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Well-posedness for a Thir-Order PDE with Dissipation

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Abstract

In this work, we prove that the Cauchy problem associated with a third-order equation with dissipation in periodic Sobolev spaces admits a unique solution. We also show that the solution depends continuously on the initial data. Our approach combines both an intuitive method, based on Fourier theory, and a more abstract framework using semigroup theory. Furthermore, by employing an alternative method, we demonstrate the uniqueness of the solution through its dissipative nature, drawing inspiration from the contributions of Iorio [1] and Santiago [2]. To deepen and enrich our study, we investigate the infinite dimensional space in which differentiability occurs and its connection to the initial data. Finally, we extend our results to equations of arbitrary nth order.

Keywords . Semigroups theory, third-order equation, dissipative property of problem, nth order equation, Periodic Sobolev spaces, Fourier Theory.

1. Introduction. We will begin studying the following problem:

$$(P_1): \qquad u_t + u_{xxx} + au = 0 \text{ in } H^{s-3}_{per} \text{ with } u(0) = \varphi \in H^s_{per},$$

considering $a > 0$, s a real number and denoting H^s_{per} as the periodic Sobolev space. We will prove that (P_1) problem is well-posed. Note that, by perturbing the third-order conservative system studied in [3] we will obtain that (P_1) is a dissipative system. In addition, we will give a family of operators that becomes a semigroup, achieving beautiful results through operators and differential calculus in Banach spaces. To deepen and enrich our study, we will investigate the infinite-dimensional space in which differentiability occurs and its connection to the initial data. Finally, we will generalize the results to the nth-order equation.

We can cite [3], where we find some results related to the conservative part of model (P_1). And we cite [1] for being a source of inspiration for this work. We also mention some works on existence of solutions by semigroups [4], [5], [6] and take support in some results of [7] and [8]. The structure of our article is as follows. In section 2, we outline the methodology used and provide the citations for the references consulted. In section 3, we prove that problem (P_1) is well posed. Moreover, we introduce a family of operators that form a semigroup of class C_o to state the result Theorem 3.3 and prove it in an abstract version. In section 4, we study the generalization to n-th order equation. Here we use the semigroups theory of contraction, and obtain important results of approximation, existence and regularity. In section 5, we obtain other results related to the dissipative property of (P_1) and some estimates of it, through the use of differential calculus on H^s_{per}. Also, we get their generalization and some remarks. In section 6, to deepen and enrich our study, we investigate the infinite-dimensional space in which differentiability occurs and its connection to the initial data. Finally, in section 7, we present the conclusions of our study.

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2. Methodology . In this article, we mainly employ [9] as the theoretical framework. In addition, we use the references [3], [1], [10] and [2] for the Fourier theory in H^s_{per}, and differential and integral calculus in Banach spaces. We'll quickly present some definitions and results to make it easier to read. Let be

$$\begin{aligned} P &:= C^\infty_{per}([-\pi,\pi]) \text{ and} \\ P' &:= \text{ the Topological Dual of } P\,. \end{aligned}$$

For $s \in R$ we define

$$H^s_{per}([-\pi,\pi]) := \left\{ f \in P' \text{ such that } \sum_{k=-\infty}^{+\infty} (1+k^2)^s |\widehat{f}(k)|^2 < \infty \right\}.$$

Then H^s_{per} is a Hilbert space with inner product

$$< f, g >_s = 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s \widehat{f}(k)\overline{\widehat{g}(k)} \;\text{ for }\; f, g \in H^s_{per}\,.$$

So, H^s_{per} is known as the periodic Sobolev space and satisfies the following result.

Proposition 2.1. *Let $s, r \in R$ such that $s \geq r$ then $H^s_{per} \subset H^r_{per}$. That is , H^s_{per} is imbedded continuously and densely in H^r_{per} and $\|u\|_r \leq \|u\|_s$, $\forall u \in H^s_{per}$. In particular, if $s \geq 0$ then $H^s_{per} \subset L^2([-\pi,\pi])$. Moreover, it is valid the "isometrically isomorphic" identication, that is $(H^s_{per})' \equiv H^{-s}_{per}$ $\forall s \in R$, where the duality is implemented by the pair*

$$< f, g >_* = 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k)\widehat{g}(k)\,, \forall f \in H^{-s}_{per},\ \forall g \in H^s_{per}\,.$$

Proof: We cite [1]. □

3. The (P_1) problem is well-posed. We will prove that (P_1) is well-posed. Also, we will introduce a family of operators that form a contraction semigroup of class C_o, as we will make it in Theorem 3.2.

Finally, we will state Theorem 3.3 whose content is a fine version of Theorem 3.1 based on the semi-group $\{S(t)\}_{t\geq 0}$.

