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Curves with Prescribed Curvature and Associated Surfaces

Curvas con curvatura prescrita y superficies asociadas

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Abstract
In this work, we study surfaces with a canonical principal direction (CPD surfaces) under the simultaneous prescription of two extrinsic geometric invariants: one principal curvature and the support function. Based on these prescriptions, we reformulate the first and second fundamental forms, which enables us to construct explicit examples of such surfaces. Finally, we show that when the curvature of the profile curve is constant, that is, when one of the principal curvatures is constant, the corresponding CPD surface is a tubular surface.
Keywords . CPD surfaces, prescribed curvature, catenaries, principal direction.

Resumen
En este trabajo se estudian las superficies con una dirección principal canónica (superficies CPD) bajo la prescripción simultánea de dos invariantes geométricos extrínsecos: una curvatura principal y la función soporte. A partir de dichas prescripciones, se reescriben las primeras y segundas formas fundamentales, lo que permite presentar ejemplos explícitos de este tipo de superficies. Finalmente, se demuestra que, cuando la curvatura de la curva de perfil es constante, es decir, cuando una de las curvaturas principales es constante, la superficie CPD resulta ser una superficie tubular.
Palabras clave. Superficies CPD, curvatura prescrita, catenarias, dirección principal.

1. Introduction. The study of prescribed curvature for planar curves is both fundamental and powerful: given a sufficiently regular curvature function k, the Fundamental Theorem of Plane Curves ensures the existence and uniqueness (up to isometries) of a curve realizing it. Expressing the curvature in terms of the tangent angle reduces the problem to a first-order separable ODE. This standard viewpoint supports many explicit families of curves, including the classical catenary (see [1]). We adopt this approach to derive explicit formulas and global criteria for curvature prescriptions relevant to our constructions.

A canonical way to associate surfaces with planar curves is through surfaces of revolution, since the geometry of a surface of revolution is governed by the geometry of its generating curve. Classical results include the theorems of Euler, Delaunay, and Darboux, which classify surfaces of revolution with zero mean curvature (minimal surfaces of revolution), constant mean curvature, and constant Gaussian curvature, see [2, 3, 4]. More recently, in [5], the authors studied surfaces of revolution with prescribed curvatures (principal, mean, or Gaussian) as functions of the distance to the axis of revolution.

In another direction, in [6], the authors classified the surfaces with a canonical principal direction (CPD surfaces) in $\mathbb{R}^3$, providing canonical coordinates and explicit expressions for the shape operator. The

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geometry of these CPD surfaces is governed by two curves, one directrix and one profile curve, note that the surfaces of revolution appear as particular cases of CPD surfaces.

In [7], the authors extended the concept of CPD surfaces to hypersurfaces, presenting different ways of constructing hypersurfaces that possess a canonical principal direction relative to a vector field X. The authors proved that CPD surfaces with constant mean curvature are Delaunay surfaces.

In [8], the authors classified all CPD surfaces by prescribing the mean curvature as an affine function of the height function, the angle function, or the support function.

Motivated by [5, 8, 7], we introduce the notion of generalized catenaries, a family of planar curves defined by a special curvature prescription, we provide three families of this curves, two of them depending on three parameters and one on two parameters where when one of the parameters (common to these three families) is zero, the curve reduces to the classical catenary. Finally, we provide families of four-parameter CPD surfaces obtained from the prescription of a principal curvature, and of the support function of the directrix curve. Moreover, we prove that, under a certain prescription of a principal curvature, CPD surfaces are tubular surfaces.

2. Preliminary. This section contains the definitions and the basic theory of curves. Next, we provide the classical definition of the catenary

Definition 2.1. *A **catenary** is the curve described by a perfectly flexible and homogeneous chain suspended by its endpoints and subject only to its own weight (gravity) [9].*

In Cartesian coordinates, its equation is given by

$$y(x) = a\cosh\left(\frac{x}{a}\right),$$

where $a > 0$ is a real constant (related to the tension of the chain).

Given a plane curve $\beta(t) = (x(t), y(t))$, the tangent vector and the unit normal vector are given, respectively, by

$$\beta'(t) = (x'(t), y'(t)) \quad \text{and} \quad n(t) = \frac{(-y'(t),\, x'(t))}{|\beta'(t)|}.$$

Under these conditions, we have

$$n'(t) = \lambda(t)\,\beta'(t), \quad \text{where} \quad k_\beta = -\lambda(t).$$

