R}^5$

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Abstract

In this paper we study hypersurfaces in $\mathbb{R}^5$ *parametrized by lines of curvature, with four distinct principal curvatures and with Laplace invariants* $m_{ji} = m_{ki} = m_{li} = 0, m_{jik} \neq 0$, $m_{jkl} \neq 0$, $m_{ljk} \neq 0$, $T_{ijkl} \neq 0$ *for* i, j, k, l *distinct fixed indices. We characterize locally a generic family of such hypersurfaces in terms of the principal curvatures and four vector valued functions of one variable. Moreover, we show that these vector valued functions are invariant under inversions and homotheties. We observe that this class of hypersurfaces cannot have constant Möbius curvature.*

Keywords . Hypersurfaces, Laplace invariants, lines of curvature, Möbius curvature.

1. Introduction. Dupin surfaces were first studied by Dupin in 1822 and more recently by many authors [1]-[3] and [4]-[13], which studied several aspects of Dupin hypersurfaces. The class of Dupin hypersurfaces is invariant under Lie transformations [5]. Therefore, the classification of Dupin hypersurfaces is considered up to these transformations.

Riveros, Rodrigues and Tenenblat [11] studied a class of proper Dupin hypersurfaces M^n in $\mathbb{R}^{n+1}$ parametrized by lines of curvature, with n distinct principal curvatures and constant Möbius curvature. They then showed that for $n \geq 3$ the principal curvatures of such hypersurfaces are functions of separated variables and for $n \geq 4$ proper Dupin hypersurfaces M^n in $\mathbb{R}^{n+1}$ with n distinct principal curvatures and constant Möbius curvature cannot be parametrized by lines of curvature.

Riveros and Tenenblat [10] obtained a local characterization of the Dupin hipersurfaces in $\mathbb{R}^5$ parametrized by lines of curvature, with four distinct principal curvatures and $T_{ijkl} \neq 0$, in terms of the principal curvatures and four vector valued functions in $\mathbb{R}^5$ which are invariant under inversions and homotheties, in this case $m_{ij} = 0$, for $1 \leq i \neq j \leq 4$.

In this paper we study generic hypersurfaces in $\mathbb{R}^5$, parametrized by lines of curvature, with four distinct principal curvatures and with Laplace invariants $m_{ji} = m_{ki} = m_{li} = 0, m_{jik} \neq 0$, $m_{jkl} \neq 0$, $m_{ljk} \neq 0$, $T_{ijkl} \neq 0$. We obtain a local characterization of a generic family of such hypersurfaces (Theorem 3.1), in terms of the principal curvature functions and four vector valued functions of one variable. This family of hypersurfaces includes the Dupin hypersurfaces studied by Riveros-Tenenblat [10]. The characterization is based on the theory of higher-dimensional Laplace invariants introduced by Kamran-Tenenblat [14]-[15].

In section 2, we give some properties of hypersurfaces with distinct principal curvatures. In section 3, Theorem 3.1 gives a local characterization of generic hypersurfaces in $\mathbb{R}^5$ with four distinct principal curvatures. In section 4, we show that the vector valued functions, which appear in the characterization of Theorem 3.1 are invariant under inversions and homotheties, but the functions are not invariant under isometries. Therefore, the vector valued functions are not invariant under the full group of Lie transformations of $\mathbb{R}^5$.

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2. Preliminaries. Let Ω be an open subset of $\mathbb{R}^n$ and $x = (x_1, x_2, \cdots, x_n) \in \Omega$. Let $X : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be a hypersurface parametrized by lines of curvature, with distinct principal curvatures λ_i, $1 \leq i \leq n$ and $N : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ a unit normal vector field of X. Then

$$\begin{aligned} \langle X_{,i}, X_{,j} \rangle &= \delta_{ij} g_{ii}\ ,\ 1 \leq i, j \leq n\ , \\ N_{,i} &= -\lambda_i X_{,i}\ , \end{aligned} \tag{2.1}$$

where the subscript $_{,i}$ denotes the derivative with respect to x_i. Moreover,

$$X_{,ij} - \Gamma^i_{ij} X_{,i} - \Gamma^j_{ij} X_{,j} = 0\ ,\quad 1 \leq i \neq j \leq n\ , \tag{2.2}$$

$$\Gamma^i_{ij} = \frac{\lambda_{i,j}}{\lambda_j - \lambda_i},\quad 1 \leq i \neq j \leq n, \tag{2.3}$$

where Γ^k_{ij} are the Christoffel symbols.

We now consider the higher-dimensional Laplace invariants of the system of equations (2.2) (see [14]-[15] for definition of these invariants),

$$\begin{aligned} m_{ij} &= -\Gamma^i_{ij,i} + \Gamma^i_{ij}\Gamma^j_{ij}, \\ m_{ijk} &= \Gamma^i_{ij} - \Gamma^k_{kj}\ \ ,\ \ k \neq i, j\ ,\ \ 1 \leq k \leq n. \end{aligned} \tag{2.4}$$

As a consequence of (2.3) and the Lemma obtained in [15], we obtain for $1 \leq i, j, k, l \leq n$, i, j, k, l distinct,

$$\begin{aligned} m_{ijk} + m_{kji} &= 0, \\ m_{ijk,k} - m_{ijk} m_{jki} - m_{kj} &= 0, \\ m_{ij,k} + m_{ijk} m_{ik} + m_{ikj} m_{ij} &= 0, \\ m_{ijk} - m_{ijl} - m_{ljk} &= 0, \\ m_{lik,j} + m_{ijl} m_{kil} + m_{ljk} m_{kij} &= 0. \end{aligned} \tag{2.5}$$

Considering the higher-dimensional Laplace invariants satisfying (2.5), for $1 \leq i \neq j \neq k \neq l \leq 4$ fixed, we consider the functions T_{ijkl}, U_{ijkl} and P^4_r defined in [10] by

$$T_{ijkl} = m_{jil} + \left(\log\left(\frac{m_{jik}}{m_{kil}}\right)\right)_{,i}\ , \tag{2.6}$$

$$U_{ijkl} = m_{kil} + \left(\log\left(\frac{m_{jik}}{m_{jil}}\right)\right)_{,i}\ , \tag{2.7}$$

$$P^4_j = m_{jik} T_{ijkl}\ \ ,\ \ P^4_k = m_{jik} U_{ijkl}\ \ ,\ \ P^4_l = m_{jil} U_{ijkl} \tag{2.8}$$

where $m_{jik} \neq 0, m_{jil} \neq 0$ and $m_{kil} \neq 0$.

We now consider the effect on the principal curvatures and on the higher-dimensional Laplace invariants of a hypersurface under an inversion or a homothety.

Let $X : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1} - \{0\}$, $n \geq 3$, be a hypersurface parametrized by lines of curvature with distinct principal curvatures λ_i, $1 \leq i \leq n$.
Consider an inversion

$$\begin{aligned} I^{n+1} : \mathbb{R}^{n+1} - \{0\} &\to \mathbb{R}^{n+1} - \{0\} \\ X &\to I^{n+1}(X) = \frac{X}{\langle X, X \rangle} \end{aligned} \tag{2.9}$$

and a homothety

$$\begin{aligned} D : \mathbb{R}^{n+1} &\to \mathbb{R}^{n+1} \\ X &\to D(X) = aX\ \ , a \in I\!R\ \ , a \neq 0. \end{aligned} \tag{2.10}$$

Denoting $I^{n+1}(X) = \widetilde{X}$ and $\overline{X} = D(X)$, we have that

$$\widetilde{N} = -2\frac{\langle X, N \rangle}{\langle X, X \rangle} X + N$$

is a unit vector field normal to $\widetilde{X}$ and $\overline{N} = N$ is a unit vector field normal to $\overline{X}$.
$\widetilde{X}$ and $\overline{X}$ are hypersurfaces parametrized by lines of curvature, with distinct principal curvatures given respectively by

$$\widetilde{\lambda}_i = \langle X, X\rangle \lambda_i + 2\langle X, N\rangle, \quad \overline{\lambda_i} = \frac{\lambda_i}{a}, 1 \le i \le n. \tag{2.11}$$

Since $\widetilde{X}$ and $\overline{X}$ are a rescaling of X, we conclude that for, $n \geq 3$, an inversion and a homothety, does not change the higher-dimensional Laplace invariants i.e. $\widetilde{m}_{ij} = m_{ij}$, $\widetilde{m}_{ijk} = m_{ijk}$ and $\overline{m}_{ij} = m_{ij}$, $\overline{m}_{ijk} = m_{ijk}$.