Theorem 3.1. *Let s be a fixed real number, $a > 0$ and*

$$(P_1) \quad \left| \begin{array}{l} u \in C([0,+\infty), H^s_{per}), \\ \partial_t u + \partial_x^3 u + au = 0 \;\in H^{s-3}_{per}, \\ u(0) = \psi \in H^s_{per}\,. \end{array} \right.$$

then (P_1) is globally well-posed, that is, $\exists u \in C([0,\infty), H^s_{per}) \cap C((0,\infty), H^{s-3}_{per})$ satisfying equation (P_1) so that the application: $\psi \to u$, which to every initial data ψ assigns the solution u of the IVP (P_1), is continuous. That is, for ψ and $\tilde{\psi}$ initial data close in H^s_{per}, their corresponding solutions u and $\tilde{u}$, respectively, are also close in the solution space.

In addition,

$$\|u(t) - \tilde{u}(t)\|_s \leq \|\psi - \tilde{\psi}\|_s\,,\ \forall t \in [0,+\infty),$$

and

$$\sup_{t>0} \|u(t) - \tilde{u}(t)\|_s \leq \|\psi - \tilde{\psi}\|_s,$$

are verified.

Moreover, the solution u satisfies $u(t) \in H^r_{per}$, $\forall t \geq 0$, $\forall r \leq s$ with $\|u(t)\|_s \leq \|\psi\|_s$ and $\|u(t)\|_r \leq \|\psi\|_r$, $\forall r < s$, $\forall t \geq 0$.

Also, the application: $\psi \to \partial_t u$, which to every initial data ψ assigns the derivate of solution u of the IVP (P_1), is continuous. That is, for ψ and $\tilde{\psi}$ initial data close in H^s_{per}, their corresponding $\partial_t u$ and $\partial_t \tilde{u}$, respectively, are also close in the solution space.

Additionally, the following inequalities are satisfied

$$\|\partial_t u(t) - \partial_t \tilde{u}(t)\|_{s-3} \leq \sqrt{\max\{1, a^2\}}\|\psi - \tilde{\psi}\|_s\,,\ \forall t \in (0,+\infty),$$

and

$$\sup_{t>0} \|\partial_t u(t) - \partial_t \tilde{u}(t)\|_{s-3} \leq \sqrt{\max\{1, a^2\}} \|\psi - \tilde{\psi}\|_s \, .$$

Moreover, $\|\partial_t u(t)\|_{s-3} \leq \sqrt{\max\{1, a^2\}} \|\psi\|_s \, , \ \forall t > 0.$

Proof: We have organized the demonstration in the following way:

1. First, we will obtain the candidate for the solution. To achieve this, we apply the Fourier transform to

$$\partial_t u = -\partial_x^3 u - au$$

and obtain

$$\partial_t \widehat{u} = -(ik)^3 \widehat{u} - a\widehat{u} = (ik^3 - a)\widehat{u} \, ,$$

which for every $k \in Z$ is an ODE with initial data $\widehat{u}(k, 0) = \tilde{\psi}(k)$.
Therefore, solving the IVP's

$$(\Omega_k) \quad \left| \begin{array}{l} u \in C([0, +\infty), l_s^2(Z)), \\ \partial_t \widehat{u}(k, \tau) = ik^3 \widehat{u}(k, \tau) - a\widehat{u}(k, \tau), \\ \widehat{u}(k, 0) = \widehat{\psi}(k), \end{array} \right.$$

we obtain

$$\widehat{u}(k, \tau) = e^{ik^3\tau} e^{-a\tau} \widehat{\psi}(k) \, ,$$

from which we get our candidate for the solution:

$$u(\tau) = \sum_{k=-\infty}^{+\infty} \widehat{u}(k, \tau)\varphi_k = \sum_{k=-\infty}^{+\infty} e^{ik^3\tau} e^{-a\tau} \widehat{\psi}(k)\varphi_k \, ; \tag{3.1}$$

here we are denoting $\varphi_k(x) = e^{ikx}$ for $x \in R$.

2. Second, we will prove

$$u(\tau) \in H_{per}^s \text{ and } \|u(\tau)\|_s \leq \|\psi\|_s \, , \ \forall \tau \geq 0 \, . \tag{3.2}$$

Indeed, let $\tau > 0$, $\psi \in H_{per}^s$, using $|e^{ik^3\tau}| = 1$ and $0 < e^{-2a\tau} < 1$, we have

$$\begin{aligned} \|u(\tau)\|_s^2 &= 2\pi \sum_{k=-\infty}^{+\infty} (1 + k^2)^s \cdot |e^{ik^3\tau} e^{-a\tau} \widehat{\psi}(k)|^2, \\ &= 2\pi \sum_{k=-\infty}^{+\infty} (1 + k^2)^s e^{-2a\tau} |\widehat{\psi}(k)|^2 < \infty, \\ &\leq \|\psi\|_s^2 \, . \end{aligned} \tag{3.3}$$

It is evident that (3.2) is satisfied for $\tau = 0$.