Definition 2.2. *Given a plane curve $\mathcal{C}$, let $\gamma : I \to \mathbb{R}^2$ be a parametrization of $\mathcal{C}$ such that the unit normal vector is n. The support function of γ is given*

$$f(u) = \langle \gamma(u), n(u) \rangle.$$

Proposition 2.1. *Let $n(\theta) = (\cos\theta, \sin\theta)$ be a unit normal field. Then, up to plane isometries, the curve $\gamma(\theta)$ such that $n(\theta)$ is its unit normal vector is given by*

$$\gamma(\theta) = \big(f(\theta)\cos\theta - f'(\theta)\sin\theta,\ f(\theta)\sin\theta + f'(\theta)\cos\theta\big),$$

where $f(\theta) = \langle n(\theta), \beta(\theta)\rangle$ is the support function. Moreover, the curvature of $\beta(\theta)$ is given by

$$k(\theta) = \frac{1}{f(\theta) + f''(\theta)}.$$

Definition 2.3 (CPD Surface). *Let $M^2 \subset \mathbb{R}^3$ be a regular surface with principal curvatures k_1, k_2. We say that M is a* Canonical Principal Direction surface *(CPD surface) if there exists a fixed nonzero vector $v \in \mathbb{R}^3$ such that, for every point $p \in M$, the tangential projection $v^T(p)$ of v onto the tangent plane T_pM is a principal direction of M at p.*

3. Curves in $\mathbb{R}^2$ with Prescribed Curvature. In this section, we characterize families of curves of graph type $y = h(x)$ whose curvature is prescribed.

Let $\alpha(t) = (t, h(t), 0)$ be a curve in the plane such that its curvature $k_\alpha = -G'(h)$, where G is a differentiable function. Hence,

$$\frac{h''}{(1+h'^2)^{3/2}} = -G'(h), \quad \text{i.e.} \quad \frac{1}{\sqrt{1+h'^2}} = G(h) + C.$$

Note that, if φ is the angle of inclination of the tangent line to the graph of $y = h(x)$ (with $\tan\varphi = h'$), then the above ODE becomes

$$\cos\varphi = G(h) + C.$$

Set $u(h) = G(h) + C$ with $0 < u(h) \le 1$. From the previous equation,

$$h' = \pm\frac{\sqrt{1-u^2}}{u}, \quad t - t_0^{\pm} = \pm\int \frac{u}{\sqrt{1-u^2}}dh. \tag{3.1}$$

Remark 3.1. *Consider* $\Phi(h) = \int \frac{u}{\sqrt{1-u^2}}dh$*, where* $u(h) = G(h) + C$ *with* $0 < u(h) \le 1$*. As the two orientations (*$h' > 0$ *and* $h' < 0$*) solve (3.1), it follows that*

$$t - t_0^+ = +\Phi(h) \quad (h' > 0), \qquad t - t_0^- = -\Phi(h) \quad (h' < 0),$$

Hence the graphs of the branches $h' > 0$ *and* $h' < 0$ *are mirror images with respect to the vertical line*

$$t = t^*, \qquad t^* = \frac{t_0^+ + t_0^-}{2}.$$

For arbitrary integration constants, the two branches glue at a point (t^*, h^*) *with* $u(h^*) = 1$ *if and only if*

$$t_0^- - t_0^+ = 2\,\Phi(h^*), \qquad t^* = t_0^+ + \Phi(h^*) = t_0^- - \Phi(h^*).$$

Without loss of generality, suppose $t_0^- = t_0^+ = t_0$*. If there is* $h^* = \lim_{t\to t^*} h(t) > 0$*, such that* $h'(t^*) = 0$ *with finite* $\Phi(h^*)$*, then then after a horizontal translation of one branch (equivalently, by recentering* Φ *so that* $\Phi(h^*) = 0$*), the two solutions glue on the symmetry axis, i.e. the gluing occurs at* $t = t_0$*.*

Remark 3.2. *We provide some solutions* h *for (3.1) after choosing* $u(h)$*.*

Case 1. *If* $u(h) = ah + b$ *such that* $0 < ah + b \le 1$*, with* $a \ne 0$ *and* $b \in \mathbf{R}$*, then the solution set of (3.1) is the circle centered at* $(t_0, -b/a)$ *with radius* $1/|a|$*. As graphs* $h = h(t)$*, there are exactly two branches (the upper and lower semicircles). Each branch passes through* $t = t_0$ *with horizontal tangent (*$h'(t_0) = 0$*), and* h' *changes sign when crossing* t_0*. Explicitly*

$$h_{\pm}(t) = -\frac{b}{a} \pm \sqrt{\frac{1}{a^2} - (t-t_0)^2}, \qquad t \in \left[t_0 - \tfrac{1}{|a|},\, t_0 + \tfrac{1}{|a|}\right]. \tag{3.2}$$

