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Stochastic Analysis on Efficiency Vaccination in a SIR Model

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Abstract

In this paper, we model the stochastic effect of vaccination on an epidemic which like epimdemic model, it can be susceptible to environmental changes. This study is carried out using a SIR model in which population remain susceptible after having been vaccinated. One important point of our research is that we apply a whit noise on recuperate rate of the infected individuals. Further, we show some stability results and some numerical simulations.

Keywords . SIR model, epidemiology model, stochastic model, Covid-19, white noise, efficiency vaccination.

1. Introduction. The study of epidemic processes is an important direction in using the method of mathematical modelling to study living systems. Examples of various mathematical models that arise in the problems of epidemiology can be seen in [1, 2]. The Covid-19 pandemic has led to active development of mathematical models reflecting the dynamics of the number infected, sick and deceased from Covid-19 individuals. Recently, Nakamura et al. [3] define a SIR model ehich studied effect of vaccination on the number of hospital admissions. They showed that any delay in the vaccination campaign results in an increase of hospitalisations, and if one tries to palliate for the delay by increasing the vaccination rate, this results in an in action of the number of necessary doses. Further, they proved that it is advantageous to prioritise the vaccination of the older groups (upholding thus the current practice). Nakamura et al. SIR model was defined as follows:

$$\begin{cases} dS = (-aSI - \eta)dt, \\ dT = (\eta f - bTI)dt, \\ dI = (aSI + bTI - \lambda I)dt, \\ dR = (\lambda I + \eta(1 - f))dt, \end{cases} \tag{1.1}$$

where

1. $a =$ Infection rate,
2. $\eta =$ Constant-rate vaccination
3. $f =$ Vaccine efficiency,
4. $b =$ Infection rate of vaccine individuals,

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5. λ = Recuperate rate of the infected individuals.

Where, S, T, I and R denote susceptible, susceptible after having been vaccinated, infected and recovered, respectively. This situation can be better seen in the following diagram:

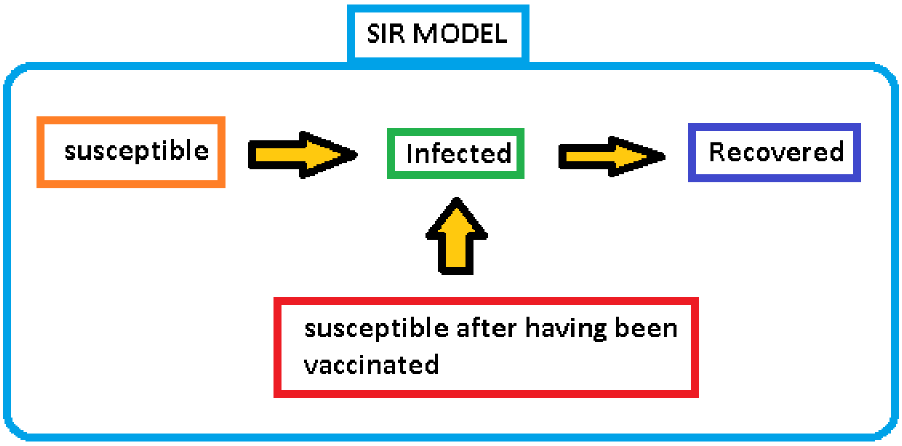

Diagram 1: SIR model described in (1.1).

Nonetheless, the results presented in that article showed that the effectiveness of the vaccine was 100%. However, it should be noted that the 1.1 model must be exposed to certain environmental noises that cannot be controlled in order to verify said effectiveness. For that reason, the analysis presented in this paper raises several questions. For instance, is it possible to predict the end of the pandemic based on simulations when some parameters are susceptible to a environmental variability? When the system maintain persistent and extinction on variables. As can be seen in [3], there can be many random situations that could affect vaccine efficiency. For that reason, it is necessary tu study model (1.1) under random conditions, in this case, under environmental variability. In order to define the stochastic model, we will consider that Recuperate rate of the infected individuals (λ) since this rate can be susceptible of a white noise (environmental variability) [4]. Therefore, stochastic predator-prey model is given as follows:

$$\begin{cases} dS = (-aSI - \eta)dt, \\ dT = (\eta f - bTI)dt, \\ dI = (aSI + bTI - \lambda I)dt - \sigma I dB(t), \\ dR = (\lambda I + \eta(1-f))dt. \end{cases} \tag{1.2}$$

2. Theoretical Results. In this section, we show some principal results, proofs of each theorems can be found in Appendix.