3. We will prove that $u(\cdot)$ is continuous in $[0, +\infty)$. Indeed, let $t', t \in (0, \infty)$, we obtain

$$\begin{aligned} \|u(t) - u(t')\|_s^2 &= 2\pi \sum_{k=-\infty}^{+\infty} (1 + k^2)^s \cdot |(e^{ik^3t} e^{-at} - e^{ik^3t'} e^{-at'})\widehat{\psi}(k)|^2, \\ &= 2\pi \sum_{k=-\infty}^{+\infty} (1 + k^2)^s |H(t)|^2 |\widehat{\psi}(k)|^2, \end{aligned} \tag{3.4}$$

where $H(t) := e^{ik^3t} e^{-at} - e^{ik^3t'} e^{-at'}$.

We see that $\lim_{t\to t'} H(t) = 0$.
To ensure the interchange of limits we need the uniform convergence of series (3.4). For this we will bound the k-th term of the series, that is

$$\begin{aligned} I_{k,t}: &= 2\pi(1+k^2)^s|\widehat{\psi}(k)|^2\left|e^{ik^3t}e^{-at} - e^{ik^3t'}e^{-at'}\right|^2 \\ &\leq 8\pi(1+k^2)^s|\widehat{\psi}(k)|^2\,, \end{aligned}$$

where we have used the triangle inequality, the equality $|e^{i\theta}| = 1$ for $\theta \in R$ and $0 < e^{-at} \leq 1$ for $a > 0$, $t \in [0,\infty)$. Thus,

$$\sum_{k=-\infty}^{+\infty} I_{k,t} \leq 4\|\psi\|_s^2 < \infty\,,$$

and using the Weierstrass M-Test, we obtain that the series (3.4) converges uniformly. So, we can interchange limits and get

$$\lim_{t\to t'}\|u(t) - u(t')\|_s^2 = \sum_{k=-\infty}^{+\infty}\underbrace{\lim_{t\to t'} I_{k,t}}_{=0} = 0,$$

and then we conclude

$$\lim_{t\to t'}\|u(t) - u(t')\|_s = 0\,.$$

It is evident that u is continuous to the right of zero. To prove this, we use the same technique, considering $t' = 0$.

4. Let $t > 0$ and $t + h > 0$, we will prove

$$\left\|\frac{u(t+h) - u(t)}{h} + \partial_x^3 u(t) + au(t)\right\|_{s-3} \longrightarrow 0 \text{ when } h \to 0\,.$$

Indeed, let $t + h > 0$,

$$\begin{aligned} &\left\|\frac{u(t+h) - u(t)}{h} + \partial_x^3 u(t) + au(t)\right\|_{s-3}^2, \\ &= 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{s-3}|\widehat{\psi}(k)|^2\left|\frac{e^{ik^3(t+h)}e^{-a(t+h)} - e^{ik^3t}e^{-at}}{h} + [(ik)^3 + a]e^{ik^3t}e^{-at}\right|^2, \\ &= 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{s-3}|\widehat{\psi}(k)|^2|e^{ik^3t}e^{-at}.M(h)|^2, \end{aligned} \tag{3.5}$$

where $M(h) := \left\{\frac{e^{ik^3h}e^{-ah}-1}{h} - ik^3 + a\right\}$.
Applying L'Hôpital's rule we obtain $M(h) \to 0$ when $h \to 0$.
To ensure the interchange of limits, we need the uniform convergence of the series (3.5). For this we will bound the k-th term of the series. Previously, for $h > 0$, we analyse

$$\begin{aligned} \frac{e^{ik^3h}e^{-ah} - 1}{h} &= \int_0^h \frac{1}{h}\frac{\partial}{\partial r}\{e^{ik^3r}e^{-ar}\}dr, \\ &= \int_0^h \frac{1}{h}[ik^3 - a]e^{ik^3r}e^{-ar}dr, \end{aligned}$$

and taking norm, we have

$$\begin{aligned} \left|\frac{e^{ik^3h}e^{-ah} - 1}{h}\right| &\leq \frac{1}{h}|ik^3 - a|\int_0^h |e^{ik^3r}|e^{-ar}dr \\ &\leq \frac{1}{h}(|k|^6 + a^2)^{\frac{1}{2}}h \\ &= (|k|^6 + a^2)^{\frac{1}{2}}\,. \end{aligned} \tag{3.6}$$

Using the inequality (3.6), we are going to bound $|M(h)|^2$ in the following way:

$$\begin{aligned} |M(h)|^2 &\leq 4\max\{1,a^2\}(k^6+1), \\ &= 4\max\{1,a^2\}(1+(k^2)^3), \\ &\leq 4\max\{1,a^2\}(1+k^2)^3 . \end{aligned} \tag{3.7}$$