Case 2. *If* $u(h) = \sin(ah + b)$ *with* $a \ne 0$ *and* $b \in \mathbf{R}$*, then the solutions of (3.1) are graphs on the half line* $a(t - t_0) \ge 0$*, implicitly determined by*

$$|\cos(ah + b)| = e^{-a(t-t_0)}.$$

Equivalently, for $\varepsilon^2 = 1$ *and* $k \in \mathbb{Z}$*,*

$$h_{\varepsilon,k}(t) = \frac{1}{a}\Big(\varepsilon\,\arccos\big(e^{-a(t-t_0)}\big) - b + 2k\pi\Big), \qquad a(t - t_0) \ge 0. \tag{3.3}$$

Along each branch,

$$u(t) = \sin\big(a\,h_{\varepsilon,k}(t) + b\big) = \varepsilon\sqrt{1 - e^{-2a(t-t_0)}}\,,$$

$|h'(t)| \to \infty$ *as* $t \to t_0^{\pm}$ *(vertical tangent at* $t = t_0$*), and* $h'(t) \to 0$ *as* $t \to \operatorname{sgn}(a)\,\infty$*. Thus there is no finite gluing point (the limit* $u \to 1$ *occurs only as* $|t| \to \infty$*).*

Case 3. *If* $u(h) = \tanh(ah + b)$ *such that* $0 < \tanh(ah + b) \le 1$ *with* $a \ne 0$*, then the solutions of (3.1) are graphs on the ray* $a(t - t_0) \ge 1$*, implicitly given by*

$$\cosh(ah + b) = a\,(t - t_0).$$

Equivalently, for $\varepsilon^2 = 1$*,*

$$h_{\varepsilon}(t) = \frac{\varepsilon}{a}\,\ln\Big(a(t-t_0) + \sqrt{a^2(t-t_0)^2 - 1}\Big) - \frac{b}{a}, \qquad a(t - t_0) \ge 1. \tag{3.4}$$

Along each branch,

$$u(t) = \tanh\big(a\,h_{\varepsilon}(t) + b\big) = \varepsilon\sqrt{1 - \frac{1}{a^2(t-t_0)^2}}\,, \qquad h_{\varepsilon}'(t) = \frac{\varepsilon}{\sqrt{a^2(t-t_0)^2 - 1}}.$$

Thus $|h'(t)| \to \infty$ *at* $a(t - t_0) = 1$ *(vertical tangent) and* $h'(t) \to 0$ *as* $a(t - t_0) \to \infty$*; there is no finite gluing point.*

Case 4. *Let* $G(h) = \dfrac{C_1}{h}$*, so that* $u(h) = \dfrac{C_1}{h} + C$ *with* $|u(h)| < 1$*. From Remark 3.1, in this case, we have gluing at the point* (t_0, h^*) *if and only if* $h^* = \frac{C_1}{1-C}$ *(when defined). At points where* $u = 0$ *(equivalently* $h = -C_1/C$*, when defined) one has* $|h'| \to \infty$ *(vertical tangent), so no gluing occurs there. Therefore, we shall evaluate the solutions of (3.1) in the following subcases, where we consider* $Q(h) = (1 - C^2)h^2 - 2CC_1h - C_1^2$*.*

a) If $C = 0$ *and* $C_1 \neq 0$*, the solutions of (3.1) are given by the catenaries*

$$h(t) = |C_1| \cosh\left(\frac{t - t_0}{|C_1|}\right), \qquad t \in \mathbb{R}. \tag{3.5}$$

These solutions are globally defined for all $t \in \mathbf{R}$.

b) For $C_1 > 0$*, we have two subcases:*

(i) For $|C| < 1$*, we can have both bounded and unbounded solutions* h*, because* $h \geq \dfrac{C_1}{1-C}$ *if* $0 \leq C < 1$ *and* $\dfrac{C_1}{1-C} \leq h < \dfrac{-C_1}{C}$ *if* $-1 < C < 0$*. In this case,* h *is given implicitly by the equation*

$$t - t_0 = \frac{\varepsilon C}{1 - C^2}\sqrt{Q(h)} + \frac{\varepsilon C_1}{\sqrt{(1 - C^2)^3}} \ln\left(\sqrt{1 - C^2}\sqrt{Q(h)} + (1 - C^2)h - CC_1\right), \tag{3.6}$$

where $\varepsilon^2 = 1$ *and the two orientation branches glue at* $t = t_0$ *at* $h = h^*$*.*

(ii) If $C \leq -1$ *we have bounded solution* h*,* $\dfrac{C_1}{1-C} \leq h < \dfrac{-C_1}{C}$*. In this case, if* $C < -1$*,* h *is given implicitly by the equation*

$$t - t_0 = \frac{\varepsilon C}{1 - C^2}\sqrt{Q(h)} - \frac{\varepsilon C_1}{\sqrt{(C^2 - 1)^3}} \arcsin\left(\frac{(C^2 - 1)h}{C_1} + C\right). \tag{3.7}$$