The coefficients of model (1.2) are continuos and locally Lipschitz. For instance, we can show that in a finite time, the solution does not diverge, thus it has a positive $\mathbb{R}^4_+ = \{(a, b, c, d,) \in \mathbb{R}^4 : a > 0, b > 0, c > 0, d > 0\}$ as an invariant set. Therefore, the following theorem come up with:

First at all, we will prove that the system has unique solution;

Theorem 2.1. *For initial values $(S(0), T(0), I(0), R(0)) \in \mathbb{R}^4_+$, the system (1.2) has an unique solution $(S(t), T(t), I(t), R(t))$ for all $t \geq 0$ and the solution remains in $\mathbb{R}^4_+$ with probability one.*

Proof: Let $q = (q_1, q_2, q_3, q_4)$ y $u = (u_1, u_2, u_3, u_4)$, where

$$\begin{aligned} q_1 &= -aSI - \eta, \\ q_2 &= \eta f - bTI, \\ q_3 &= aSI + bTI - \lambda I, \\ q_4 &= \lambda I + \eta - \eta f, \\ u_1 &= u_2 = u_4 = 0, \\ u_3 &= -\sigma I. \end{aligned}$$

Lyapunov's operator associated (1.2) is given by

$$L = q_1 \frac{\partial}{\partial S} + q_2 \frac{\partial}{\partial T} + q_3 \frac{\partial}{\partial I} + q_4 \frac{\partial}{\partial R} + \frac{1}{2} u_1^2 \frac{\partial^2}{\partial S^2} + \frac{1}{2} u_2^2 \frac{\partial^2}{\partial T^2} + \frac{1}{2} u_3^2 \frac{\partial^2}{\partial I^2} + \frac{1}{2} u_4^2 \frac{\partial^2}{\partial R^2}.$$

Now, let's define $V : \mathbb{R}^4_+ \times [0,\infty) \to [0,\infty)$ by

$$V(M,E,L,S,P,R) = S - 1 - ln(S) + T - 1 - ln(T) + I - 1 - ln(I) + R - 1 - ln(R).$$

Therefore, we obtain

$$q_1 \frac{\partial V}{\partial S} = -aSI - \eta + aI + \frac{\eta}{S},$$
$$q_2 \frac{\partial V}{\partial T} = \eta f - bTI - \frac{\eta f}{T} + bI,$$
$$q_3 \frac{\partial V}{\partial I} = aSI + bTI - \lambda I - aS - bT,$$
$$q_4 \frac{\partial V}{\partial R} = \lambda I + \eta - \eta f - \frac{\lambda I}{R} - \frac{\eta}{R} + \frac{\eta f}{R},$$
$$\frac{u_1^2}{2}\frac{\partial^2 V}{\partial S^2} = 0,$$
$$\frac{u_2^2}{2}\frac{\partial^2 V}{\partial T^2} = 0,$$
$$\frac{u_3^2}{2}\frac{\partial^2 V}{\partial I^2} = \sigma^2 \frac{1}{2},$$
$$\frac{u_4^2}{2}\frac{\partial^2 V}{\partial R^2} = 0.$$

Hence, $LV(S,T,I,R)$ is given by

$$LV(S,T,I,R) \le aI + bI + 0.5\sigma^2 \le \max\{a+b, 1\}(S+T+I+R) + 0.5\sigma^2 \le K_1 V(S,T,I,R) + K_2.$$

By this, we conclude that the solution remains in $\mathbb{R}^4_+$ with probability one. □

Next, we show some stability results.

1. The trivial solution of a stochastic differential equation is said to be stable in probability if for all $t \ge 0$ and for any $\varepsilon \in (0,1)$ and $r > 0$, there exists $\delta = \delta(\varepsilon, r) > 0$ such that

$$\mathbb{P}(\|x(t)\| < r \text{ for all } t \ge 0) \ge \varepsilon,$$

for any $\|x(0)\| < \delta$.

