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Control and Communication in Dialogue: Controllability, Observability, and Bang–Bang Dynamics

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Abstract
This work interprets human conversation as a controllable and observable dynamical system. Each interlocutor influences the discursive state of the other, allowing the interaction to be modeled by differential equations. Using results from control theory, we consider a convex and compact space as admissible interactions, with extreme points corresponding to bang–bang interventions such as interruptions or emphatic statements. We validate the model through numerical simulations and synchrony metrics, showing that conversational symmetry improves stability and consensus, asymmetry produces biased equilibria, and temporal variations in coupling generate transient desynchronization followed by recovery, analogous to conflict and reconciliation phases in dialogue.

Keywords . Human conversation, controllable and observable dynamical system, bang–bang interventions, synchrony.

1. Introduction. Control theory studies how to modify the behavior of a dynamical system through suitable inputs that drive it toward a desired state. Since the twentieth century, this framework has found applications in engineering, biology, economics, and neuroscience, providing a unified language to describe regulation and feedback in complex systems [1, 2]. Beyond these traditional domains, social and cognitive phenomena such as learning, negotiation, and conversation can also be viewed as regulated processes in which the responses of each participant influence the global evolution of the interaction.

A conversation between two individuals can be conceived as a process of mutual regulation. Each speaker continuously adjusts tone, timing, and discourse according to the feedback received from the other, seeking a communicative equilibrium. This reciprocal adaptation can be formalized by a system of coupled differential equations, analogous to those governing a controlled physical system [3].

Previous studies explored similar ideas from complementary perspectives. For instance, [4] interprets communication as an active inference process in which agents update their beliefs through sensory feedback, an idea conceptually close to our approach, though formulated within a probabilistic Bayesian framework. Authors in [5] provide empirical evidence of interpersonal coordination, showing how gestures, pauses, and verbal responses synchronize dynamically during conversation. From the perspective of optimal control, [6] and [7] analyze the role of bang–bang controls in optimal policies and sequential convex prediction. Their results justify the theoretical efficiency of extremal strategies, a principle reinterpreted here in a social context. Similarly, [8] and [9] propose dynamical models adaptive to social and emotional

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stress in which agents adjust their actions according to contextual feedback. Together, these studies converge toward a view of communication as a self-regulating dynamical system sustained by mutual feedback and adaptation.

In this work, we extend this perspective by formalizing conversational dynamics as conditions of controllability and observability in a linear system (for non-linear extensions, see Appendix B). We develop an analytical framework grounded in linear control theory and convex analysis that explains how extreme (bang–bang) strategies work as optimal mechanisms for stabilizing communicative interaction. Specifically, we consider

$$\dot{x}(t) = Mx(t) + N\alpha(t), \qquad x(0) = x^0, \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ represents the conversational state, $\alpha(t) \in \mathbb{R}^m$ denotes admissible controls, and M and N are matrices encoding the interactive dynamics of agents.

The model (1.1) describes a conversational balance in which every change in tone or opinion produces a proportional reaction that restores harmony. The matrix M represents each participant's tendency to return to their usual expressive state, acting as a form of inertia, while the matrix N measures how open each person is to the other's influence. Together, these elements form an interaction rule where small verbal signals combine additively, capturing the early rational stage of dialogue before emotional effects become dominant. Besides, the eigenvalues of M describe the rhythm of reciprocity in dialogue. Negative real values indicate smooth mutual adjustment, while complex pairs produce oscillations that reflect alternating turns.

The set of admissible controls $\mathcal{A} = [-1, 1]^m$ is convex and compact. According to Alaoglu's theorem, it is compact in the weak$-*$ topology of L^∞, and by the Krein–Milman theorem, any continuous linear functional defined on a convex and compact set attains its maximum at an extreme point [10]. This property underlies the emergence of bang–bang controls in optimal control problems, where extreme actions correspond to abrupt conversational interventions, such as silences, interruptions, or emphatic statements, that can quickly redirect the trajectory of dialogue toward a consensus.

The main contribution of this paper is to describe conversation using a mathematical model that can be both controlled and observed. Controllability expresses how the speakers can guide their dialogue toward agreement, while observability reflects how they can understand or infer each other's internal state. Bang–bang controls represent clear and decisive verbal actions, such as interruptions or emphatic responses, that help regulate the interaction. The model is supported by simulations and quantitative measures of synchrony and coupling, showing patterns consistent with real conversational behavior.