If $C = -1$*,* h *is given by*

$$h = \frac{\varepsilon C_1}{2} + 2\varepsilon C_1 \cos^2\left(\frac{1}{3}\arccos\left(\frac{-3(t - t_0)}{C_1}\right)\right), \quad |t - t_0| \leq \frac{C_1}{3}. \tag{3.8}$$

where $\varepsilon^2 = 1$ *and the two orientation branches glue at* $t = t_0$ *at* $h = h^*$*.*

c) For $C_1 < 0$ *we have two subcases:*

i) Case $0 < C < 1$*, we have unbounded solutions* h*,* $h > \dfrac{-C_1}{C}$*. In this case,* h *is given by (3.6).*

The solutions of this case do not glue. Indeed, the gluing would occur at t^* *such that* $h'(t^*) = 0$*, equivalently* $h^* = \dfrac{C_1}{1-C}$*. However, since* $C_1 < 0$ *and* $0 < C < 1$*, we have* $h^* < 0$*, contradicting the assumption* $h > 0$*. Thus, no gluing occurs.*

ii) Case $C > 1$*, we have bounded solutions* h*,* $\dfrac{-C_1}{C} < h < \dfrac{-C_1}{C-1}$*. In this case,* h *is given by (3.7), where* $\varepsilon^2 = 1$*. In this case, the solutions glue at* $u(h^*) = 1$*.*

4. Generalized Catenaries.. In this section, using a geometric invariant that allows us to construct plane curves with prescribed curvature, we make a choice in the prescription of the curvature such that all the resulting curves are concave upwards, with a unique global minimum point. These curves are called *Generalized Catenaries*.

Definition 4.1. *We call a* Generalized Catenary *with parameters* C *and* $C_1 > 0$ *any curve*

$$\delta_C(t) = (t, h_C(t)),$$

such that h *satisfies*

$$\frac{1}{\sqrt{1 + (h_C')^2}} = \frac{C_1}{h} + C.$$

i) If $C = 0$*, then* δ_0 *is a horizontal translate of* $(t, \cosh t)$*, i.e., the classical catenary. Its revolution about the* x*-axis yields the catenoid, the unique minimal surface of revolution.*

ii) If $C \neq 0$*, then* δ_C *is still a Generalized Catenary, but its revolution about the* x*-axis does not produce a minimal surface.*

Proposition 4.1. *Let $\alpha(t) = (t, h(t))$ be a curve with $h > 0$ satisfying (3.1). Then the mean and Gaussian curvatures of the surface obtained by revolving α around the x-axis are given by*

$$H = \frac{hG'(h) + G(h) + C}{2h}, \qquad K = \frac{G'(h)\big(G(h) + C\big)}{h}. \tag{4.1}$$

Proof: Let $X(t, \theta) = (t,\ h\cos\theta,\ h\sin\theta)$ be the surface obtained by revolving α around the x-axis. Then the mean curvatures of X are given by

$$\kappa_1 = \frac{-h''}{\big(1 + (h')^2\big)^{\frac{3}{2}}}, \qquad \kappa_2 = \frac{1}{h\big(1 + (h')^2\big)^{\frac{1}{2}}}.$$

By the definition of a Generalized Catenary, we obtain

$$\kappa_1 = G'(h), \qquad \kappa_2 = \frac{G(h) + C}{h}.$$

Thus, the mean curvature $H = \dfrac{\kappa_1 + \kappa_2}{2}$ and Gaussian $k = \kappa_1\kappa_2$, are given by (4.2). □

Corollary 4.1. *Under the assumptions of Proposition 4.1, if α is a Generalized Catenary, then*

$$H = \frac{C}{2h}, \qquad K = \frac{C_1\big(1 + Ch\big)}{h^4}. \tag{4.2}$$

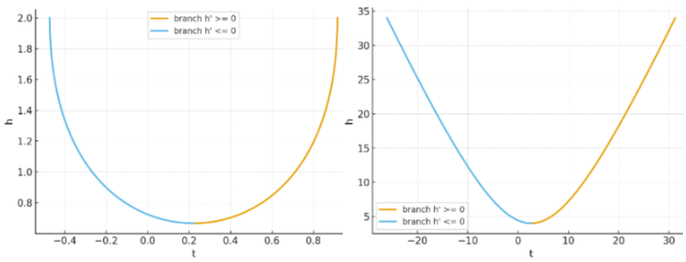

Figure 4.1: In these figures, we have the curve $\alpha(t) = (t, h(t)$ given by (3.6) and (3.7). The first figure, we have $C_1 = 1$, $C = \frac{-1}{2}$ and $t_0 = 0$ and in second, we have $C_1 = 2$, $C = \frac{1}{2}$ and $t_0 = 0$.