For $h<0$ such that $0<t+h$ we analyse

$$\begin{aligned} \frac{e^{ik^3t}e^{-at}-e^{ik^3(t+h)}e^{-a(t+h)}}{h} &= \int_{t+h}^{t}\frac{1}{h}\frac{\partial}{\partial r}\{e^{ik^3r}e^{-ar}\}dr, \\ &= \int_{t+h}^{t}\frac{1}{h}[ik^3-a]e^{ik^3r}e^{-ar}dr, \end{aligned}$$

and taking norm, we have

$$\begin{aligned} \left|\frac{e^{ik^3t}e^{-at}-e^{ik^3(t+h)}e^{-a(t+h)}}{h}\right| &\leq \frac{1}{|h|}|ik^3-a|\int_{t+h}^{t}|e^{ik^3r}|e^{-ar}dr, \\ &\leq \frac{1}{|h|}(k^6+a^2)^{\frac{1}{2}}(-h)=(k^6+a^2)^{\frac{1}{2}}, \\ &\leq \sqrt{2\max\{1,a^2\}}\cdot(1+k^2)^{\frac{3}{2}}. \end{aligned} \tag{3.8}$$

Let us bound the k-th term of the series (3.5) using estimations (3.7) for $h>0$ and (3.8) for $h<0$, and denoting $8\max\{1,a^2\}$ by C_a :

$$\begin{aligned} (1+k^2)^{s-3}|\widehat{\psi}(k)|^2e^{-2at}|M(h)|^2 &\leq (1+k^2)^{s-3}|\widehat{\psi}(k)|^2C_a(1+k^2)^3 \\ &= C_a(1+k^2)^s|\widehat{\psi}(k)|^2 . \end{aligned}$$

Therefore, since

$$2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^s|\widehat{\psi}(k)|^2=\|\psi\|_s^2<\infty,$$

for $\psi\in H^s_{per}$ and using the Weierstrass M-Test we get that the series (3.5) converges uniformly. So, it is possible to interchange of limits and obtain

$$\left\|\frac{u(t+h)-u(t)}{h}+\partial_x^3u(t)+au(t)\right\|_{s-3}^2 \longrightarrow 0 \text{ when } h\to 0 . \tag{3.9}$$

5. We will demonstrate the continuous dependence of the solution with respect to the initial data, that is, let ψ and $\tilde{\psi}$ be close data in H^s_{per}, then their corresponding solutions u and $\tilde{u}$, respectively, are also close in the solution space. Let $t>0$,

$$\begin{aligned} \|u(t)-\tilde{u}(t)\|_s^2 &= 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^s\left|e^{ik^3t}e^{-at}\left\{\widehat{\psi}(k)-\widehat{\tilde{\psi}}(k)\right\}\right|^2, \\ &\leq 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^s\left|\widehat{\psi}(k)-\widehat{\tilde{\psi}}(k),\right|^2 \\ &= \|\psi-\tilde{\psi}\|_s^2 . \end{aligned} \tag{3.10}$$

Taking supremum over $(0,\infty)$ we have

$$\sup_{t\in(0,\infty)}\|u(t)-\tilde{u}(t)\|_s\leq\|\psi-\tilde{\psi}\|_s . \tag{3.11}$$

Hence, we have: if $\psi\to\tilde{\psi}$ then $u\to\tilde{u}$.

6. Uniqueness of solution. - Inequalities (3.11) or (3.10) will enable us to demonstrate that the solution is unique. Effectively, let $\psi\in H^s_{per}$ and suppose there are two solutions u and $\tilde{u}$, then using (3.11) we have,

$$\|u(\tau)-\tilde{u}(\tau)\|_s\leq\sup_{t\in(0,\infty)}\|u(t)-\tilde{u}(t)\|_s\leq\|\psi-\psi\|_s=0\,,\ \forall\tau>0 .$$

from where we conclude that $u = \tilde{u}$.
Thus, the (P_1) problem is well-posed, and its unique solution is

$$u(t) = \sum_{k=-\infty}^{+\infty} e^{ik^3 t} e^{-at} \widehat{\psi}(k) \varphi_k,$$

which depends continuously on the initial data.

7. Now, we analyse the case $r < s$. Under this condition we have $H^s_{per} \subset H^r_{per}$ and since the initial data $\psi \in H^s_{per}$, then $\psi \in H^r_{per}$ and satisfies

$$\|\psi\|_r \leq \|\psi\|_s \,. \tag{3.12}$$

From (3.3) for r and using (3.12) we get

$$\|u(t)\|_r^2 \leq \|\psi\|_r^2 \leq \|\psi\|_s^2 < \infty \,.$$

That is,

$$u(t) \in H^r_{per} \,,\ \forall r \in (-\infty, s) \,. \tag{3.13}$$

The case $r = s$ has already been proved in item 2.
Therefore, from (3.2) and (3.13), we conclude for $t \in [0, +\infty)$ that

$$u(t) \in H^r_{per} \,,\ \forall r \in (-\infty, s] \,.$$

8. We will demonstrate that $\partial_t u(\cdot)$ is continuous in $(0, +\infty)$. Let $t > 0$ and $t' > 0$, using the inequality $\|\partial_x^m u(t)\|_{s-m} \leq \|u(t)\|_s$ and continuity of $u(\cdot)$ we obtain