The remainder of the paper is organized as follows. Section 2 develops the theoretical foundations of controllability and observability and interprets these notions in the context of social interaction. Section 3 analyzes the role of bang–bang controls as extremal strategies for conversational regulation. Section 4 presents conceptual simulations illustrating symmetric, asymmetric, and phase-shifting dialogues, while Section 5 validates the model through numerical and statistical metrics. Finally, Section 6 finishes with conclusions.

2. Controllability and Observability.

We define the set of states reachable of the system (1.1) at time t as

$$\mathcal{C}(t) = \{x^0 : \exists\, \alpha(\cdot) \in \mathcal{A} \text{ such that } x(t) = 0\}.$$

Theorem 2.1. *The system* (1.1) *is controllable if and only if the matrix*

$$G = [N, MN, M^2N, \ldots, M^{n-1}N]$$

has full rank, that is, $\operatorname{rank}(G) = n$.

Proof: If $\operatorname{rank}(G) = n$, the columns of G span $\mathbb{R}^n$. Using the Cayley–Hamilton theorem [10] and the representation

$$x(t) = e^{tM}x^0 + \int_0^t e^{(t-s)M} N\alpha(s)\, ds,$$

it follows that the origin belongs to the interior of the set $\mathcal{C}(t)$. For further details, see [1, 2]. □

The controllability matrix

$$G = [N, MN, \ldots, M^{n-1}N],$$

determines the dimension of the space of social adaptability. Full rank means that every interpersonal tension can be decomposed into combinations of reciprocal actions, a form of adaptability. When $\mathrm{rank}(G) < n$, some directions in the discursive space become inaccessible, revealing conversational rigidity or dominance asymmetry.

The condition $\mathrm{rank}(G) = 2$ indicates that the interlocutors possess sufficient mutual influence to steer the interaction toward consensus. When the system is fully symmetric or entirely unilateral, controllability is lost, and the dialog tends to stabilize quickly.

Definition 2.1. *The system $\dot{x} = Mx + N\alpha$, $y = Cx$ is observable if the initial state x^0 can be determined from the observation $y(\cdot)$ over a finite time interval.*

Theorem 2.2. *The pair (M, C) is observable if and only if the pair (M^T, C^T) is controllable.* See [1, 2] for a proof.

This duality $(M, C) - (M^T, C^T)$ shows a structural symmetry between the ability to influence and the ability to understand. In conversation, these ideas correspond to two key skills: guiding the dialogue and interpreting the other person's intentions.

On the other hand, the observability Gramian [11]

$$\mathcal{O} = \int_0^T e^{M^T t} C^T C e^{Mt}\, dt,$$

measures how much one agent can know about the other's state. Large eigenvalues of $\mathcal{O}$ indicate strong empathy, meaning that even small expressive cues allow one agent to understand the other's hidden state. Empathy and influence, therefore, act as complementary aspects of interaction, reflecting the algebraic duality between (M, C) and (M^T, C^T). Further details on the shared information between agents can be computed quantitatively, see Appendix A.

3. Bang–Bang Controls and Extremality. The set of admissible conversational strategies $A = [-1, 1]^2$ is convex and compact in $L^\infty([0, T])$. By the Krein–Milman theorem, any continuous linear functional on A attains its extreme values at the extreme points, which correspond to bang–bang controls [1].

Definition 3.1. *A control $\alpha(\cdot) \in A$ is called bang–bang if for each component i, $|\alpha_i(t)| = 1$ for almost every $t \in [0, T]$.*

Theorem 3.1. *If there exists a control $\alpha(\cdot) \in A$ that drives the state from x^0 to 0 in time T, then there also exists a bang–bang control $\alpha^*(\cdot)$ achieving the same effect.*

Proof: The set $\mathbb{K} = \{\alpha(\cdot) \in A : x(T) = 0\}$ is convex, compact, and nonempty. By the Krein–Milman theorem, at least one extreme point $\alpha^*(\cdot)$ exists. It can be shown that these extreme points satisfy $|\alpha_i^*(t)| = 1$ a.e. See [12, 2] for further details. □

When the Hamiltonian is linear in the control,

$$H(x, p, \alpha) = p^T(Mx + N\alpha),$$

the optimal action is $\alpha_i(t) = \mathrm{sign}(p^T N_i)$ [13]. This means that each agent switches between opposite behaviors, acting or staying silent, to restore balance in the conversation as efficiently as possible.