2. (See Theorem 2.2 in [4]) If there exists a function $V(x,t) \in C^{2,1} \in (\mathbb{R}^d_+ \times \mathbb{R}_+, \mathbb{R}_+)$ positive-definitive such that $LV(x,t) \le 0$ for all $(x,t) \in \mathbb{R}^d_+ \times \mathbb{R}_+$, then the trivial solution of the stochastic differential equation is stable in probability.

From the above conditions, we carry out the following result:

Theorem 2.2. *Let's consider the stochastic model (1.2). Then, the trivial solution of (1.2) is stable in probability.* *Proof:* Let's define $V(S,T,I,R) = S+T+I+R \in C^{2,1} \in (\mathbb{R}^4_+ \times \mathbb{R}_+, \mathbb{R}_+)$; we can see that $V(S,T,I,R)$ is positive-definitive. Therefore, Lyapunov's operator associated to (1.2) over $C^{2,1}(S,T,I,R)$, is given by

$$L = q_1 \frac{\partial}{\partial S} + q_2 \frac{\partial}{\partial T} + q_3 \frac{\partial}{\partial I} + q_4 \frac{\partial}{\partial R} + \frac{1}{2}u_1^2 \frac{\partial^2}{\partial S^2} + \frac{1}{2}u_2^2 \frac{\partial^2}{\partial T^2} + \frac{1}{2}u_3^2 \frac{\partial^2}{\partial I^2} + \frac{1}{2}u_4^2 \frac{\partial^2}{\partial R^2},$$

where,

$$q_1 = -aSI - \eta,$$
$$q_2 = \eta f - bTI,$$
$$q_3 = aSI + bTI - \lambda I,$$
$$q_4 = \lambda I + \eta - \eta f,$$
$$u_1 = u_2 = u_4 = 0,$$
$$u_3 = -\sigma I.$$

By mentioned above, we can see that

$$LV = 0.$$

Therefore, trivial solution of (1.2) is stable in probability. □
Now, from the following conditions

1. [4] The trivial solution of a stochastic differential equation is called stochastically asymptotically stable in probability if for every $\varepsilon \in (0,1)$ there exists $\delta = \delta(\varepsilon)$ such that

$$\mathbb{P}\left(\lim_{t\to\infty} x(t) = 0\right) \geq 1 - \varepsilon,$$

whenever $\|x(0)\| < \delta$.

2. (See Theorem 2.4 in [4]) If there exists a positive-define decreasing unbounded fucntion $V(x,t) \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$ such that $Lv(x,t)$ is negative-definitive, the the trivial solution of the stochastic differential equation is stochastically stable in the large.

3. (See Theorem 7.1 in [5]) If the trivial solution of the linear system associated to a stochastic differential equation is stochastically asymptotically stable, then the trivial solution with respect to the stochastic differential equation is stochastically stable.

From the above results, we developed the following result:
Theorem 2.3. *Consider the stochastic model (1.2), then the trivial solution of (1.2) is stochastically asymptotically stable in probability if the following conditions holds:*

1. $\sigma^2 < \lambda$,
2. $\eta > \eta f$,
3. $(\sigma^2 - \lambda)(\eta - \eta f) < \dfrac{\lambda^2}{2}$.

Proof:
According to results showed previously , the model (1.2) should be linearised around the trivial solution, thus

$$\begin{cases} dS = -\eta dt, \\ dT = \eta f dt, \\ dI = -\lambda I dt - \sigma I dB(t), \\ dR = (\lambda I + \eta - \eta f)dt. \end{cases} \tag{2.1}$$

Now, let

$$\begin{aligned} q_1 &= -\eta, \\ q_2 &= \eta f, \\ q_3 &= -\lambda I, \\ q_4 &= \lambda I + \eta - \eta f, \\ u_1 &= 0, \\ u_2 &= 0, \\ u_3 &= -\sigma I, \\ u_4 &= 0. \end{aligned}$$