$$\begin{aligned} \|\partial_t u(t) - \partial_t u(t')\|_{s-3} &= \| - \partial_x^3 u(t) - au(t) + \partial_x^3 u(t') + au(t')\|_{s-3}, \\ &\leq \|\partial_x^3 [u(t) - u(t')]\|_{s-3} + a\|u(t) - u(t')\|_{s-3}, \\ &\leq (1 + a)\|u(t) - u(t')\|_s \to 0, \end{aligned} \tag{3.14}$$

when $t \to t'$. That is, $\partial_t u \in C((0, \infty), H^{s-3}_{per})$.

9. Let $\psi \in H^s_{per}$, if we define

$$W(t)\psi = \sum_{k=-\infty}^{+\infty} (k^3 i - a) e^{ik^3 t} e^{-at} \widehat{\psi}(k) \varphi_k,$$

then $W(t)\psi \in H^{s-3}_{per}$ and $\|W(t)\psi\|_{s-3} \leq [\max\{1, a^2\}]^{\frac{1}{2}} \|\psi\|_s$, $\forall t > 0$. That is, $W(t) \in L(H^s_{per}, H^{s-3}_{per})$ with $\|W(t)\| \leq [\max\{1, a^2\}]^{\frac{1}{2}}$.
In effect, using, $|k^3 i - a|^2 = (k^6 + a^2) \leq \max\{1, a^2\}(k^6 + 1) \leq \max\{1, a^2\}(1 + k^2)^3$, $\forall k \in Z$, the equality $|e^{i\theta}| = 1$, $\forall \theta \in R$ and $0 < e^{-at} < 1$, for $a > 0$, $t > 0$, we have

$$\begin{aligned} \|W(t)\psi\|_{s-3}^2 &= 2\pi \sum_{k=-\infty}^{+\infty} (1 + k^2)^{s-3} |(k^3 i - a) e^{ik^3 t} e^{-at} \widehat{\psi}(k)|^2, \\ &\leq 2\pi \sum_{k=-\infty}^{+\infty} (1 + k^2)^{s-3} |k^3 i - a|^2 |\widehat{\psi}(k)|^2, \\ &\leq \max\{1, a^2\} 2\pi \sum_{k=-\infty}^{+\infty} (1 + k^2)^s |\widehat{\psi}(k)|^2 < \infty, \\ &= \max\{1, a^2\} \|\psi\|_s^2 \,. \end{aligned}$$

10. By considering items 4 and 9, we conclude $\partial_t u(t) = W(t)\psi$.

□

An immediate consequence is the following Corollary.
Corollary 3.1. *The unique solution of (P_1) is*

$$u(t) = \sum_{k=-\infty}^{+\infty} e^{ik^3 t} e^{-at} \widehat{\psi}(k) \varphi_k$$

where $\varphi_k(x) = e^{ikx}$ for $x \in R$.
Also, we obtain the following result.
Corollary 3.2. *Based on the hypothesis of the previous Theorem, we obtain*

1. $u \in C([0,\infty), H^r_{per}) \cap C^1((0,\infty), H^{r-3}_{per})$, $\forall r < s$.
2. *u satisfies*

$$\begin{aligned} \|u(t)\|_r &\leq \|\psi\|_s\,,\ \forall t \geq 0\,,\ \forall r < s\,. && (3.15)\\ \|\partial_t u(t)\|_{r-3} &\leq \sqrt{\max\{1,a^2\}}\|\psi\|_s\,,\ \forall t > 0\,,\ \forall r < s\,. && (3.16)\end{aligned}$$

3. *That is,*

$$\begin{aligned} \|u(t)-\tilde{u}(t)\|_r &\leq \|\psi-\tilde{\psi}\|_s\,,\ \forall t\geq 0\,,\ \forall r<s\,,\\ \sup_{t\in(0,\infty)}\|u(t)-\tilde{u}(t)\|_r &\leq \|\psi-\tilde{\psi}\|_s\,,\ \forall r<s\,.\end{aligned}$$

4. *Moreover*

$$\begin{aligned} \|\partial_t u(t)-\partial_t\tilde{u}(t)\|_{r-3} &\leq \sqrt{\max\{1,a^2\}}\|\psi-\tilde{\psi}\|_s\,,\ \forall t>0\,,\ \forall r\leq s\,,\\ \sup_{t\in(0,\infty)}\|\partial_t u(t)-\partial_t\tilde{u}(t)\|_{r-3} &\leq \sqrt{\max\{1,a^2\}}\|\psi-\tilde{\psi}\|_s\,,\ \forall r<s\,.\end{aligned}$$

Proof: Through the use of the continuous Sobolev embedding, we obtain the inequality (3.15). We will use the continuous Sobolev embedding and item 9 for prove that if $\psi \in H^s_{per}$ then $W(t)\psi \in H^{r-3}_{per}$ and $\|W(t)\psi\|_{r-3} \leq \sqrt{\max\{1,a^2\}}\|\psi\|_s\,,\ \forall t > 0\,,\ \forall r < s$. That is, $W(t) \in L(H^s_{per}, H^{r-3}_{per})$ with $\|W(t)\| \leq \sqrt{\max\{1,a^2\}}$, $\forall r < s$.