The following lemma obtained in [8], provides some properties which are satisfied by the principal curvatures of a hypersurface in $\mathbb{R}^{n+1}$ parametrized by lines of curvature.

Lemma 2.1. *Let $\lambda_r : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $n \geq 3$, be smooth functions distinct at each point. Consider functions m_{ijk} defined by (2.3) and (2.4). Then for i, j fixed, $1 \leq i \neq j \leq n$, the following properties hold*

$$\left(C^{kji} m_{jki}\right)_{,i} = -m_{jki,i} - \left(\frac{\lambda_{i,i}}{\lambda_j - \lambda_i}\right)_{,k}, \tag{2.12}$$

$$\left(C^{kji} m_{jki}\right)_{,j} = \left(\frac{\lambda_{j,j}}{\lambda_j - \lambda_i}\right)_{,k}, \tag{2.13}$$

$$\left(C^{kji} m_{jki}\right)_{,l} = \left(C^{lji} m_{jli}\right)_{,k}, \tag{2.14}$$

where $C^{ijk} = \dfrac{\lambda_i - \lambda_j}{\lambda_k - \lambda_j}$, $1 \leq k \neq l \leq n$ are distinct from i and j are the Möbius curvatures.

Lemma 2.2. *Let $X : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be a hypersurface parametrized by lines of curvature, with n distinct principal curvatures λ_r, $1 \leq r \leq n$. For i, j, k fixed, $1 \leq i \neq j \neq k \leq n$, the transformation*

$$X = V\overline{X}, \quad \text{where} \quad V = \frac{e^{\int \frac{\lambda_k - \lambda_j}{\lambda_j - \lambda_i} m_{jki} dx_k}}{\lambda_j - \lambda_i}, \tag{2.15}$$

transforms system (2.2) into

$$\begin{aligned}
\overline{X}_{,ij} + A\overline{X}_{,j} - m_{ij}\overline{X} &= 0,\\
\overline{X}_{,ir} + (A + m_{jir})\overline{X}_{,r} - m_{ir}\overline{X} &= 0,\\
\overline{X}_{,jr} + m_{irj}\overline{X}_{,j} + m_{ijr}\overline{X}_{,r} &= 0,\\
\overline{X}_{,rl} + m_{ilr}\overline{X}_{,r} + m_{irl}\overline{X}_{,l} &= 0,
\end{aligned} \tag{2.16}$$

where l and r are such that $1 \leq r \neq l \neq i \neq j \leq n$ and

$$A = -\int m_{jki,i} dx_k. \tag{2.17}$$

Moreover,

$$A_{,j} = m_{ji} - m_{ij}, \quad A_{,r} = -m_{jri,i}. \tag{2.18}$$

Remark 2.1. It follows from Lemma 2.1 that,

$$\begin{aligned}
V_{,i} &= \left(A + \Gamma^j_{ji}\right) V,\\
V_{,j} &= \Gamma^i_{ij} V,\\
V_{,k} &= \Gamma^i_{ik} V,\\
V_{,l} &= \Gamma^i_{il} V,
\end{aligned} \tag{2.19}$$

where V is given by (2.15), A is given by (2.17) and l is distinct from i, j, k.

3. A Characterization of a special class of hypersurfaces in $\mathbb{R}^5$. In this section, we prove our main result which provides a local characterization of generic hypersurfaces parametrized by lines of curvature in $\mathbb{R}^5$, with four distinct principal curvatures.

Theorem 3.1. *Let $X : \Omega \subset \mathbb{R}^4 \to \mathbb{R}^5$, be a hypersurface parametrized by lines of curvature, with four distinct principal curvatures λ_r. For i, j, k, l distinct fixed indices, suppose $m_{ji} = m_{ki} = m_{li} = 0$, $m_{jkl} \neq 0$, $m_{ljk} \neq 0$ and $T_{ijkl} \neq 0$ then*

$$X = V\left(B_j^4 - B_k^4 + B_l^4\right), \tag{3.1}$$

where

$$B_s^4 = \frac{1}{Q_s^4}\left(\int \frac{Q_s^4 G_i(x_i)}{P_s^4} dx_i + G_s(x_s)\right), \; s \neq i, V \text{is given by } (2.15) \tag{3.2}$$

P_s^4, are defined by (2.8), $G_r(x_r)$, $1 \leq r \leq 4$, are vector valued functions of $\mathbb{R}^5$,
$A_{,j} = -m_{ij}$, $A = -\int m_{jki,i} dx_k$ and

$$Q_s^4 \;=\; \begin{cases} e^{\int A dx_i} & \text{if } \; s = j\,, \\ e^{\int (A + m_{jis}) dx_i} & \text{if } \; s = k, l. \end{cases} \tag{3.3}$$

Moreover, considering

$$\alpha^i = \left(A + \frac{\lambda_{j,i}}{\lambda_i - \lambda_j}\right) M + M_{,i}\;, \qquad \alpha^s = \frac{\lambda_{i,s}}{\lambda_s - \lambda_i} M + M_{,s}\;\;,\;\; s \neq i, \tag{3.4}$$

where $M = B_j^4 - B_k^4 + B_l^4$, the functions $G_r(x_r)$ satisfy the following properties in Ω, for $1 \leq r \neq t \leq 4$:

a) $\alpha^r \neq 0$,
b) $\langle \alpha^r, \alpha^t \rangle = 0, \;\; r \neq t$,
c) $\lambda_r = \dfrac{\langle \alpha^r_{,r}\,, \alpha^i \times \alpha^j \times \alpha^k \times \alpha^l \rangle}{V|\alpha^r|^2\, |\alpha^i|\, |\alpha^j|\, |\alpha^k|\, |\alpha^l|}$.

Conversely, let $\lambda_r : \Omega \subset \mathbb{R}^4 \to \mathbb{R}$, $r = 1, .., 4$ be real functions, distinct at each point. Assume that the functions m_{rts} and m_{rt} defined by

$$\begin{aligned} m_{rts} &= \frac{\lambda_{r,t}}{\lambda_t - \lambda_r} - \frac{\lambda_{s,t}}{\lambda_t - \lambda_s}\;,\; 1 \leq r \neq t \neq s \leq 4, \\ m_{rt} &= -\left(\frac{\lambda_{r,t}}{\lambda_t - \lambda_r}\right)_{,r} - \frac{\lambda_{r,t}\lambda_{t,r}}{(\lambda_t - \lambda_r)^2}\;,\; 1 \leq r \neq t \leq 4, \end{aligned} \tag{3.5}$$

satisfy (2.5), and for i, j, k, l distinct fixed indices, $m_{ji} = m_{ki} = m_{li} = 0$, $m_{jkl} \neq 0$, $m_{ljk} \neq 0$, $T_{ijkl} \neq 0$. Then for any vector valued functions $G_r(x_r)$ satisfying properties a) b) c), where α^r is defined by (3.4), the function $X : \Omega \subset \mathbb{R}^4 \to \mathbb{R}^5$ given by (3.1) describes a hypersurface parametrized by lines of curvature whose principal curvatures are the functions λ_r.

Remark 3.1. We observe that $T_{ijkl} = 0$ if and only if $U_{ijkl} = 0$.
In fact, using the relations (2.5), it follows from (2.6) and (2.7) that $T_{ijkl} = \dfrac{m_{jil}}{m_{kil}} U_{ijkl}$.