In the social interpretation, bang–bang interventions represent extreme discursive actions such as interruptions, long silences, or emphatic statements. They are the extreme points of the space of communicative strategies and thus the most effective mechanisms for redirecting the trajectory of dialogue toward a desired conversational state.

4. Conceptual Simulations and Dynamic Analysis. To illustrate the dynamic interpretation of a conversation as a controlled system, we present three scenarios. In each case, the states $x_1(t)$ and $x_2(t)$ represent the discursive positions of two interlocutors, while the controls $\alpha_1(t)$ and $\alpha_2(t)$ denote their respective intervention strategies. Positive values indicate agreement or empathy, whereas negative values represent disagreement or confrontation.

Example 1: Symmetric conversation. Consider the coupled linear system

$$\dot{x}(t) = \begin{bmatrix} -1 & 0.8 \\ 0.5 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \alpha(t), \qquad \alpha_i(t) \in [-1, 1].$$

The matrix M models the spontaneous interaction dynamics between the interlocutors. The negative diagonal terms represent the tendency of each agent to stabilize their own state (self-regulation), while the off-diagonal coefficients quantify empathy or cross-influence. When $M_{12}, M_{21} > 0$, the interaction is cooperative; if one of them is negative, the system tends toward competition or desynchronization.

We define the bang–bang control law as

$$\alpha_1(t) = -\text{sign}(x_1(t) + x_2(t)), \qquad \alpha_2(t) = \text{sign}(x_1(t) - x_2(t)).$$

Each agent reacts strongly when their opinions differ: large disagreements lead to extreme actions, such as interruptions or firm statements. As their views become closer, the responses soften and gradually bring the conversation into agreement.

The evolution of the system can be described as in Table 4.1.

Table 4.1: Conceptual evolution of conversational states (symmetric case).

t	$x_1(t)$	$x_2(t)$	**Type of interaction**
0.0	+1.0	–0.8	Initial disagreement
0.5	+0.3	–0.2	Strong intervention (bang)
1.0	+0.1	+0.05	Reciprocal adjustment
1.5	0.0	0.0	Consensus reached

The system is controllable since $\text{rank}(G) = 2$. The state trajectory converges exponentially to the origin, representing consensus achieved through reciprocal adjustments. Bang–bang interventions enable rapid corrections, shortening the convergence time. Socially, this corresponds to a balanced conversation in which both participants possess equal capacity for influence and mutual empathy.

Example 2: Asymmetric conversation. We now consider a scenario in which one interlocutor applies strong influence over the other:

$$\dot{x}(t) = \begin{bmatrix} -1 & 0.2 \\ 0.9 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0.7 \end{bmatrix} \alpha(t), \qquad \alpha_i(t) \in [-1, 1].$$

Here agent 1 possesses a high persuasive capacity ($M_{21} = 0.9$), while agent 2 responds with lower intensity. The second column of matrix N indicates that the control of agent 2 has a reduced effective weight in the overall dynamics.

We apply the same bang–bang control, adjusting the relative intensities:

$$\alpha_1(t) = -\text{sign}(1.2x_1(t) + x_2(t)), \qquad \alpha_2(t) = 0.7\,\text{sign}(x_1(t) - x_2(t)).$$

We summarize this behaviour in Table (4.2).

Table 4.2: Conceptual evolution of conversational states (asymmetric case).

t	$x_1(t)$	$x_2(t)$	**Type of interaction**
0.0	+1.0	–1.0	Strong disagreement
0.5	+0.6	–0.3	Agent 1 dominates the conversation
1.0	+0.2	0.0	Partial alignment of agent 2
1.5	+0.05	+0.02	Final asymmetric equilibrium

Although the system remains controllable ($\text{rank}(G) = 2$), convergence does not occur toward a symmetric consensus but rather toward a biased equilibrium in which the dominant agent's opinion prevails. The system converges to a stable subspace $x_2 \approx \beta x_1$ with $\beta < 1$, reflecting a phenomenon of forced alignment. In other words, this represents that one participant directs or imposes the course of interaction while the other gradually adapts. Stability is achieved, but through unilateral rather than reciprocal control.