In effect, using $|k^3 i - a|^2 \leq \max\{1,a^2\}(1+k^2)^3$, $\forall k \in Z$ and $|e^{i\theta}| = 1$, $\forall \theta \in R$ we have

$$\begin{aligned} \|W(t)\psi\|^2_{r-3} &= 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{r-3}|(k^3 i-a)e^{ik^3t}e^{-at}\widehat{\psi}(k)|^2\\ &\leq 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{r-3}|k^3 i-a|^2|\widehat{\psi}(k)|^2\\ &\leq \max\{1,a^2\}2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{r}|\widehat{\psi}(k)|^2\\ &\leq \max\{1,a^2\}2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{s}|\widehat{\psi}(k)|^2 < \infty\\ &= \max\{1,a^2\}\|\psi\|^2_s\,. && (3.17)\end{aligned}$$

□

In this point, we will introduce a family of operators which will meet the requirement of being a contraction semigroup of class C_o.

Theorem 3.2. *Let $s \in R$ and $a > 0$. The application*

$$\begin{aligned} \mathcal{S} : [0,\infty) &\to L(H^s_{per}),\\ t &\to \mathcal{S}(t), \end{aligned}$$

such that $\mathcal{S}(t) = e^{-(\partial_x^3 + aI)t}$, that is, applies

$$\mathcal{S}(t)\varphi = \left[\left(e^{(ik^3-a)t}\widehat{\varphi}(k)\right)_{k\in Z}\right]^{\vee}\,,\ \varphi \in H^s_{per}\,,$$

then $\{\mathcal{S}(t)\}_{t\geq 0}$ is a contraction semigroup of class C_o on H^s_{per}.
Additionally, the following statements hold:

1. *If $\varphi \in H^s_{per}$ then $\mathcal{S}(\cdot)\varphi \in C([0,\infty), H^s_{per})$.*
2. *The application: $\varphi \to \mathcal{S}(\cdot)\varphi$ is continuous and verifies:*

$$\|\mathcal{S}(t)\psi_1 - \mathcal{S}(t)\psi_2\|_s \leq \|\psi_1-\psi_2\|_s\,, \forall t \in [0,\infty),$$

and

$$\sup_{t\in[0,\infty)}\|\mathcal{S}(t)\psi_1 - \mathcal{S}(t)\psi_2\|_s \leq \|\psi_1-\psi_2\|_s$$

with $\psi_j \in H^s_{per}$ for $j \in \{1,2\}$.

3. *If $\varphi \in H^s_{per}$ then $\partial_t \mathcal{S}(t)\varphi \in H^{s-3}_{per}$ and $\|\partial_t \mathcal{S}(t)\varphi\|_{s-3} \leq \sqrt{\max\{1, a^2\}}\|\varphi\|_s$, $\forall t \in (0, \infty)$. That is, $\partial_t \mathcal{S}(t) \in L(H^s_{per}, H^{s-3}_{per})$, $\forall t \in (0, \infty)$, where*

$$\partial_t \mathcal{S}(t)\varphi = \left[\left((ik^3 - a)e^{(ik^3-a)t}\widehat{\varphi}(k)\right)_{k\in Z}\right]^{\vee} \in H^{s-3}_{per}\,,\ \forall \varphi \in H^s_{per}\,.$$

4. *If $\varphi \in H^s_{per}$ then $\partial_t \mathcal{S}(\cdot)\varphi \in C((0, \infty), H^{s-3}_{per})$.*
5. *The application: $\psi \to \partial_t \mathcal{S}(\cdot)\psi$ is continuous and verifies:*

$$\|\partial_t \mathcal{S}(t)\psi_1 - \partial_t \mathcal{S}(t)\psi_2\|_{s-3} \leq \sqrt{\max\{1, a^2\}}\|\psi_1 - \psi_2\|_s\,, \forall t \in (0, \infty)$$

and

$$\sup_{t\in[0,\infty)} \|\partial_t \mathcal{S}(t)\psi_1 - \partial_t \mathcal{S}(t)\psi_2\|_{s-3} \leq \sqrt{\max\{1, a^2\}}\|\psi_1 - \psi_2\|_s$$

with $\psi_j \in H^s_{per}$ for $j \in \{1, 2\}$.