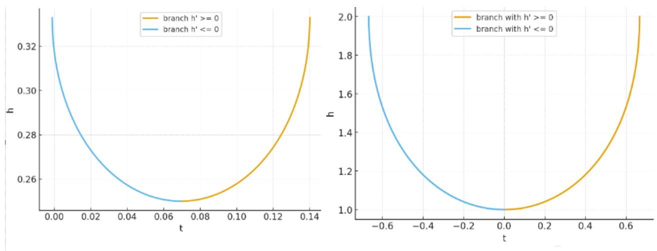

Figure 4.2: In these figures, we have the curve $\alpha(t) = (t, h(t)$ given by (3.7) and (3.8). The first figure, we have $C_1 = 1$, $C = -3$ and $t_0 = 0$ and in second, we have $C_1 = 2$, $C = -1$ and $t_0 = 0$.

Remark 4.1. *From the solutions given in Case 4, item (b), that is, the curves $\alpha(t) = (t, h(t))$ with curvature $k = \frac{C_1}{h^2}$, where $C_1 > 0$, we obtain two families of generalized catenaries depending on three parameters and one family depending on two parameters.*

5. Associated Surfaces with Prescribed Principal Curvature. In this section, we consider surfaces with a canonical principal direction (CPD surfaces) defined by the prescription of a principal curvature.

Given a plane curve $\mathcal{C}$ parametrized by $\gamma(u) = (x(u), y(u), 0)$ with unit normal vector $n(u)$, for every graph type curve $\alpha(v) = (h(v), v)$, the CPD surface M can be parametrized, up to an isometry, by

$$X(u, v) = \gamma(u) + h(v)n(u) + ve_3. \tag{5.1}$$

Remark 5.1.

We say that this surface is a CPD surface with a fixed vector e_3. The curve γ is named the directrix of M, and the curve $\alpha(v) = (h(v), v)$ is the profile curve of M. Moreover, the function $h = h(v)$ controls the distance between the directrix curve γ and the curve shifted along the normal of γ. If $h(v) = 0$, then M is a cylinder, and if $h(v) \neq 0$, at each level v the curve γ is translated in the normal direction by an amount $h(v)$.

If $\mathcal{C}$ is a circle, then the CPD surface is a surface of revolution obtained by rotating the curve $(1 + h(v), 0, v)$ around the z-axis.

The unit normal vector of this CPD surface is given by

$$N = \frac{1}{\sqrt{1+h'^2}}\big(h'e_3 - n\big).$$

The first and second fundamental forms of X are given by

$$I = |\gamma'(u)|^2(1-k_\gamma h)^2du^2 + (1+h'^2)dv^2, \tag{5.2}$$

$$II = \frac{-|\gamma'(u)|^2(1-k_\gamma h)k_\gamma}{\sqrt{1+h'^2}}du^2 - \frac{h''}{\sqrt{1+h'^2}}dv^2. \tag{5.3}$$

Thus, the principal curvatures are given by

$$\kappa_1 = \frac{-k_\gamma}{(1-k_\gamma h)\sqrt{1+h'^2}}, \quad \text{and} \quad \kappa_2 = \frac{-h''}{(1+h'^2)^{3/2}}. \tag{5.4}$$

Theorem 5.1. *Let M be a CPD surface parametrized as (5.1). For each smooth function $G : \mathbb{R} \to \mathbb{R}$, consider*

$$\kappa_2(h) = G'(h), \qquad \kappa_1 = \frac{G(h)+C}{L(u)+h(v)}, \tag{5.5}$$

where h is a smooth function, satisfying $\dfrac{1}{\sqrt{1+h'^2}} = G(h(v)) + C$, $0 < G(h(v)) + C < 1$ and $L = L(u)$ be a nonvanishing smooth function and $C \in \mathbb{R}$ is constant. Then there exists a curve γ such that, up to rigid motions, the principal curvatures of the CPD surface (5.1) are given by k_1 and k_2.

Proof: By the Fundamental Theorem of Plane Curves, there exists a curve γ such that its curvature is $k_\gamma = \frac{-1}{L(u)}$. Denote by n the normal vector of this curve and consider the CPD surface given by (5.1). Then the first and second fundamental forms of X are given by (5.2) and (5.3). Moreover, the principal curvatures are given by (5.4).