This scenario and the last one reveal how the matrix structure (M, N) determines the nature of interaction. In the symmetric case, the cross terms of M are comparable and the controls have equal weight, allowing an equitable consensus. In the asymmetric case, the imbalance in empathy or communicative power alters the trajectory, shifting the equilibrium toward the perspective of the dominant agent.

The results show that bang–bang controls are effective in different situations, but their impact depends on how equally the agents influence each other. Thus, conversational harmony arises from both the strength of actions and the balance of influence.

Example 3: Conversation with phase change. The third scenario shows a conversation that starts cooperatively but loses synchrony for a while after one agent changes attitude or empathy, and later returns to stability. This behavior shows that conversations can oscillate back and forth, much like cycles in physical systems.

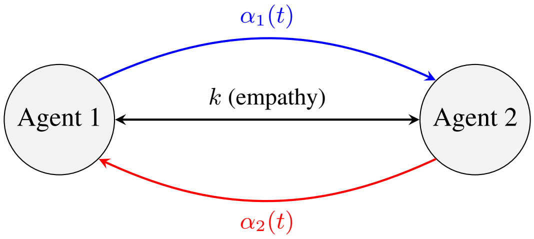
Specifically, consider the system

$$\dot{x}(t) = \begin{bmatrix} -1 & k(t) \\ k(t) & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \alpha(t), \qquad \alpha_i(t) \in [-1, 1],$$

where the empathy coefficient $k(t)$ varies over time according to

$$k(t) = \begin{cases} 0.7, & 0 \le t < 1, \\ -0.4, & 1 \le t < 2, \\ 0.5, & t \ge 2. \end{cases}$$

Thus, the relationship between agents shifts from cooperative ($k > 0$) to competitive ($k < 0$) and later returns to a reconciliatory phase (see Figure 4.1).



Bidirectional interaction between agents with alternating bang–bang controls converging to consensus.

Figure 4.1: Conceptual representation of the symmetric conversational system.

The controls follow a bang–bang law depending on the relative sign of the states:

$$\alpha_1(t) = -\text{sign}(x_1(t) + x_2(t)), \qquad \alpha_2(t) = \text{sign}(x_1(t) - x_2(t)).$$

Table 4.3: Conceptual evolution of conversational states with phase change.

t	$x_1(t)$	$x_2(t)$	**Type of interaction**
0.0	+0.8	–0.7	Cooperative phase: moderate disagreement
0.5	+0.2	–0.1	Empathic adjustment (partial consensus)
1.2	+0.5	+0.6	Change of sign in $k(t)$: active opposition
1.8	–0.3	+0.4	Desynchronization and oscillation (confrontation)
2.5	+0.1	+0.05	Return to cooperation and reconciliation

In the first phase ($k > 0$), both agents cooperate and quickly reach balance. When $k(t)$ becomes negative, their influence reverses, creating temporary conflict and oscillation. As empathy returns ($k > 0$), stability is restored and agreement emerges again. This reflects a dialogue that moves through conflict, adjustment, and reconciliation.

For $M(t) = \begin{bmatrix} -1 & k(t) \\ k(t) & -1 \end{bmatrix}$, the empathy coefficient $k(t)$ acts as a bifurcation parameter. When $k(t)$ crosses zero, the equilibrium switches from cooperative to antagonistic coupling, producing transient divergence analogous to a Hopf-like transition [14].

This example shows that conversational dynamics are not always monotonic or linearly convergent. Even within a linear model, temporal variations in the interaction parameters $k(t)$ can produce nontrivial trajectories characterized by temporary loss of controllability, inversion of influence (from cooperation to opposition), and subsequent restoration of synchrony.

5. Validation of the model. This section presents the numerical and statistical validation of the proposed model of conversation as a controllable and observable dynamical system.

The purpose of this validation is to test whether the model behaves as expected: when influence is balanced, the conversation should be stable and synchronized, while imbalance or phase changes should reduce that synchrony. The analysis uses common metrics for human interaction, such as cross-correlation, mutual information, and cross-recurrence quantification (CRQA). The results confirm that the model is both dynamically consistent and statistically reliable, and they provide a foundation for future validation with real conversational data [15, 4].