It is enough to define a function $V(S,T,I,R,t) \in C^{2,1}(\mathbb{R}^4_+ \times \mathbb{R}_+; \mathbb{R}_+)$ positive-definite. Now, we define Lyapunov's operator associated to (1.1) is given by:

$$L = q_1 \frac{\partial}{\partial S} + q_2 \frac{\partial}{\partial T} + q_3 \frac{\partial}{\partial I} + q_4 \frac{\partial}{\partial R} + \frac{1}{2} u_1^2 \frac{\partial^2}{\partial S^2} + \frac{1}{2} u_2^2 \frac{\partial^2}{\partial T^2} + \frac{1}{2} u_3^2 \frac{\partial^2}{\partial I^2} + \frac{1}{2} u_4^2 \frac{\partial^2}{\partial R^2}.$$

Let's define

$$V(S,T,I,R,t) = c_1 S^2 + c_2 T^2 + c_3 I^2 + c_4 R^2,$$

where will be represented by $c_i, i = 1,2,3,4$, are non-negative constants. Applying L in V, we have

$$LV = -2c_1 S\eta + 2c_2\eta f T - 2c_3\lambda I^2 + 2c_4\lambda IR + 2c_4\eta R - 2c_4\eta f R + 2c_3\sigma^2 I^2.$$

$$LV = (-2c_3\lambda + 2c_3\sigma^2)I^2 + 2c_4\lambda IR + (2c_4\eta - 2c_4\eta f)R - 2c_1\eta S + 2c_2\eta f T.$$

Taking $c_1 = \dfrac{1}{2\eta}$, $c_2 = \dfrac{1}{2\eta f}$, $c_3 = \dfrac{1}{2}$ and $c_4 = \dfrac{1}{2}$, we obtain

$$LV = (\sigma^2 - \lambda)I^2 + \lambda IR + (\eta - \eta f)R - S + T.$$

Therefore, LV can be represented by $LV = AI_1^2 + BRI + CR + DS + ET$, where $A = \sigma^2 - \lambda$, $B = \lambda$, $C = \eta - \eta f$, $D = -1$ and $E = 1$. Now, consider the vector $y := (S, T, I, R)$ and coefficient matrix Q

$$\begin{pmatrix} D & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & 2A & B \\ 0 & 0 & B & C \end{pmatrix},$$

we can rewrite $LV = AI_1^2 + BRI + CR + DS + ET$ in its quadratic form $LV = \dfrac{1}{2}y^T Q y$. Now, let's define the following sub-matrices on Q,

$$Q_1 = \begin{pmatrix} 2A & B \\ B & C \end{pmatrix}, Q_2 = 2A, Q_3 = \eta - \eta f, Q_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2A \end{pmatrix}.$$

Thus, LV is negative-definitive since $det(Q_1) < 0$ if $(\sigma^2 - \lambda)(\eta - \eta f) < \frac{\lambda^2}{2}$; $det(Q_2) < 0$ if $\sigma^2 < \lambda$; $det(Q_3) > 0$ if $\eta > \eta f$ and $det(Q_4) < 0$ if $\sigma 2 < \lambda$. Therefore, LV is negative-definitive for the trajectories in $\mathbb{R}^4_+$, except in the point $(0, 0, 0, 0)$. □

3. Numerical Simulations. In this section, we show some simulations cases taking into account conditions obtained in the previous section and their variations. In the following simulations, $S \equiv x$, $T \equiv y$, $I \equiv z$ and $R \equiv w$.

Simulation 1: At $t = 30$ with initial values $(S, T, I, R) \equiv (30, 15, 10, 20)$ and with parameters $a = 0.2$, $\eta = 7$, $f = 0.1$, $b = 0.02$, $\lambda = 0.1$ and $\sigma = 0.05$. We have $(S, T, I, R) \equiv (-38, 55, 7, 307)$.

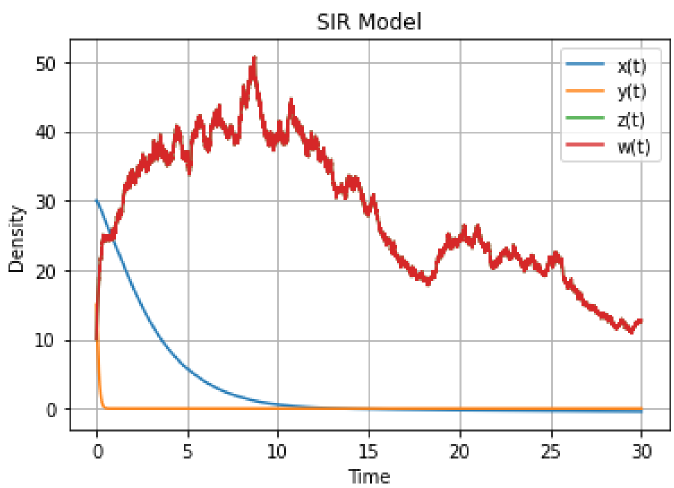

Figure 1: Stochastic trajectories with white noise on the system are described by (1.2).