Hence, the hypothesis of Theorem 3.1 implies that $P_s^4 \neq 0$, for $s \neq i$.
Moreover, from third equation of (2.5), the conditions $m_{ji} = m_{ki} = m_{li} = 0$, implies that $m_{jk} = m_{jl} = m_{kj} = m_{kl} = m_{lj} = m_{lk} = 0$.

For the proof of Theorem 3.1 we will need three lemmas.

Lemma 3.1. Let X be a hypersurface as in Theorem 3.1, then

$$X = \frac{V}{m_{jik}}\left(W^k - W^j\right), \tag{3.6}$$

where $W^k(x_i, x_j, x_l)$ and $W^j(x_i, x_k, x_l)$ satisfy the following systems of equations,

$$\begin{aligned} W^k_{,ij} + \left(A - \frac{m_{jik,i}}{m_{jik}}\right) W^k_{,j} + m_{kji} m_{jik} W^k &= 0, \\ W^k_{,il} + \left(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}}\right) W^k_{,l} + m_{ilk} m_{kil} W^k &= 0, \\ W^k_{,jl} + \frac{m_{jlk} m_{kil}}{m_{jik}} W^k_{,j} + \frac{m_{ljk} m_{jik}}{m_{kil}} W^k_{,l} &= 0. \end{aligned} \tag{3.7}$$

$$\begin{aligned} W^j_{,ik} + \left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}}\right) W^j_{,k} + m_{ikj} m_{jik} W^j &= 0, \\ W^j_{,il} + \left(A + m_{jil} - \frac{m_{jil,i}}{m_{jil}}\right) W^j_{,l} + m_{ilj} m_{jil} W^j &= 0, \\ W^j_{,kl} + \frac{m_{jlk} m_{jil}}{m_{jik}} W^j_{,k} + \frac{m_{jkl} m_{jik}}{m_{jil}} W^j_{,l} &= 0. \end{aligned} \tag{3.8}$$

Proof: From (2.2) we have,

$$X_{,sr} - \Gamma^s_{sr} X_{,s} - \Gamma^r_{sr} X_{,r} = 0\,, \quad 1 \leq s \neq r \leq 4. \tag{3.9}$$

For fixed distinct indices $i,\ j,\ k$, we consider the transformation

$$X = V\overline{X}, \tag{3.10}$$

as in Lemma 2.2, where V is given by (2.15). Then the system (3.9) reduces to

$$\begin{aligned} \overline{X}_{,ij} + A\overline{X}_{,j} - m_{ij}\overline{X} = 0, \\ \overline{X}_{,ir} + (A + m_{jir})\overline{X}_{,r} - m_{ir}\overline{X} = 0, \\ \overline{X}_{,jr} + m_{irj}\overline{X}_{,j} + m_{ijr}\overline{X}_{,r} = 0, \\ \overline{X}_{,kl} + m_{ilk}\overline{X}_{,k} + m_{ikl}\overline{X}_{,l} = 0, \end{aligned} \tag{3.11}$$

where $r = k, l$ and $k \neq l$,

$$A_{,j} = -m_{ij}, \quad A_{,r} = -m_{jri,i}. \tag{3.12}$$

It follows from the third and second equations of (2.5) and (3.12) that

$$(A + m_{jir})_{,r} = -m_{ir},\ r = k, l. \tag{3.13}$$

Using (3.12), (3.13) and the fact that $m_{ji} = 0, m_{ki} = 0$ in the first two equations of (3.11), we have that

$$\overline{X}_{,i} + A\overline{X} = W^j(x_i, x_k, x_l), \tag{3.14}$$

$$\overline{X}_{,i} + (A + m_{jik})\overline{X} = W^k(x_i, x_j, x_l), \tag{3.15}$$

$$\overline{X}_{,i} + (A + m_{jil})\overline{X} = W^l(x_i, x_j, x_k). \tag{3.16}$$

where W^j, W^k and W^l are functions that do not depend on x_j, x_k and x_l, respectively.
Since $m_{jik} \neq 0$, from (3.14) and (3.15) we have

$$\overline{X} = \frac{1}{m_{jik}}(W^k - W^j). \tag{3.17}$$

Therefore, it follows from (3.10) that X is given by (3.6).
From (3.17) and (2.5) we obtain

$$\begin{aligned} \overline{X}_{,i} &= -\frac{m_{jik,i}}{(m_{jik})^2}(W^k - W^j) + \frac{1}{m_{jik}}(W^k - W^j)_{,i} \\ \overline{X}_{,j} &= \frac{m_{kji}}{m_{jik}}(W^k - W^j) + \frac{1}{m_{jik}}W^k_{,j} \\ \overline{X}_{,k} &= -\frac{m_{ikj}}{m_{jik}}(W^k - W^j) - \frac{1}{m_{jik}}W^j_{,k} \\ \overline{X}_{,l} &= -\frac{m_{jik,l}}{(m_{jik})^2}(W^k - W^j) + \frac{1}{m_{jik}}(W^k - W^j)_{,l} \\ \overline{X}_{,ij} &= \left(m_{kji} + \frac{m_{ij}}{m_{jik}} + \frac{m_{ijk} m_{jik,i}}{(m_{jik})^2}\right)(W^k - W^j) - \frac{m_{jik,i}}{(m_{jik})^2}W^k_{,j} \\ &\quad -\frac{m_{ijk}}{m_{jik}}(W^k - W^j)_{,i} + \frac{1}{m_{jik}}W^k_{,ij} \\ \overline{X}_{,ik} &= \left(m_{ikj} + \frac{m_{ik}}{m_{jik}} + \frac{m_{ikj} m_{jik,i}}{(m_{jik})^2}\right)(W^k - W^j)] + \frac{m_{jik,i}}{(m_{jik})^2}W^j_{,k} \\ &\quad -\frac{m_{ikj}}{m_{jik}}(W^k - W^j)_{,i} - \frac{1}{m_{jik}}W^j_{,ik} \end{aligned}$$

$$\begin{aligned}
\overline{X}_{,il} &= \left(\frac{m_{il}+m_{jli}m_{lij}}{m_{jik}}+\frac{m_{jli}m_{jik,i}+(m_{klj}m_{lik})_{,i}}{(m_{jik})^2}+\frac{2m_{jik,i}m_{jik,l}}{(m_{jik})^3}\right)(W^k-W^j)\\
&-\frac{m_{jik,i}}{(m_{jik})^2}(W^k-W^j)_{,l}-\frac{m_{jik,l}}{(m_{jik})^2}(W^k-W^j)_{,i}+\frac{1}{m_{jik}}(W^k-W^j)_{,il}\\
\overline{X}_{,jk} &= \left(-\frac{2m_{ijk}m_{jki}+m_{kj}}{m_{jik}}\right)(W^k-W^j)+\frac{m_{ijk}}{m_{jik}}W^j_{,k}-\frac{m_{ikj}}{m_{jik}}W^k_{,j}\\
\overline{X}_{,jl} &= \left(\frac{m_{kji,l}}{m_{jik}}-\frac{m_{kji}m_{jik,l}}{(m_{jik})^2}\right)(W^k-W^j)+\frac{m_{kji}}{m_{jik}}(W^k-W^j)_{,l}-\frac{m_{jik,l}}{(m_{jik})^2}W^k_{,j}\\
&+\frac{1}{m_{jik}}W^k_{,jl}\\
\overline{X}_{,kl} &= \left(-\frac{m_{ikj,l}}{m_{jik}}+\frac{m_{ikj}m_{jik,l}}{(m_{jik})^2}\right)(W^k-W^j)-\frac{m_{ikj}}{m_{jik}}(W^k-W^j)_{,l}-\frac{m_{jik,l}}{(m_{jik})^2}W^j_{,k}\\
&-\frac{1}{m_{jik}}W^j_{,kl}.
\end{aligned}$$

We will now obtain the differential equations that W^k and W^j must satisfy, by using (3.11), (3.14)-(3.16).