Proof: Initially, we observe $\mathcal{S}(0)\varphi = \left[(\widehat{\varphi}(k))_{k\in Z}\right]^{\vee} = [\widehat{\varphi}]^{\vee} = \varphi\,,\ \forall \varphi \in H^s_{per}$; that is, $\mathcal{S}(0) = I$. From the linearity of the Fourier transform and its inverse, we obtain the linearity of $\mathcal{S}(t)$. Indeed, let $\sigma \in \mathbb{C}$, $\varphi, \psi \in H^s_{per}$, we have

$$\begin{aligned}
\mathcal{S}(t)(\sigma\varphi + \psi) &= \left[\left(e^{(ik^3-a)t}[\sigma\varphi + \psi]^{\wedge}(k)\right)_{k\in Z}\right]^{\vee} \\
&= \left[\left(e^{(ik^3-a)t}\left[\sigma\widehat{\varphi}(k) + \widehat{\psi}(k)\right]\right)_{k\in Z}\right]^{\vee} \\
&= \left[\sigma\left(e^{(ik^3-a)t}\widehat{\varphi}(k)\right)_{k\in Z} + \left(e^{(ik^3-a)t}\widehat{\psi}(k)\right)_{k\in Z}\right]^{\vee} \\
&= \sigma\left[\left(e^{(ik^3-a)t}\widehat{\varphi}(k)\right)_{k\in Z}\right]^{\vee} + \left[\left(e^{(ik^3-a)t}\widehat{\psi}(k)\right)_{k\in Z}\right]^{\vee}, \\
&= \sigma\mathcal{S}(t)(\varphi) + \mathcal{S}(t)(\psi)\,,
\end{aligned}$$

for $t > 0$.

If $\varphi \in H^s_{per}$and $t > 0$, we will demonstrate that $\mathcal{S}(t)\varphi \in H^s_{per}$ and $\|\mathcal{S}(t)\varphi\|_s \leq \|\varphi\|_s$; that is $\|\mathcal{S}(t)\| \leq 1$.

In effect, similar to (3.3) we have

$$\begin{aligned}
\|\mathcal{S}(t)\varphi\|^2_s &= 2\pi \sum_{k=-\infty}^{+\infty} (1 + k^2)^s \left|e^{(ik^3-a)t}\widehat{\varphi}(k)\right|^2, \\
&\leq 2\pi \sum_{k=-\infty}^{+\infty} (1 + k^2)^s \left|\widehat{\varphi}(k)\right|^2, \\
&= \|\varphi\|^2_s < \infty\,.
\end{aligned}$$

Then $\mathcal{S}(t)\varphi \in H^s_{per}$ and $\|\mathcal{S}(t)\varphi\|_s \leq \|\varphi\|_s$, $\forall t \geq 0$.
Therefore,

$$\|\mathcal{S}(t)\varphi\|_s \leq \|\varphi\|_s,\ \forall t \geq 0,\ \forall \varphi \in H^s_{per}\,. \tag{3.18}$$

That is,

$$\mathcal{S}(t) \in L(H^s_{per}) \text{ with } \|\mathcal{S}(t)\| \leq 1, \forall t \in [0, \infty)\,. \tag{3.19}$$

At this point, we will demonstrate that $\mathcal{S}(t + r) = \mathcal{S}(t) \circ \mathcal{S}(r)$, $\forall t, r \in [0, \infty)$. Indeed, let $f \in H^s_{per}$ and $t, r \in (0, \infty)$,

$$\begin{aligned}
\mathcal{S}(t + r)f &= \left[\left(e^{(ik^3-a)(t+r)}\widehat{f}(k)\right)_{k\in Z}\right]^{\vee}, \\
&= \left[\left(e^{(ik^3-a)t}e^{(ik^3-a)r}\widehat{f}(k)\right)_{k\in Z}\right]^{\vee}.
\end{aligned} \tag{3.20}$$

We know that if $f \in H^s_{per}$ then $\widehat{f} \in l^2_s$, that is

$$\sum_{k=-\infty}^{+\infty} (1+k^2)^s |\widehat{f}(k)|^2 < \infty\,. \tag{3.21}$$

We affirm that

$$\left(e^{(ik^3-a)r}\widehat{f}(k)\right)_{k\in Z} \in l^2_s\,, \forall r \in [0,\infty)\,. \tag{3.22}$$

In effect, when r is zero, it is obvious that the statement is true. Thus, we will demonstrate the case $r > 0$. For this, using (3.21) we obtain

$$\begin{aligned}\sum_{k=-\infty}^{+\infty} (1+k^2)^s |e^{(ik^3-a)r}\widehat{f}(k)|^2 &= \sum_{k=-\infty}^{+\infty} (1+k^2)^s \underbrace{|e^{i2k^3r}|}_{=1}\underbrace{e^{-2ar}}_{<1} |\widehat{f}(k)|^2,\\ &\le \sum_{k=-\infty}^{+\infty} (1+k^2)^s |\widehat{f}(k)|^2 < \infty\,.\end{aligned}$$