For each smooth function G, suppose that $\dfrac{1}{\sqrt{1+h'^2}} = G(h(v)) + C$ where $h = h(v)$ is a smooth function. Thus, differentiating with respect to v, we obtain

$$\frac{-h'h''}{(1+h'^2)^{3/2}} = G'(h)h'.$$

Since h is nonconstant, we have $G'(h) = \dfrac{-h''}{(1+h'^2)^{3/2}}$, where from (5.4), κ_2 is a principal curvature of (5.1). From the previous equations, as in (3.1) we have

$$t - t_0 = \pm \int \frac{G(h)+C}{\sqrt{1-(G(h)+C)^2}}dh. \tag{5.6}$$

Since G is a smooth function, there exists a primitive $\Phi(h) = \pm \displaystyle\int \frac{G(h)+C}{\sqrt{1-(G(h)+C)^2}}dh$, and the functions h such that $\kappa_2 = G'(h)$ are given implicitly by equation (5.6).

Furthermore, as $\dfrac{1}{\sqrt{1+h'^2}} = G(h(v)) + C$ and $L(u) = \frac{-1}{k_\gamma}$, we have

$$\kappa_1 = \frac{G(h)+C}{L(u)+h(v)} = \frac{-k_\gamma}{(1-k_\gamma h)\sqrt{1+h'^2}}.$$

Therefore, from (5.4), we have that κ_1 is the other principal curvature.

Moreover, the first and second fundamental forms of X can be rewritten as

$$\begin{aligned} I &= \frac{(L+h)^2}{L^2}\left(|\gamma'(u)|du\right)^2 + \frac{1}{(G(h)+C)^2}dv^2, \\ II &= \frac{L+h}{L^2}(G(h)+C)\left(|\gamma'(u)|du\right)^2 + \frac{G'(h)}{(G(h)+C)^2}dv^2. \end{aligned}$$

Therefore, for each given curve γ, parametrized by arc length, whose curvature is $k_\gamma = \frac{-1}{L(u)}$, by the Fundamental Theorem of Surfaces, up to isometries, the CPD surface whose principal curvatures are given by (5.5) is the surface (5.1). □

Remark 5.2. *Let M be a CPD surface parametrized by (5.1). Considering the support function of γ, $f(u) = \langle \gamma(u), n(u)\rangle$, then by Proposition 2.1 the curve γ can be parametrized by*

$$\gamma(u) = (f(u)\cos u - f'(u)\sin u,\ f(u)\sin u + f'(u)\cos u,\ 0) \tag{5.7}$$

with unit normal vector given by $n(u) = (\cos u, \sin u, 0)$. Moreover, the curvature of γ is given in terms of the support function by

$$k_\gamma = \frac{1}{f''(u) + f(u)}.$$

Therefore, as in Theorem 5.1, we can prescribe the support function of γ and the principal curvature κ_2, and characterize the CPD surfaces with these prescriptions.

Theorem 5.2. *Let $G : \mathbb{R} \to \mathbb{R}$ be a smooth function and let $A, B, C \in \mathbb{R}$ be constants. Suppose that h and f are smooth functions such that*

$$k_1(u,v) = \frac{G(h(v)) + C}{h(v) + Af(u) + B}, \qquad k_2(u,v) = G'(h(v)).$$

Then the CPD surface $X(u,v) = \gamma(u) + h(v)n(u) + ve_3$, has principal curvatures k_1, k_2 if and only if

$$\frac{1}{\sqrt{1+h'(v)^2}} = G(h(v)) + C,$$

and the support function f of the γ is given by one of the following cases:

If $A = 1$, then

$$f(u) = \tfrac{B}{2}u^2 + B_1 u + B_2. \tag{5.8}$$

If $A > 1$, then

$$f(u) = A_1 \cosh(\sqrt{A-1}\,u) + A_2 \sinh(\sqrt{A-1}\,u) + \tfrac{B}{1-A}. \tag{5.9}$$

If $A < 1$, then

$$f(u) = A_1 \sin(\sqrt{1-A}\,u + A_2) + \tfrac{B}{1-A}. \tag{5.10}$$

Proof: Let $X(u,v) = \gamma(u) + h(v)n(u) + ve_3$ be a CPD surface. Since f is the support function of γ, it follows from (5.7) that $\gamma(u) = (f(u)\cos u - f'(u)\sin u,\ f(u)\sin u + f'(u)\cos u,\ 0)$ and $n(u) = (\cos u, \sin u, 0)$. By Theorem (5.1), we have that

$$f'' + f = Af + B.$$

Therefore, the proof is concluded.

Remark 5.3. *Under the assumptions of the last theorem, the cases $A = 1$, $A < 1$ and $A > 1$ represent, geometrically, the parabolic, elliptic, and hyperbolic cases, respectively, of the second order linear differential equation $f'' + f = Af + B$.*

Next, using the Remark 3.2, we provide some examples of families of CPD surfaces obtained by prescribing principal curvatures.

Proposition 5.1. *Under the conditions of Theorem 5.2. For every support function f of γ.*

1. *If $G(h) + C = ah + b$, $a \neq 0$ and $b \in \mathbf{R}$ are constants, then (5.1) is a four-parameter family of CPD surfaces whose principal curvatures are prescribed by*

$$k_1 = \frac{ah+b}{Af+B+h}, \qquad k_2 = a \neq 0,$$

where h is given by (3.2) and f is given by (5.8), (5.9) and (5.10).