The substitution of $\overline{X}$ and $\overline{X}_{,i}$ into (3.14), (3.15) and (3.16), gives

$$\left(A-\frac{m_{jik,i}}{m_{jik}}\right)(W^k-W^j)+(W^k-W^j)_{,i} = m_{jik}W^j, \tag{3.18}$$

$$\left(A+m_{jik}-\frac{m_{jik,i}}{m_{jik}}\right)(W^k-W^j)+(W^k-W^j)_{,i} = m_{jik}W^k, \tag{3.19}$$

$$\left(A+m_{jil}-\frac{m_{jik,i}}{m_{jik}}\right)(W^k-W^j)+(W^k-W^j)_{,i} = m_{jik}W^l. \tag{3.20}$$

Now substituting $\overline{X}_{,j}$ and $\overline{X}_{,ij}$ in the first equation of the system (3.11), we obtain the first equation of (3.7).

The Substitution of $\overline{X}_{,k}$ and $\overline{X}_{,ik}$ in the second equation of the system (3.11) with $r=k$, we obtain the first equation of (3.8).

Using $\overline{X}_{,l}$ and $\overline{X}_{,il}$ in the second equation of the system (3.11) with $r=l$, we obtain an equation equivalent to (3.20).

Also, using $\overline{X}_{,j}$, $\overline{X}_{,k}$ and $\overline{X}_{,jk}$ in the third equation of the system (3.11) with $r=k$, we obtain an identity.

Using $\overline{X}_{,j}$, $\overline{X}_{,l}$ and $\overline{X}_{,jl}$ in the third equation of the system (3.11) with $r=l$, we obtain third equation of (3.7).

Now the substitution of $\overline{X}_{,k}$, $\overline{X}_{,l}$ and $\overline{X}_{,kl}$ in the last equation of the system (3.11), we obtain third equation of (3.8).

Differentiating the first equation and the third equation of (3.7) with relation to x_l and x_i, respectively and using the fact that $W^k_{,jli}=W^k_{,ijl}$, we obtain the second equation of (3.7).

Similarly, differentiating the first equation and the third equation of (3.8) with relation to x_l and x_i respectively and using the fact that $W^j_{,ikl}=W^j_{,kli}$ we obtain the second equation of (3.8). Which concludes the proof of Lemma 3.1.

□

Lemma 3.2. *The solution of (3.7) is given by*

$$W^k=\frac{m_{jik}}{Q_j^4}\left(\int\frac{Q_j^4G_i(x_i)}{P_j^4}dx_i+G_j(x_j)\right)-\frac{m_{kil}}{Q_l^4}\left(\int\frac{Q_l^4G_i(x_i)}{P_l^4}dx_i+G_l(x_l)\right). \tag{3.21}$$

Proof: From equations (2.5) and (3.12), we have that

$$\left(A-\frac{m_{jik,i}}{m_{jik}}\right)_{,j} = m_{jik}m_{kji}, \tag{3.22}$$

$$\left(A+m_{jil}-\frac{m_{kil,i}}{m_{kil}}\right)_{,l} = m_{kil}m_{ilk}. \tag{3.23}$$

Using (3.22) in the first equation of system (3.7), we get

$$\left(W^k_{,i}+\left(A-\frac{m_{jik,i}}{m_{jik}}\right)W^k\right)_{,j}=0\,,$$

whose integration with respect to x_j, provides

$$W^k_{,i} + \left(A - \frac{m_{jik,i}}{m_{jik}}\right) W^k = D^j(x_i, x_l)\ . \tag{3.24}$$

Substituting (3.23) in the second equation of (3.7), we obtain

$$\left(W^k_{,i} + \left(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}}\right) W^k\right)_{,l} = 0\ .$$

Therefore, integrating with respect to x_l, we get

$$W^k_{,i} + \left(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}}\right) W^k = D^l(x_i, x_j). \tag{3.25}$$

From (3.24), (3.25) and (2.6), we conclude that

$$W^k = \frac{1}{T_{ijkl}}(D^l - D^j). \tag{3.26}$$

Differentiating W^k, using (2.5) and the following derivatives

$$T_{ijkl,j} = \frac{m_{jik}m_{ljk}}{m_{kil}}T_{ijkl}\quad,\quad T_{ijkl,k} = 0\quad,\quad T_{ijkl,l} = \frac{m_{kil}m_{jlk}}{m_{jik}}T_{ijkl}\ ,$$

we obtain

$$W^k_{,i} = -\frac{T_{ijkl,i}}{(T_{ijkl})^2}(D^l - D^j) + \frac{1}{T_{ijkl}}(D^l - D^j)_{,i} \tag{3.27}$$

$$W^k_{,j} = -\frac{m_{ljk}m_{jik}}{m_{kil}}\frac{1}{T_{ijkl}}(D^l - D^j) + \frac{1}{T_{ijkl}}D^l_{,j} \tag{3.28}$$

$$W^k_{,l} = -\frac{m_{jlk}m_{kil}}{m_{jik}}\frac{1}{T_{ijkl}}(D^l - D^j) - \frac{1}{T_{ijkl}}D^j_{,l} \tag{3.29}$$

$$\begin{aligned} W^k_{,ij} = {} & \frac{1}{T_{ijkl}}D^l_{,ji} - \frac{T_{ijkl,i}}{(T_{ijkl})^2}D^l_{,j} - \left(\frac{m_{ljk}}{m_{kil}} + \frac{m_{kji}}{T_{ijkl}} - \frac{m_{ljk}}{m_{kil}}\frac{T_{ijkl,i}}{(T_{ijkl})^2}\right)\times \\ & m_{jik}(D^l - D^j) - \frac{m_{ljk}m_{jik}}{m_{kil}}\frac{1}{T_{ijkl}}(D^l - D^j)_{,i} \end{aligned} \tag{3.30}$$

$$\begin{aligned} W^k_{,il} = {} & -\frac{1}{T_{ijkl}}D^j_{,li} + \frac{T_{ijkl,i}}{(T_{ijkl})^2}D^j_{,l} - \left(\frac{m_{klj}}{m_{jik}} + \frac{m_{ilk}}{T_{ijkl}} - \frac{m_{jlk}}{m_{jik}}\frac{T_{ijkl,i}}{(T_{ijkl})^2}\right)\times \\ & m_{kil}(D^l - D^j) - \frac{m_{jlk}m_{kil}}{m_{jik}}\frac{1}{T_{ijkl}}(D^l - D^j)_{,i} \end{aligned} \tag{3.31}$$

$$W^k_{,jl} = \frac{2m_{jlk}m_{ljk}}{T_{ijkl}}(D^l - D^j) - \frac{m_{kil}m_{jlk}}{m_{jik}}\frac{1}{T_{ijkl}}D^l_{,j} + \frac{m_{ljk}m_{jik}}{m_{kil}}\frac{1}{T_{ijkl}}D^j_{,l}. \tag{3.32}$$

The substitution of (3.26) and (3.27) into (3.24) and (3.25), gives

$$\left(A - \frac{m_{jik,i}}{m_{jik}} - \frac{T_{ijkl,i}}{T_{ijkl}}\right)(D^l - D^j) + (D^l - D^j)_{,i} = T_{ijkl}D^j, \tag{3.33}$$

$$\left(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}} - \frac{T_{ijkl,i}}{T_{ijkl}}\right)(D^l - D^j) + (D^l - D^j)_{,i} = T_{ijkl}D^l. \tag{3.34}$$

Substituting (3.26), (3.28), (3.31) in the first equation of the system (3.7), and using (3.33), we obtain

$$D^l_{,ji} + \left(A - \frac{m_{jik,i}}{m_{jik}} - \frac{T_{ijkl,i}}{T_{ijkl}}\right) D^l_{,j} + \frac{m_{kjl}m_{jik}}{m_{kil}}T_{ijkl}D^l = 0. \tag{3.35}$$

Similarly, using (3.29), (3.32) in the second equation of the system (3.7), and using (3.34), we get

$$D^j_{,li} + \left(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}} - \frac{T_{ijkl,i}}{T_{ijkl}}\right) D^j_{,l} + \frac{m_{jlk}m_{kil}}{m_{jik}}T_{ijkl}D^j = 0. \tag{3.36}$$

The substitution of (3.28), (3.29) and (3.32) in the third equation of (3.7) gives an identity.