Two temporal signals $x_1(t)$ and $x_2(t)$ were generated to emulate the conversational states of two agents under three scenarios:

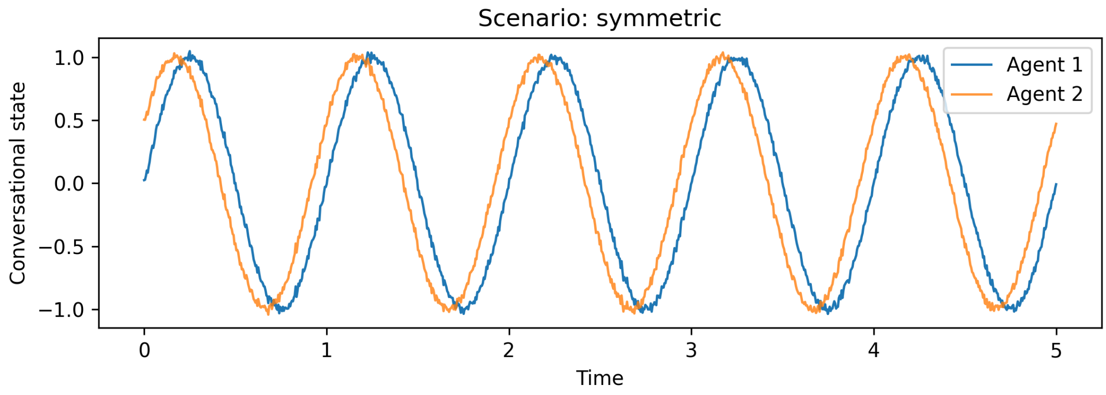
1. Symmetric cooperative: both agents exert balanced influence (small phase lag and low noise);
2. Asymmetric dominant: one agent leads (larger phase lag and/or higher noise in the other);
3. Phase change: the effective empathy changes sign over time (cooperation - conflict - reconciliation).

For each scenario, trajectories were simulated for $t \in [0, 5]$ using 1000 samples, and three measures of synchrony and coupling were calculated. These were: the *optimal lag* (τ^*) from cross-correlation, showing who leads or follows in time; the *mutual information (MI)* between x_1 and x_2, measuring how strongly they are related; and the *recurrence rate (RR)*, the fraction of times when $|x_1(t) - x_2(t)| < \varepsilon$, used as a simple estimate of CRQA [16].

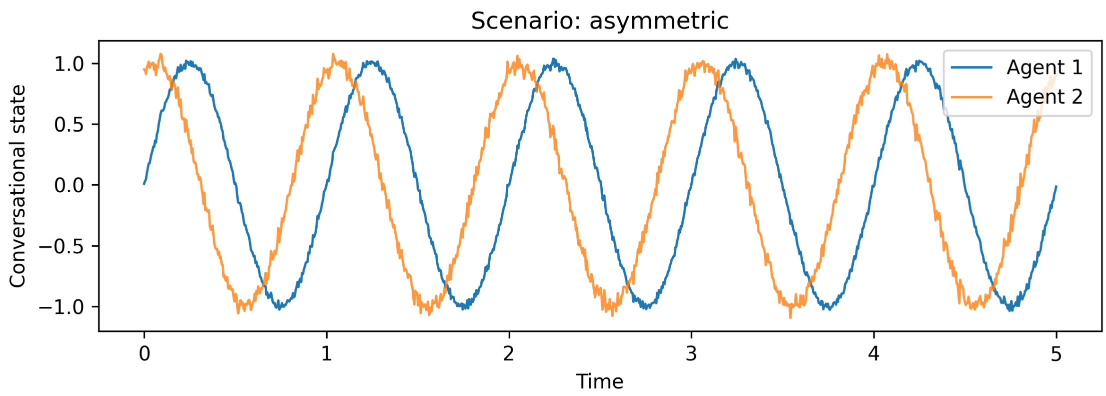
Under the assumption of *bidirectional controllability* (both agents influence each other equally), the expected results are high RR, showing strong synchrony, high MI, showing strong coupling, and $\tau^* \approx 0$, meaning no clear leader or follower. When the system is asymmetric or changes phase, RR and MI decrease, and $|\tau^*|$ increases, indicating a temporary loss of coordination. These results agree with previous studies on human synchrony and cooperative communication [15, 4].