2. *If* $G(h)+C=\sin(ah+b)$, $a\neq 0$ *and* $b\in\mathbf{R}$ *are constants, then (5.1) is a four-parameter family of CPD surfaces whose principal curvatures are prescribed by*

$$k_1=\frac{\sin(ah+b)}{Af+B+h},\qquad k_2=a\cos(ah+b),$$

where h *is given by (3.3) and* f *is given by (5.8), (5.9) and (5.10).*

3. *If* $G(h)+C=\tanh(ah+b)$, $a\neq 0$ *and* $b\in\mathbf{R}$ *are constants, then (5.1) is a four-parameter family of CPD surfaces whose principal curvatures are prescribed by*

$$k_1=\frac{\tanh(ah+b)}{Af+B+h},\qquad k_2=\frac{a}{\cosh^2(ah+b)},$$

where h *is given by (3.4) and* f *is given by (5.8), (5.9) and (5.10).*

4. *If* $G(h)=\dfrac{C_1}{h}$, $C_1\in\mathbf{R}$ *is a constant, then (5.1) is a four-parameter family of CPD surfaces whose principal curvatures are prescribed by*

$$k_1=\frac{Ch+C_1}{h(Af+B+h)},\qquad k_2=\frac{-C_1}{h^2},$$

where
   - *a)* h *is given by (3.5), if* $C=0$ *and* $C_1\neq 0$,
   - *b)* h *is given by (3.6), if* $|C|<1$ *and* $C_1>0$,
   - *c)* h *is given by (3.7), if* $C<-1$ *and* $C_1>0$,
   - *d)* h *is given by (3.8), if* $C=-1$ *and* $C_1>0$,
   - *e)* h *is given by (3.6), if* $0<C<1$ *and* $C_1<0$,
   - *e)* h *is given by (3.7), if* $C>1$ *and* $C_1<0$,

and f *is given by (5.8), (5.9) and (5.10).*

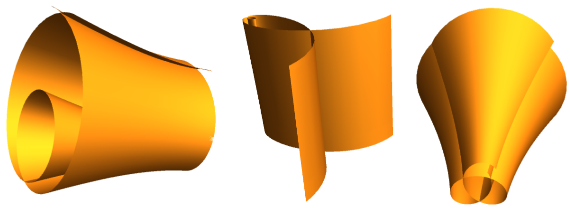

Figure 5.1: In these figures, we have CPD surfaces given by Theorem 5.2 with $G(h)=\frac{C_1}{h}$, where h given by (3.6) with $C_1=1$, $C=1/2$ and $t_0=0$, $\varepsilon=1$. The support function f is given by: (5.8) with $A=B_1=B_2=0$ and $B=1$ in first image, (5.9) with $A=2$, $A_1=1$ and $B=A_2=0$ in second image, (5.10) with $A=-1$, $A_1=1$ and $B=A_2=0$ in third image.

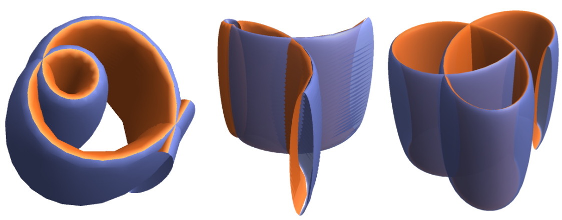

Figure 5.2: In these figures, we have CPD surfaces given by Theorem 5.2 with $G(h)=ah+2C$, where h given by (3.2) with $t_0=0$. The support function f is given by: (5.8) with $A=0$, $B_2=3$ and $B=0,7$, with $a=1/3$ and $C=0$ in first image, (5.9) with $A=2$, $A_1=0,2$, $B=3$ and $A_2=0$, with $a=0,2$ and $C=0,1$ in second image, (5.10) with $A=-1$, $A_1=3$, $B=6$ and $A_2=0$, with $a=1$ and $C=0$ in third image.

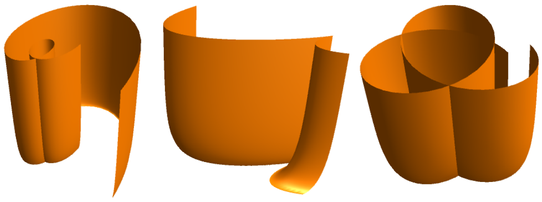

Figure 5.3: In these figures, we have CPD surfaces given by Theorem 5.2 with $G(h) + C = \sin(ah + b)$, where h given by (3.3) with $a = 1$, $b = 0$ and $t_0 = 0$, $\varepsilon = 1$. The support function f is given by: (5.8) with $A = 0$, $B_1 = -2$, $B_2 = 3$ and $B = 1$ in first image, (5.9) with $A = 2$, $A_1 = 1$, $A_2 = -3$ and $B = 1$ in second image, (5.10) with $A = -1$, $A_1 = 1$, $A_2 = 0$ and $B = 1$ in third image.