Simulation 2: At $t = 30$ with initial values $(S, T, I, R) \equiv (30, 15, 10, 20)$ and with parameters $a = 0.2$, $\eta = 7$, $f = 0.1$, $b = 0.02$, $\lambda = 0.1$ and $\sigma = 0.05$. We have $(S, T, I, R) \equiv (-0.04, 0, 392, 1398)$.

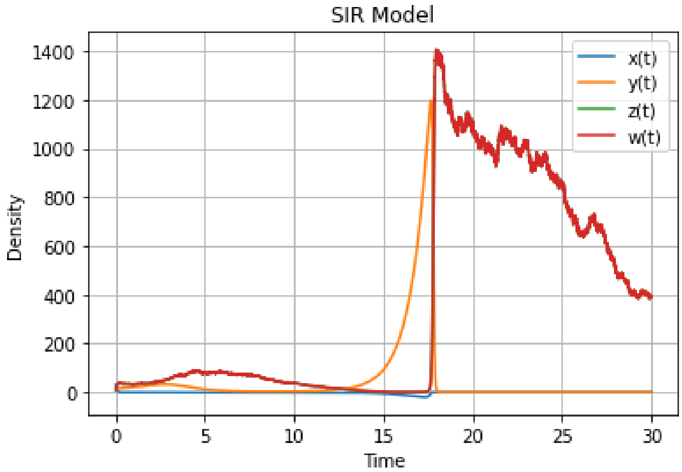

Figure 2: Stochastic trajectories with white noise on the system are described by (1.2) with extinction of population remain susceptible after having been vaccinated.

Simulation 3: At $t = 30$ with initial values $(S, T, I, R) \equiv (30, 15, 10, 20)$ and with parameters $a = 0.2$, $\eta = 10$, $f = 0.1$, $b = 0.4$, $\lambda = 0.5$ and $\sigma = 0.09$. We have $(S, T, I, R) \equiv (-274, 93, 0, 325)$.

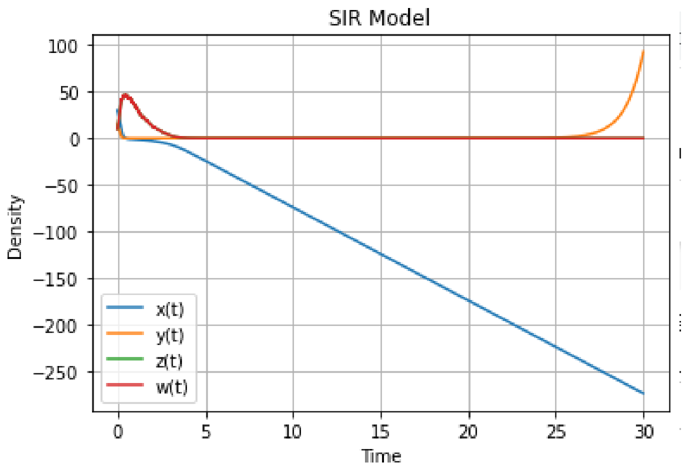

Figure 3: Stochastic trajectories with white noise on the system are described by (1.2) with extinction of population which is infected .

Simulation 4: At $t = 30$ with initial values $(S, T, I, R) \equiv (30, 15, 10, 20)$ and with parameters $a = 0.3$, $\eta = 7$, $f = 0.7$, $b = 0.1$, $\lambda = 0.1$ and $\sigma = 0.5$. We have $(S, T, I, R) \equiv (-0.2, 0, 98, 997)$.