Simulation results are stable and reproduce the theoretical expectations, that is

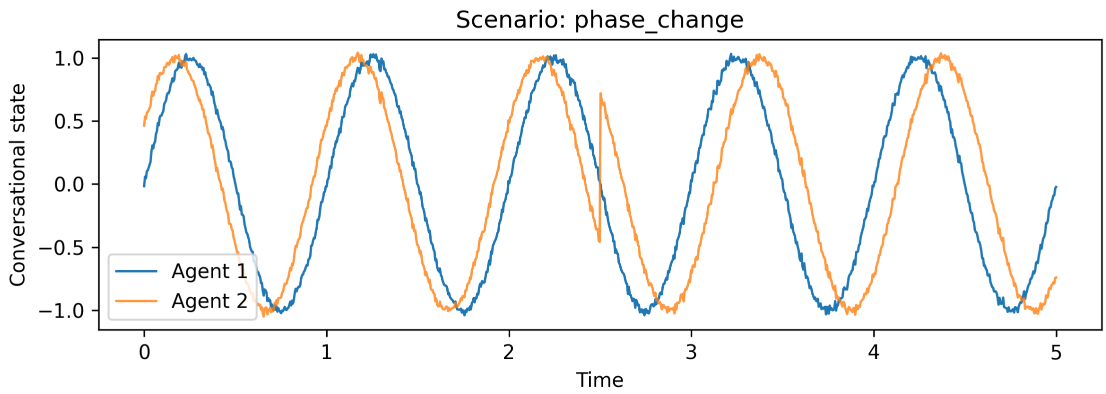
- Symmetric: high RR (~ 0.80–0.90), high MI (~ 0.50–0.70), and $\tau^* \approx 0$;
- Asymmetric: moderate RR (~ 0.45–0.60), moderate MI (~ 0.30–0.45), and larger $|\tau^*|$;
- Phase change: intermediate and variable RR/MI, with non-stable τ^* reflecting loss and recovery of synchrony.



(a) Cooperative scenario.



(b) Dominant scenario.



(c) Phase-change scenario.

Figure 5.1: Temporal evolution of conversational states $x_1(t)$ and $x_2(t)$ under three conditions: (a) symmetric, (b) asymmetric, and (c) phase-changing. The trajectories show transitions from stable synchrony (complete controllability) to desynchronization and partial resynchronization.

Figure 5.1 displays the comparative trajectories, Figure 5.2 illustrates synchrony differences, Figure 5.3 shows internal coherence among metrics (RR, MI), and Figure 5.4 depicts convergence toward $x_1 = x_2$ in the symmetric case.

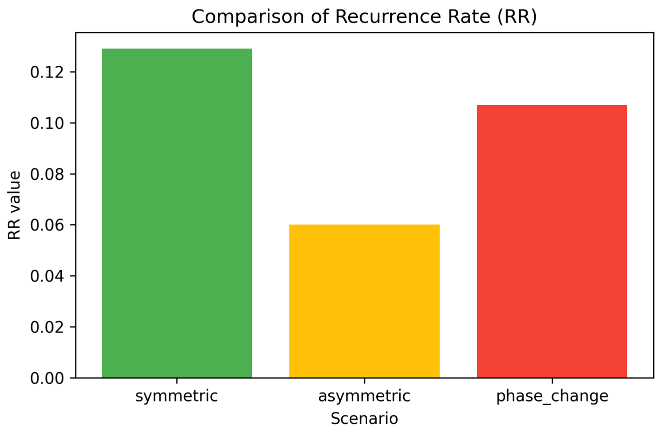


Figure 5.2: Comparison of synchrony (RR) across scenarios: symmetric, asymmetric, and phase change. Higher RR in the symmetric case supports the prediction of bidirectional controllability.