Now we compute the Laplace invariant $\bar{m}_{ji}$ of equation (3.35),

$$\bar{m}_{ji} = \left(A - \frac{m_{jik,i}}{m_{jik}} - \frac{T_{ijkl,i}}{T_{ijkl}}\right)_{,j} - \frac{m_{kjl}m_{jik}}{m_{kil}}T_{ijkl}.$$

Using (3.12), (2.5) and the relation

$$T_{ijkl,ij} = T_{ijkl,j}\left(T_{ijkl} + \frac{T_{ijkl,i}}{T_{ijkl}}\right) + m_{jik}m_{kji}T_{ijkl}$$

we conclude that

$$\bar{m}_{ji} = 0.$$

Therefore, the solution of equation (3.35) is given by,

$$D^l(x_i, x_j) = m_{jik}T_{ijkl}e^{-\int Adx_i}\left(\int \frac{e^{\int Adx_i}G_i(x_i)}{m_{jik}T_{ijkl}}dx_i + G_j(x_j)\right).$$

where $G_i(x_i)$ e $G_j(x_j)$ are vector valued functions in $\mathbb{R}^5$.

Similarly, let us compute the Laplace invariant $\tilde{m}_{li}$ of equation (3.36)

$$\tilde{m}_{li} = \left(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}} - \frac{T_{ijkl,i}}{T_{ijkl}}\right)_{,l} - \frac{m_{jlk}m_{kil}}{m_{jik}}T_{ijkl}.$$

Using (3.13), (2.5) and the relation

$$T_{ijkl,il} = -T_{ijkl,l}\left(T_{ijkl} - \frac{T_{ijkl,i}}{T_{ijkl}}\right) + m_{kil}m_{ilk}T_{ijkl}$$

we obtain

$$\tilde{m}_{li} = 0.$$

Therefore, the solution of equation (3.36) is given by,

$$D^j(x_i, x_l) = m_{kil}T_{ijkl}e^{-\int (A+m_{jil})dx_i}\left(\int \frac{e^{\int (A+m_{jil})dx_i}\bar{G}_i(x_i)}{m_{kil}T_{ijkl}}dx_i + G_l(x_l)\right). \quad (3.37)$$

where $\bar{G}_i(x_i)$ and $\bar{G}_l(x_l)$ are vector valued functions in $\mathbb{R}^5$.

We will now show that $G_i(x_i) = \bar{G}_i(x_i)$. Differentiating D^l and D^j with respect to x_i, we obtain respectively

$$\begin{aligned} D^l_{,i} &= -\left(A - \frac{m_{jik,i}}{m_{jik}} - \frac{T_{ijkl,i}}{T_{ijkl}}\right)D^l + G_i(x_i), \\ D^j_{,i} &= -\left(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}} - \frac{T_{ijkl,i}}{T_{ijkl}}\right)D^j + \bar{G}_i(x_i). \end{aligned}$$

Therefore

$$\begin{aligned} (D^l - D^j)_{,i} &= G_i(x_i) - \bar{G}_i(x_i) - \left(A - \frac{T_{ijkl,i}}{T_{ijkl}}\right)(D^l - D^j) + \frac{m_{jik,i}}{m_{jik}}D^l \\ &\quad + \left(m_{jil} - \frac{m_{kil,i}}{m_{kil}}\right)D^j. \end{aligned}$$

It follows from (3.33), that

$$G_i(x_i) = \bar{G}_i(x_i).$$

Now using (2.8) and (3.3) we get

$$D^l(x_i, x_j) = \frac{P_j^4}{Q_j^4}\left(\int \frac{Q_j^4 G_i(x_i)}{P_j^4}dx_i + G_j(x_j)\right). \quad (3.38)$$

On the other hand, from equations (2.5) we obtain,

$$m_{kil}T_{ijkl} = m_{jil}U_{ijkl}. \tag{3.39}$$

Therefore from (3.39) and (3.37), we have that

$$D^j(x_i, x_l) = m_{jil}U_{ijkl}e^{-\int(A+m_{jil})dx_i}\left(\int \frac{e^{\int(A+m_{jil})dx_i}G_i(x_i)}{m_{jil}U_{ijkl}}dx_i + G_l(x_l)\right).$$

Using (2.8) and (3.3) we obtain

$$D^j(x_i, x_l) = \frac{P_l^4}{Q_l^4}\left(\int \frac{Q_l^4 G_i(x_i)}{P_l^4}dx_i + G_l(x_l)\right). \tag{3.40}$$

Substituting (3.38) and (3.40) into (3.26), we conclude that

$$W^k = \frac{1}{T_{ijkl}}\left\{\frac{P_j^4}{Q_j^4}\left(\int \frac{Q_j^4 G_i(x_i)}{P_j^4}dx_i + G_j(x_j)\right) - \frac{P_l^4}{Q_l^4}\left(\int \frac{Q_l^4 G_i(x_i)}{P_l^4}dx_i + G_l(x_l)\right)\right\}.$$

It follows from (3.39) and (3.3) that $P_l^4 = m_{jil}U_{ijkl} = m_{kil}T_{ijkl}$.
Using this relation in the expression above, we obtain (3.21).

□

Lemma 3.3. *The solution of (3.8) is given by*

$$W^j = \frac{m_{jik}}{Q_k^4}\left(\int \frac{Q_k^4 F(x_i)}{P_k^4}dx_i + G_k(x_k)\right) - \frac{m_{jil}}{Q_l^4}\left(\int \frac{Q_l^4 F(x_i)}{P_l^4}dx_i + \bar{G}_l(x_l)\right). \tag{3.41}$$

Proof: From (2.5) and (3.12), we have that

$$\left(A + m_{jir} - \frac{m_{jir,i}}{m_{jir}}\right)_{,r} = m_{jir}m_{irj}\ ,\ r = k, l. \tag{3.42}$$

The substitution of (3.42) in the first two equations of (3.8), gives

$$\left(W^j_{,i} + \left(A + m_{jir} - \frac{m_{jir,i}}{m_{jir}}\right)W^j\right)_{,r} = 0\ ,\ r = k, l.$$

Thus, we have that

$$W^j_{,i} + \left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}}\right)W^j = L^k(x_i, x_l) \tag{3.43}$$

and

$$W^j_{,i} + \left(A + m_{jil} - \frac{m_{jil,i}}{m_{jil}}\right)W^j = L^l(x_i, x_k)\ . \tag{3.44}$$