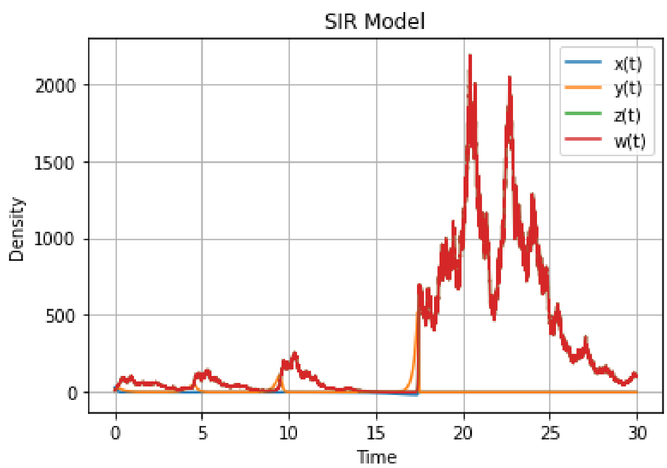

Figure 4: Stochastic trajectories with white noise on the system are described by (1.2) with extinction of population remain susceptible after having been vaccinated (situation 1).

Simulation 5: At $t = 10$ with initial values $(S, T, I, R) \equiv (30, 15, 10, 20)$ and with parameters $a = 0.3$, $\eta = 0.01$, $f = 0.01$, $b = 0.3$, $\lambda = 0.01$ and $\sigma = 0.04$. We have $(S, T, I, R) \equiv (0, 0, 48, 25)$.

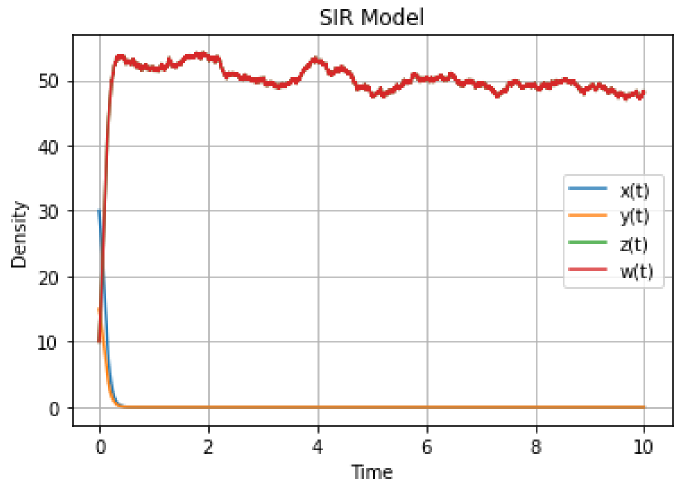

Figure 5: Stochastic trajectories with white noise on the system are described by (1.2) with extinction of population of susceptible and people who remain susceptible after having been vaccinated, and infected and recovered maintain persistent.

Simulation 6: At $t = 10$ with initial values $(S, T, I, R) \equiv (30, 15, 10, 20)$ and with parameters $a = 0.01$, $\eta = 0.8$, $f = 0.85$, $b = 0.04$, $\lambda = 0.75$ and $\sigma = 0.1$. We have $(S, T, I, R) \equiv (0, 0, 48, 25)$.

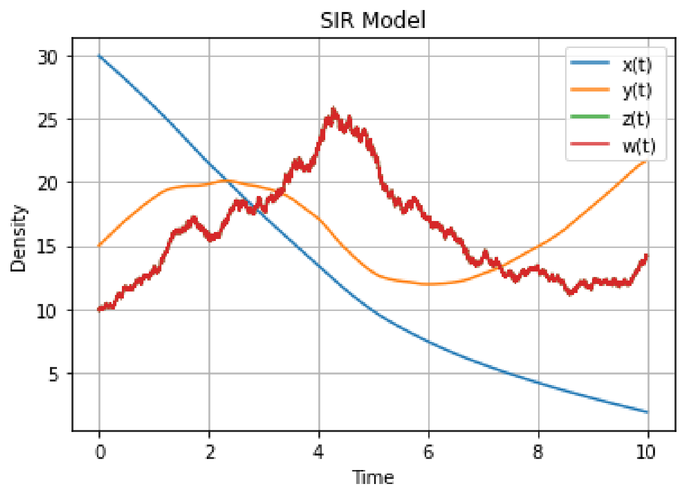

Figure 6: Stochastic trajectories with white noise on the system are described by (1.2) with extinction of population of susceptible and people who remain susceptible after having been vaccinated, but there is not persistent.