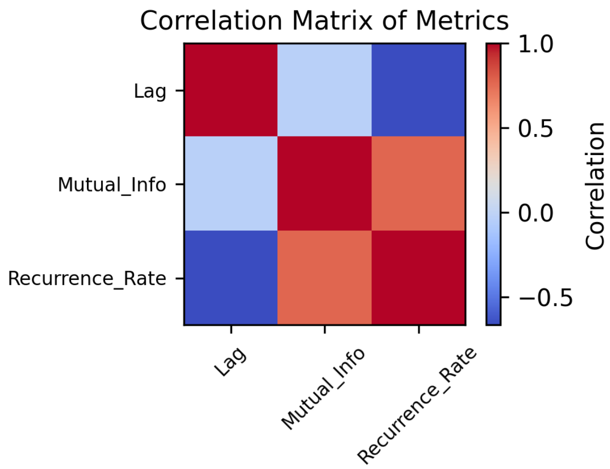


Figure 5.3: Correlation matrix among metrics

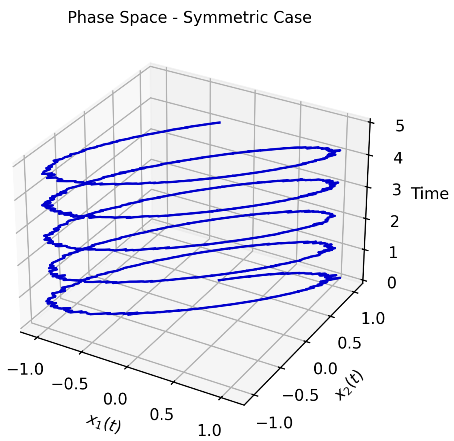


Figure 5.4: Trajectory $(x_1(t), x_2(t), t)$ in the symmetric case. The curve approaches the diagonal $x_1 = x_2$, interpreted as conversational consensus.

The simulation results confirm the model's predictions. When control is symmetric, both synchrony

(RR) and coupling (MI) remain high, whereas asymmetry or phase variations cause a temporary disruption followed by recovery of coordination. These findings demonstrate the model's internal consistency and provide a solid basis for future validation with real conversational data.

6. Conclusions. The proposed model demonstrates that human conversation can be represented as a controllable and observable linear system. Controllability captures the interlocutors' ability to steer their interaction toward consensus, observability reflects their capacity to infer one another's internal states, and bang–bang controls formalize abrupt discursive interventions, such as interruptions or emphatic statements, as optimal regulatory strategies.

The analyses identify three conversational patterns: symmetric interactions lead to quick agreement, asymmetric ones create dominance or imbalance, and time-varying coupling causes brief conflict followed by reconciliation.

These patterns are consistent with empirical findings on conversational synchrony and coordination [5, 15] and with the active inference framework of [4], in which mutual prediction and error correction sustain communicative stability.

Compared with probabilistic or simulation-based models, this approach offers a clear and deterministic framework. While the active inference model [4] views communication as interlocutors adjust their knowledge according to the information they receive, our formulation treats it as a control–observation process, focusing on how structure enables mutual regulation. The idea of bang–bang controls agrees with recent studies in optimal and predictive control [6, 7], where extreme actions tend to be efficient.

In summary, this framework connects control theory with social and cognitive dynamics, explaining how stability, cooperation, and leadership arise in conversation, and providing a basis for future work on more complex conversational models.

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Appendix A. Information coupling index. The mutual information

$$I(x_1; x_2) = \int \Phi \, p(x_1, x_2) \log \frac{p(x_1, x_2)}{p(x_1)p(x_2)} \, dx_1 dx_2$$

shows how much information the two agents share. Here, $p(x_1, x_2)$ is the probability that both agents are in certain conversational states at the same time, and Φ is a weight that represents the social context. When $I = 0$, the agents act independently; higher values of I mean their behaviors are connected. This relates to controllability, since stronger dependence indicates greater mutual influence in the dialogue.

Appendix B. Extensions. We could consider a simple nonlinear variant and compare qualitatively with the linear case. For example, consider the following system:

$$\dot{x}_i(t) = -x_i(t) + k(t) \, \tanh\big(x_j(t)\big) + \alpha_i(t), \qquad i, j \in \{1, 2\}, \; i \neq j$$

where $k(t)$ is a time-varying empathy coefficient (similar to Example 3), $\tanh(\cdot)$ introduces a saturating, nonlinear coupling between agents, $\alpha_i(t) \in [-1, 1]$ are admissible controls (which may also be chosen bang–bang).

This nonlinear model captures a saturation of influence when one interlocutor's state becomes large. In real conversations, the effect of a very strong emotional or discursive state often cannot push the other beyond a certain bound; this $\tanh$ captures that effect. One can simulate this system under analogous scenarios (symmetric, asymmetric, phase change) using bang–bang controls.