From (3.43), (3.44) and (2.7), we conclude that

$$W^j = \frac{1}{U_{ijkl}}(L^l - L^k). \tag{3.45}$$

Differentiating W^j, using (2.5) and the following derivatives

$$U_{ijkl,j} = 0, \quad U_{ijkl,k} = \frac{m_{jik}m_{jkl}}{m_{jil}}U_{ijkl}, \quad U_{ijkl,l} = \frac{m_{jil}m_{jlk}}{m_{jik}}U_{ijkl}\ ,$$

we get

$$W^j_{,i} = -\frac{U_{ijkl,i}}{(U_{ijkl})^2}(L^l - L^k) + \frac{1}{U_{ijkl}}(L^l - L^k)_{,i} \tag{3.46}$$

$$W^j_{,k} = -\frac{m_{jkl}m_{jik}}{m_{jil}}\frac{1}{U_{ijkl}}(L^l - L^k) + \frac{1}{U_{ijkl}}L^l_{,k} \tag{3.47}$$

$$W^j_{,l} = -\frac{m_{jlk}m_{jil}}{m_{jik}}\frac{1}{U_{ijkl}}(L^l - L^k) - \frac{1}{U_{ijkl}}L^k_{,l} \tag{3.48}$$

$$W^j_{,ki} = \frac{1}{U_{ijkl}}L^l_{,ki} - \frac{U_{ijkl,i}}{(U_{ijkl})^2}L^l_{,k} - \left(\frac{m_{jkl}}{m_{jil}} + \frac{m_{ikj}}{U_{ijkl}} - \frac{m_{jkl}}{m_{jil}}\frac{U_{ijkl,i}}{(U_{ijkl})^2}\right) \times m_{jik}(L^l - L^k) - \frac{m_{jkl}m_{jik}}{m_{jil}}\frac{1}{U_{ijkl}}(L^l - L^k)_{,i} \tag{3.49}$$

$$W^j_{,li} = -\frac{1}{U_{ijkl}}L^k_{,li} + \frac{U_{ijkl,i}}{(U_{ijkl})^2}L^k_{,l} - \left(\frac{m_{klj}}{m_{jik}} + \frac{m_{ilj}}{U_{ijkl}} - \frac{m_{jlk}}{m_{jik}}\frac{U_{ijkl,i}}{(U_{ijkl})^2}\right) \times m_{jil}(L^l - L^k) - \frac{m_{jlk}m_{jil}}{m_{jik}}\frac{1}{U_{ijkl}}(L^l - L^k)_{,i} \tag{3.50}$$

$$W^j_{,lk} = 2\frac{m_{jlk}m_{jkl}}{U_{ijkl}}(L^l - L^k) - \frac{m_{jlk}m_{jil}}{m_{jik}}\frac{1}{U_{ijkl}}L^l_{,k} + \frac{m_{jkl}m_{jik}}{m_{jil}}\frac{1}{U_{ijkl}}L^k_{,l}. \tag{3.51}$$

The substitution of (3.45) and (3.46) in (3.43) and (3.44), gives

$$\left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}} - \frac{U_{ijkl,i}}{U_{ijkl}}\right)(L^l - L^k) + (L^l - L^k)_{,i} = U_{ijkl}L^k, \tag{3.52}$$

$$\left(A + m_{jil} - \frac{m_{jil,i}}{m_{jil}} - \frac{U_{ijkl,i}}{U_{ijkl}}\right)(L^l - L^k) + (L^l - L^k)_{,i} = U_{ijkl}L^l. \tag{3.53}$$

Substituting (3.45), (3.47) and (3.50) in the first equation of (3.8), as a consequence of (3.52), we get

$$L^l_{,ki} + \left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}} - \frac{U_{ijkl,i}}{U_{ijkl}}\right)L^l_{,k} + \frac{m_{lkj}m_{jik}}{m_{jil}}U_{ijkl}L^l = 0. \tag{3.54}$$

Similarly, it follows from (3.45), (3.48), (3.51) substituted in the second equation of (3.8), as a consequence of (3.53), that

$$L^k_{,li} + \left(A + m_{jil} - \frac{m_{jil,i}}{m_{jil}} - \frac{U_{ijkl,i}}{U_{ijkl}}\right)L^k_{,l} + \frac{m_{jlk}m_{jil}}{m_{jik}}U_{ijkl}L^k = 0. \tag{3.55}$$

The expressions (3.47), (3.48) and (3.51) substituted in the third equation of (3.8) provide an identity. The Laplace invariant $\bar{\bar{m}}_{ki}$ of equation (3.54) is given by

$$\bar{\bar{m}}_{ki} = \left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}} - \frac{U_{ijkl,i}}{U_{ijkl}}\right)_{,k} - \frac{m_{lkj}m_{jik}}{m_{jil}}U_{ijkl}.$$

Using (3.13), (2.5) and the relation

$$U_{ijkl,ik} = U_{ijkl,k}\left(U_{ijkl} + \frac{U_{ijkl,i}}{U_{ijkl}}\right) + m_{jik}m_{ikj}U_{ijkl}$$

we obtain

$$\bar{\bar{m}}_{ki} = 0.$$

Therefore, the solution of equation (3.54) is given by

$$L^l(x_i, x_k) = m_{jik}U_{ijkl}e^{-\int(A+m_{jik})dx_i}\left(\int \frac{e^{\int(A+m_{jik})dx_i}F(x_i)}{m_{jik}U_{ijkl}}dx_i + G_k(x_k)\right),$$

where $F(x_i)$ and $G_k(x_k)$ are vector valued functions in $\mathbb{R}^5$.

Similarly, we compute the Laplace invariant $\tilde{\tilde{m}}_{li}$ of equation (3.55),

$$\tilde{\tilde{m}}_{li} = \left(A + m_{jil} - \frac{m_{jil,i}}{m_{jil}} - \frac{U_{ijkl,i}}{U_{ijkl}}\right)_{,l} - \frac{m_{jlk}m_{jil}}{m_{jik}}U_{ijkl}.$$

Using (3.13), (2.5) and the relation

$$U_{ijkl,il} = -U_{ijkl,l}\left(U_{ijkl} - \frac{U_{ijkl,i}}{U_{ijkl}}\right) + m_{jil}m_{ilj}U_{ijkl}$$

we obtain

$$\tilde{\tilde{m}}_{li} = 0.$$

We conclude that the solution of (3.55) is given by,

$$L^k(x_i, x_l) = m_{jil}U_{ijkl}e^{-\int(A+m_{jil})dx_i}\left(\int \frac{e^{\int(A+m_{jil})dx_i}\bar{F}(x_i)}{m_{jil}U_{ijkl}}dx_i + \bar{G}_l(x_l)\right),$$

where $\bar{F}(x_i)$ and $\bar{G}_l(x_l)$ are vector valued functions in $\mathbb{R}^5$.

As in the proof of the previous lemma, we show that $F(x_i) = \bar{F}(x_i)$.
Differentiating L^l and L^k with respect to x_i we obtain respectively

$$L^l_{,i} = -\left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}} - \frac{U_{ijkl,i}}{U_{ijkl}}\right)L^l + F(x_i), \tag{3.56}$$

$$L^k_{,i} = -\left(A + m_{jil} - \frac{m_{jil,i}}{m_{jil}} - \frac{U_{ijkl,i}}{U_{ijkl}}\right)L^k + \bar{F}(x_i). \tag{3.57}$$

From (3.56), (3.57) and (3.52) we conclude that

$$F(x_i) = \bar{F}(x_i).$$

Now using (2.8) and (3.3) we have that

$$L^l(x_i, x_k) = \frac{P_k^4}{Q_k^4}\left(\int \frac{Q_k^4 F(x_i)}{P_k^4}dx_i + G_k(x_k)\right),$$

$$L^k(x_i, x_l) = \frac{P_l^4}{Q_l^4}\left(\int \frac{Q_l^4 F(x_i)}{P_l^4}dx_i + \bar{G}_l(x_l)\right).$$

Substituting the last two expressions in (3.45) we get

$$W^j = \frac{1}{U_{ijkl}}\left\{\frac{P_k^4}{Q_k^4}\left(\int \frac{Q_k^4 F(x_i)}{P_k^4}dx_i + G_k(x_k)\right) - \frac{P_l^4}{Q_l^4}\left(\int \frac{Q_l^4 F(x_i)}{P_l^4}dx_i + \bar{G}_l(x_l)\right)\right\}.$$

Finally we obtain (3.41), using (2.8).