Simulation 7: At $t = 10$ with initial values $(S, T, I, R) \equiv (30, 15, 10, 20)$ and with parameters $a = 0.01$, $\eta = 0.8$, $f = 0.85$, $b = 0.8$, $\lambda = 0.75$ and $\sigma = 0.1$. We have $(S, T, I, R) \equiv (0, 0, 48, 25)$.

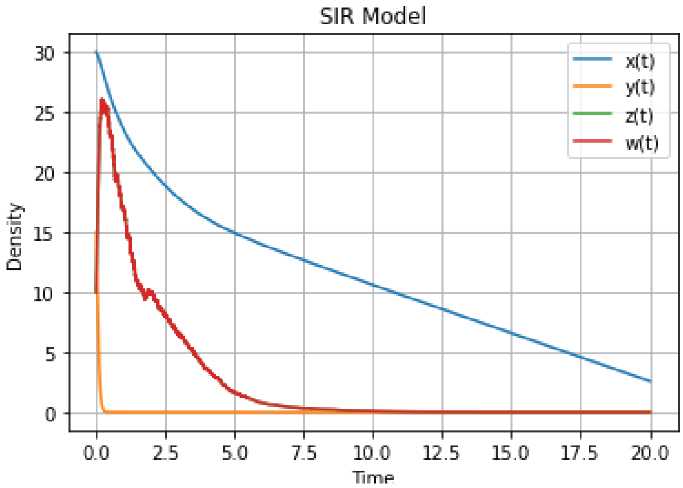

Figure 7: Stochastic trajectories with white noise on the system are described by (1.2) extinction of population of susceptible and people who remain susceptible after having been vaccinated, and extinction of infected and recovered for some $t \to \infty$.

Simulation 8: At $t = 30$ with initial values $(S, T, I, R) \equiv (30, 15, 10, 20)$ and with parameters $a = 0.01$, $\eta = 0.08$, $f = 0.05$, $b = 0.5$, $\lambda = 0.05$ and $\sigma = 0.1$. We have $(S, T, I, R) \equiv (-0.45, 0, 13, 64)$.

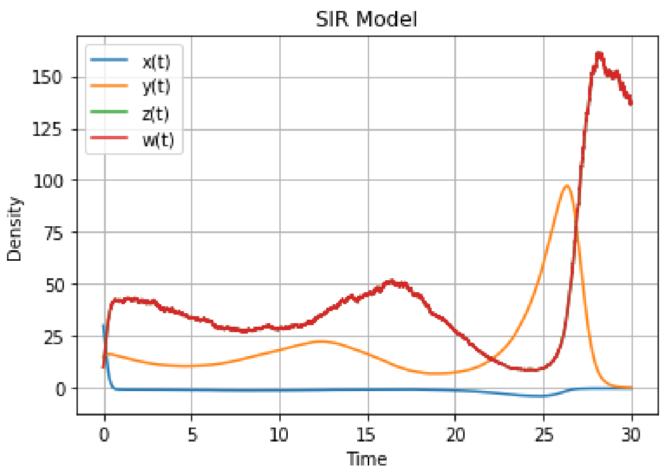

Figure 8: Stochastic trajectories with white noise on the system are described by (1.2) with extinction of population remain susceptible after having been vaccinated (situation 2).

4. Conclusion. In this paper, we have taken recuperate rate of the infected individuals (λ), and we have considered that λ is susceptible to a white noise (environment changes). This was made to aim to show what would happen with the system (1.2). The results showed that the system maintain nearby $(0, 0, 0, 0) \in \mathbb{R}^4_+$. Besides, with the simulations cases, we saw that the best condition to maintain the system is that infection rate is low as possible, and it does not matter the vaccine efficiency. But, if me make some chances in the parameters, population of people who are susceptible after having been vaccinated, get extinction and at the end we obtain a SIR common model. Moreover, other cases under which infected and whole the system get extinction.

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Conflicts of Interest. The author declares no conflict of interest.

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