Figure 5.4: In these figures, we have CPD surfaces given by Theorem 5.2 with $G(h) + C = \tanh(ah + b)$, where h given by (3.4) with $a = 1$, $b = 0$ and $t_0 = 0$. The support function f is given by: (5.8) with $A = 0$, $B_1 = B_2 = 0$ and $B = 1$ in first image, (5.9) with $A = 2$, $A_1 = 0,3$, $A_2 = 0,5$ and $B = 1$ in second image, (5.10) with $A = -1$, $A_1 = 1$, $A_2 = 0$ and $B = 0,5$ in third image.

Proposition 5.2. *Let M be a CPD surface given by (5.1). Under the conditions of the Theorem 5.1, if $G(h) = ah + 2C$, where a and C are constants, then M is a tubular surface.*
Moreover, if the support function of the directrix curve is $f(u) = A_1 \sin(u + A_2) + B$, then the CPD surface is a Dupin surface.

Proof: Since $G(h) = ah + 2C$, then using (3.2), $h = h(v)$ satisfies

$$(v - t_0)^2 + \left(h + \frac{2C}{a}\right)^2 = \frac{1}{a^2}.$$

Therefore, in the (h, v)-plane, the profile curve of the CPD surface is the circle of radius $r = \frac{1}{|a|}$ and center $\left(t_0, -\frac{2C}{a}\right)$.

Note that for fixed u, the map $v \mapsto X(u, v)$ is a circle in $\operatorname{span}\{n(u), e_3\}$.
In fact, define $c(u) = \gamma(u) - \dfrac{2C}{a} n(u) + t_0 e_3$ and consider $\Pi(u) = \{\, \gamma(u) + s\, n(u) + t\, e_3 : s, t \in \mathbb{R}\,\}$.
Thus, we can write

$$X(u, v) - c(u) = \left(h(v) + \tfrac{2C}{a}\right) n(u) + (v - t_0)e_3 \in \Pi(u).$$

From the last equation, we have

$$|X(u, v) - c(u)|^2 = \left(h + \tfrac{2C}{a}\right)^2 + (v - t_0)^2 = \frac{1}{a^2}.$$

Therefore, for each fixed u, the trace $v \mapsto X(u, v)$ is a circle of constant radius $\frac{1}{|a|}$ in the plane $\Pi(u)$, with center $c(u)$.

It remains to show that, for each u, the planes $\Pi(u)$ are the normal planes to the curve $c(u)$.
In fact, since $\gamma(u)$ and $n(u)$ lie in the plane $z = 0$, then $c'(u) = \gamma'(u) - \dfrac{2C}{a} n'(u)$ also lies in the plane $z = 0$. Thus, we have

$$< c'(u)\,,\, n(u) >= 0, \qquad \text{and} \qquad < c'(u)\,,\, e_3 >= 0.$$

Therefore, in this case, the parametrization X of the CPD surface is precisely the parametrization of a circular tube of constant radius $\frac{1}{|a|}$ around the curve c.

Finally, assuming that the support function is given by $f(u) = A_1 \sin(u + A_2) + B$, then from Theorem 5.1, the principal curvatures are given by $k_1 = \dfrac{ah(v) + 3C}{h(v) + B}$ and $\kappa_2 = a$ and hence the surface is a Dupin surface. □

Remark 5.4. *Under the conditions of the last proposition, by reparameterizing the parameter v*

$$v = t_0 + \frac{1}{|a|} \sin\theta, \qquad h = \frac{-2C}{a} + \frac{1}{|a|}\cos\theta,$$

we obtain that the parametrization (5.1) of the CPD surface in this case becomes

$$X(u, \theta) = c(u) + \frac{1}{|a|}\big(\cos\theta\, n(u) + \sin\theta\, e_3\big).$$

6. Conclusions. From the results obtained in this work, we observe the following: Using a geometric invariant, the support function together with the prescription of a principal curvature, allows us to characterize families of CPD surfaces. Such surfaces generalize surfaces of revolution.

7. Author contributions. The authors of this publication contributed equally in the following aspects:Conceptualization, Corro AMV and Ferro ML; investigation, Corro AMV and Ferro ML; formal analysis,Corro AMV and Ferro ML; metodologia, Corro AMV; validation, Corro AMV and Ferro ML; writing -original draft, Ferro ML; Writing - review and editing, Corro AMV and Ferro ML.

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