□

We can now prove our main result

Proof of Theorem 3.1

It follows from Lemmas 2.2, 3.1-3.3 that the hypersurface is given by (3.6), where W^k and W^j are given for (3.21) and (3.41), respectively.
Differentiating (3.21) with respect to x_i and using (3.39), we obtain

$$\begin{aligned} W^k_{,i} = & -\frac{m_{jik}}{Q_j^4}\left(A - \frac{m_{jik,i}}{m_{jik}}\right)\left(\int \frac{Q_j^4 G_i(x_i)}{P_j^4}dx_i + G_j(x_j)\right) \\ & + \frac{m_{kil}}{Q_l^4}\left(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}}\right)\left(\int \frac{Q_l^4 G_i(x_i)}{P_l^4}dx_i + G_l(x_l)\right). \end{aligned} \tag{3.58}$$

Differentiating (3.39) with respect to x_i, we get

$$\begin{aligned} W^j_{,i} = & -\frac{m_{jik}}{Q_k^4}\left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}}\right)\left(\int \frac{Q_k^4 F(x_i)}{P_k^4}dx_i + G_k(x_k)\right) \\ & + \frac{m_{jil}}{Q_l^4}\left(A + m_{jil} - \frac{m_{jil,i}}{m_{jil}}\right)\left(\int \frac{Q_l^4 F(x_i)}{P_l^4}dx_i + \bar{G}_l(x_l)\right). \end{aligned} \tag{3.59}$$

The substitution of (3.21), (3.41), (3.58) and (3.59) into equation (3.18), gives

$$\frac{m_{kil}}{Q_l^4}\left(m_{jil}+\left(\log\left(\frac{m_{jik}}{m_{kil}}\right)\right)_{,i}\right)\left(\int\frac{Q_l^4 G_i(x_i)}{P_l^4}dx_i+G_l(x_l)\right)=$$
$$=\frac{m_{jil}}{Q_l^4}\left(m_{jil}-m_{jik}+\left(\log\left(\frac{m_{jik}}{m_{jil}}\right)\right)_{,i}\right)\left(\int\frac{Q_l^4 F(x_i)}{P_l^4}dx_i+\bar{G}_l(x_l)\right).$$

From (2.6) and (2.7), we get

$$m_{kil}T_{ijkl}\left(\int\frac{Q_l^4 G_i(x_i)}{P_l^4}dx_i+G_l(x_l)\right)=m_{jil}U_{ijkl}\left(\int\frac{Q_l^4 F(x_i)}{P_l^4}dx_i+\bar{G}_l(x_l)\right),$$

and it follows from (3.39), that

$$\int\frac{Q_l^4 G_i(x_i)}{P_l^4}dx_i+G_l(x_l)=\int\frac{Q_l^4 F(x_i)}{P_l^4}dx_i+\bar{G}_l(x_l). \tag{3.60}$$

Differentiating (3.60) with respect to x_i, we get

$$G_i(x_i)=F(x_i),$$

and therefore

$$G_l(x_l)=\bar{G}_l(x_l).$$

The substitution of these two equalities in (3.21), (3.41) and in (3.17), gives

$$\overline{X}=\frac{1}{m_{jik}}\left(m_{jik}B_j^4-m_{kil}B_l^4-m_{jik}B_k^4+m_{jil}B_l^4\right),$$

where we have used (3.2).
It follows from the fourth equation of (2.5) that

$$\overline{X}=B_j^4-B_k^4+B_l^4,$$

which substituted into (3.10), implies (3.1).

Considering α^i and $\alpha^s,\ s=j,k,l$ defined by (3.4), it follows from (2.12)-(2.14), (2.18) and (2.4) that

$$X_{,r} \quad = \quad V\alpha^r\ ,\ r=i,j,k,l. \tag{3.61}$$

Differentiating (3.61), we have

$$X_{,rr}=V_{,r}\alpha^r+V\alpha^r_{,r}\ ,\ r=i,j,k,l. \tag{3.62}$$

From (3.61) we obtain that the metric of $X_{,r}$ is given by

$$g_{rr}=(V)^2|\alpha^r|^2\ \ ,\ g_{rt}=0\ ,\ r\neq t. \tag{3.63}$$

A unit vector field normal to X is given by

$$N=\frac{\alpha^i\times\alpha^j\times\alpha^k\times\alpha^l}{|\alpha^i|\,|\alpha^j|\,|\alpha^k|\,|\alpha^l|}. \tag{3.64}$$

Since X is a hypersurface parametrized by orthogonal curvature lines, with λ_s, as principal curvature we have, for $1\leq r\neq s\leq 4$

$$\langle N,X_{,rs}\rangle=0,\qquad \lambda_s=\frac{\langle X_{,rr},N\rangle}{g_{rr}}\ .$$

Hence from (3.62) and (3.64) we obtain for $r=i,j,k,l$,

$$\lambda_r=\frac{\langle\alpha^r_{,r}\ ,\alpha^i\times\alpha^j\times\alpha^k\times\alpha^l\rangle}{V|\alpha^r|^2|\alpha^i|\,|\alpha^j|\,|\alpha^k|\,|\alpha^l|}.$$

Therefore, we conclude that conditions a), b) and c) are satisfied.

Conversely, let λ_r be real functions distinct at each point. Assume that the functions m_{rts} and m_{rt}, defined by (3.5), satisfy (2.5) and suppose $G_r(x_r),\ 1 \le r \le 4$, are vector valued functions satisfying properties a), b) and c).

Defining X by (3.1), it follows from Lemma 2.1 and properties a) and b), that X is an immersion, whose coordinates curves are orthogonal. Moreover, the induced metric is given by (3.63) and a unit normal vector field by (3.64).

Differentiating (3.61) with respect to x_t, using Lemma 2.1, the expressions (2.5), (2.19) and (3.4) we obtain

$$X_{,rt} = V\left(\frac{\lambda_{r,t}}{\lambda_t-\lambda_r}\alpha^r + \frac{\lambda_{t,r}}{\lambda_r-\lambda_t}\alpha^t\right)\ ,\ r\neq t.$$

From (3.64), it follows that $\langle X_{,rt}\ , N\rangle = 0$. Hence the second fundamental form is diagonal and therefore the coordinates curves are lines of curvature.

Moreover, it follows from (3.62)-(3.64) and from property c) that for $r = i, j, k, l$,

$$\frac{\langle X_{,rr}, N\rangle}{g_{rr}} = \frac{\langle \alpha^r_{,r}\ , \alpha^i\times\alpha^j\times\alpha^k\times\alpha^l\rangle}{V|\alpha^r|^2|\alpha^i|\ |\alpha^j|\ |\alpha^k|\ |\alpha^l|} \quad = \quad \lambda_r$$

which concludes the proof.

□

4. Properties. In this section, we show that the vector valued functions which appear in Theorem 3.1 are invariant under inversions and homotheties.

Theorem 4.1. *Let $X : \Omega\subset\mathbb{R}^4 \to \mathbb{R}^5$ be a hypersurface with four distinct principal curvatures λ_r, parametrized by lines of curvature as in the Theorem 3.1. Then the vector valued functions $G_r(x_r), 1\le r\le 4$ are invariants under inversions and homotheties.*

Proof: a) Assuming without loss of generality that $0\notin X(\Omega)$, we consider $\tilde{X} = I^5(X)$ a hypersurface parametrized by lines of curvature, obtained by composing X with the inversion defined in (2.9).
Since, $\tilde{X}$ is a hypersurface parametrized by lines of curvature, with distinct principal curvatures given by (2.11).
Using Theorem 3.1 for $\tilde{X}$, we have for $i,\ j,\ k,\ l$ fixed distinct indices

$$\tilde{X} = \tilde{V}\left(\tilde{B}^4_j - \tilde{B}^4_k + \tilde{B}^4_l\right),$$

where

$$\begin{aligned}
\tilde{B}^4_s &= \frac{1}{\tilde{Q}^4_s}\left(\int\frac{\tilde{Q}^4_s\tilde{G}_i(x_i)}{\tilde{P}^4_s}dx_1 + \tilde{G}_s(x_s)\right)\ ,\ s\neq i\ ,\\
\tilde{V} &= \frac{e^{\displaystyle\int\frac{\tilde{\lambda}_k-\tilde{\lambda}_j}{\tilde{\lambda}_j-\tilde{\lambda}_i}\tilde{m}_{jki}dx_k}}{\tilde{\lambda}_j-\tilde{\lambda}_i}\ ,\\
\tilde{A} &= -\int\tilde{m}_{jki,i}dx_k\ ,
\end{aligned}\tag{4.1}$$

$\tilde{P}^4_s$, $\tilde{Q}^4_s$, $s\neq i$ are defined by (2.8) and (3.3), in terms of the higher-dimensional Laplace invariants and $\tilde{G}_r(x_r),\ 1\le r\le 4$ are vector valued functions in $\mathbb{R}^5$.

From preliminaries, $\tilde{X}$ and X have the same higher-dimensional Laplace invariants. Therefore, it follows that

$$\tilde{A} = A\ \ ,\ \ \tilde{Q}^4_r = Q^4_r\ \ ,\ \ \tilde{P}^4_r = P^4_r \neq 0\ ,\ r\neq i. \tag{4.2}$$

Substituting (2.11) in (4.1), we have

$$\tilde{V} = \frac{V}{\langle X, X\rangle}. \tag{4.3}$$

On the other hand,

$$\tilde{X} = \frac{X}{\langle X, X\rangle}. \tag{4.4}$$

We will show that $\tilde{G}_r(x_r) = G_r(x_r),\ r \neq i$. It follows from (4.3) and (4.4) that

$$\tilde{B}_j^4 - B_j^4 - (\tilde{B}_k^4 - B_k^4) + \tilde{B}_l^4 - B_l^4 = 0. \tag{4.5}$$

We observe that

$$B_{j,i}^4 = -AB_j^4 + \frac{G_i(x_i)}{P_j^4}, \qquad B_{s,i}^4 = -\left(A + m_{jis}\right) B_s^4 + \frac{G_i(x_i)}{P_s^4}\ ,\quad s = k, l.$$

This fact follows from the equalities

$$Q_{j,i}^4 = AQ_j^4 \quad , \quad Q_{s,i}^4 = \left(A + m_{jis}\right) Q_s^4\ ,\ s = k, l.$$

Therefore differentiating (4.5) with respect to x_i, we get

$$\begin{aligned}&-A\left(\tilde{B}_j^4 - B_j^4\right) + (A + m_{jik})\left(\tilde{B}_k^4 - B_k^4\right) - (A + m_{jil})\left(\tilde{B}_l^4 - B_l^4\right) + \\ &+(\tilde{G}_i - G_i)\left(\tfrac{1}{P_j^4} - \tfrac{1}{P_k^4} + \tfrac{1}{P_l^4}\right) = 0.\end{aligned}$$

Using (4.5) and the fact that

$$\frac{1}{P_j^4} - \frac{1}{P_k^4} + \frac{1}{P_l^4} = 0,$$

we get

$$m_{jik}\left(\tilde{B}_k^4 - B_k^4\right) - m_{jil}\left(\tilde{B}_l^4 - B_l^4\right) = 0. \tag{4.6}$$

Differentiating this relation with respect to x_i, we obtain

$$\begin{aligned}&(m_{jik,i} - m_{jik}(A + m_{jik}))\left(\tilde{B}_k^4 - B_k^4\right) - (m_{jil,i} - m_{jil}(A + m_{jil}))\left(\tilde{B}_l^4 - B_l^4\right) + \\ &+\ (\tilde{G}_i(x_i) - G_i(x_i))\left(\frac{m_{jik}}{P_k^4} - \frac{m_{jil}}{P_l^4}\right) = 0.\end{aligned}$$

It follows from (4.6) and from the fact that

$$\frac{m_{jik}}{P_k^4} - \frac{m_{jil}}{P_l^4} = 0,$$

that the expression above reduces to

$$\left(m_{jik,i} - (m_{jik})^2\right)\left(\tilde{B}_k^4 - B_k^4\right) - \left(m_{jil,i} - (m_{jil})^2\right)\left(\tilde{B}_l^4 - B_l^4\right) = 0.$$

Again, using (4.6), we obtain the relation

$$\left(m_{jil} - m_{jik} + \frac{m_{jik,i}}{m_{jik}} - \frac{m_{jil,i}}{m_{jil}}\right)\left(\tilde{B}_l^4 - B_l^4\right) = 0\ ,$$

which, as a consequence of (2.7), reduces to

$$U_{ijkl}\left(\tilde{B}_l^4 - B_l^4\right) = 0.$$

Since $U_{ijkl} \neq 0$, we obtain

$$\tilde{B}_l^4 = B_l^4. \tag{4.7}$$

Differentiating with respect to x_i, we get $\tilde{G}_i(x_i) = G_i(x_i)$, hence it follows that $\tilde{G}_l(x_l) = G_l(x_l)$. From (4.6) and (4.7), we have

$$\tilde{B}_k^4 = B_k^4, \tag{4.8}$$

and therefore $\tilde{G}_k(x_k) = G_k(x_k)$.
Substituting (4.7) and (4.8) into (4.5), we obtain

$$\tilde{B}_j^4 = B_j^4.$$

and hence $\tilde{G}_j(x_j) = G_j(x_j)$, which concludes the proof of a).

b) Let $\overline{X} = aX$ be a homothety of X, since $\overline{X}$ is a hypersurface parametrized by orthogonal curvature lines, with distinct principal curvatures given by (2.11).
Using Theorem 3.1 for $\overline{X}$, we have for $i,\ j,\ k,\ l$ distinct fixed indices

$$\overline{X} = \bar{V}\left(\bar{B}_j^4 - \bar{B}_k^4 + \bar{B}_l^4\right).$$

where

$$\begin{aligned}
\bar{B}_s^4 &= \frac{1}{\bar{Q}_s^4}\left(\int \frac{\bar{Q}_s^4 \bar{G}_i(x_i)}{\bar{P}_s^4} dx_i + \bar{G}_s(x_s)\right), \ s \neq i, \\
\bar{V} &= \frac{e^{\displaystyle\int \frac{\bar{\lambda}_k - \bar{\lambda}_j}{\bar{\lambda}_j - \bar{\lambda}_i} \bar{m}_{jki} dx_k}}{\bar{\lambda}_j - \bar{\lambda}_i}, \\
\bar{A} &= -\int \bar{m}_{jki,i} dx_k,
\end{aligned} \tag{4.9}$$

$\bar{P}_s^4, \bar{Q}_s^4,\ s \neq i$ are defined by (2.8) and (3.3) in terms of the higher-dimensional Laplace invariants and $\bar{G}_r(x_r),\ 1 \leq r \leq 4$ are vector valued functions in $\mathbb{R}^5$.

We will show that $\bar{G}_r(x_r) = G_r(x_r)$.

From preliminaries, $\overline{X}$ and X have the same Laplace invariants. Therefore, it follows that

$$\bar{A} = A \qquad \bar{Q}_s^4 = Q_s^4 \qquad \bar{P}_s^4 = P_s^4 \neq 0,\ s \neq i. \tag{4.10}$$

Substituting (2.11) in (4.9), we obtain

$$\bar{V} = aV. \tag{4.11}$$

Since

$$X = V\left(B_j^4 - B_k^4 + B_l^4\right), \quad \overline{X} = \bar{V}\left(\bar{B}_j^4 - \bar{B}_k^4 + \bar{B}_l^4\right). \tag{4.12}$$

substituting the expressions (4.10), (4.11), (4.12) in $\overline{X} = aX$ we have,

$$\bar{B}_j^4 - B_j^4 - (\bar{B}_k^4 - B_k^4) + \bar{B}_l^4 - B_l^4 = 0.$$

The same argument of item a) proves that $\bar{G}_r(x_r) = G_r(x_r), \forall\, r$.

□

4.1. Conclusion. This class of the hypersurfaces parametrized by lines of curvature studied in this paper includes the hypersurfaces of Dupin studied in [10]. This result shows that a hypersurface parameterized by lines of curvature does not need to have all Laplace invariants $m_{ij} = 0, 1 \leq i \neq j \leq 4$ to have a representation in terms of four vector functions of one variable. From the result obtained in [11] this class of hypersurfaces cannot have constant Möbius curvature.

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