Therefore, from (3.22) and taking the inverse Fourier transform, we have

$$\left[\left(e^{(ik^3-a)r}\widehat{f}(k)\right)_{k\in Z}\right]^{\vee} \in H^s_{per}\,,\ \forall r \in [0,\infty)\,.$$

This motivates us to define

$$g_r := \left[\left(e^{(ik^3-a)r}\widehat{f}(k)\right)_{k\in Z}\right]^{\vee} \in H^s_{per}\,.$$

That is,

$$g_r = \mathcal{S}(r)f\,. \tag{3.23}$$

Taking the Fourier transform to g_r, we get

$$\widehat{g}_r = \left(e^{(ik^3-a)r}\widehat{f}(k)\right)_{k\in Z}\,,$$

that is,

$$\widehat{g}_r(k) = e^{(ik^3-a)r}\widehat{f}(k)\,,\ \forall k \in Z\,. \tag{3.24}$$

Using (3.24) in (3.20) and from (3.23) we have

$$\begin{aligned}\mathcal{S}(t+r)f &= \left[\left(e^{(ik^3-a)t}\widehat{g}_r(k)\right)_{k\in Z}\right]^{\vee},\\ &= \mathcal{S}(t)g_r,\\ &= \mathcal{S}(t)\{\mathcal{S}(r)f\},\\ &= \{\mathcal{S}(t)\circ\mathcal{S}(r)\}f\,,\ \forall t>0\,,\ r>0\,.\end{aligned}$$

Thus,

$$\mathcal{S}(t+r) = \mathcal{S}(t)\circ\mathcal{S}(r)\,,\ \forall t>0\,,\ r>0\,. \tag{3.25}$$

If t or r is zero, then the equality (3.25) is also true. Thus, we have demonstrated:

$$\mathcal{S}(t+r) = \mathcal{S}(t)\circ\mathcal{S}(r)\,,\ \forall t\ge 0\,,\ r\ge 0\,. \tag{3.26}$$

At this point, we will demonstrate the continuity of $t \to \mathcal{S}(t)\varphi$, that is, for $t > 0$

$$\|\mathcal{S}(t+h)\varphi - \mathcal{S}(t)\varphi\|_s \longrightarrow 0 \text{ when } h \to 0\,. \tag{3.27}$$

and $\|\mathcal{S}(h)\varphi - \varphi\|_s \longrightarrow 0$ when $h \to 0^+$.

Indeed, by applying item 3 from the proof of the previous theorem, we obtain

$$\begin{aligned} \|\mathcal{S}(t+h)\varphi - \mathcal{S}(t)\varphi\|_s^2 &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s \left| \left(e^{(ik^3-a)(t+h)} - e^{(ik^3-a)t} \right) \cdot \widehat{\varphi}(k) \right|^2 \\ &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s \, |\widehat{\varphi}(k)|^2 \, |\tilde{H}(t+h)|^2 \end{aligned} \tag{3.28}$$

where $t+h>0$ and $\tilde{H}(t+h) := e^{(ik^3-a)(t+h)} - e^{(ik^3-a)t}$.

We observe that $\lim\limits_{h\to 0} \tilde{H}(t+h) = 0$.

To ensure the interchange of limits, we need the uniform convergence of the series (3.28). For this we will bound the k-th term of the series, that is,

$$\begin{aligned} I_{k,t,h} &:= 2\pi(1+k^2)^s \, |\widehat{\varphi}(k)|^2 \left| e^{(ik^3-a)(t+h)} - e^{(ik^3-a)t} \right|^2 \\ &\leq 8\pi(1+k^2)^s \, |\widehat{\varphi}(k)|^2 \,, \end{aligned}$$

where we have employed the triangle inequality, the equality $|e^{i\theta}| = 1$ for $\theta \in R$ and $0 < e^{-at} \leq 1$ for $t \in [0,\infty)$.

Thus,

$$\sum_{k=-\infty}^{+\infty} I_{k,t,h} \leq 4\|\varphi\|_s^2 < \infty \,, \tag{3.29}$$

and using the Weierstrass M-Test, we obtain that the series (3.29) converges uniformly. So, it is possible to interchange of limits and obtain

$$\lim_{h\to 0} \|\mathcal{S}(t+h)\varphi - \mathcal{S}(t)\varphi\|_s^2 = \sum_{k=-\infty}^{+\infty} \underbrace{\lim_{h\to 0} I_{k,t,h}}_{=0} = 0\,;$$

that is,

$$\lim_{h\to 0} \|\mathcal{S}(t+h)\varphi - \mathcal{S}(t)\varphi\|_s = 0\,.$$

Remark 3.1. *It is verified*

$$\lim_{h\to 0^+} \|\mathcal{S}(h)\varphi - \varphi\|_s = 0\,,\ \forall \varphi \in H^s_{per}\,.$$

To prove this, we use the same technique, considering $h>0$ and $t=0$.

Remark 3.2. *With the remark 3.1 we would have that $\{\mathcal{S}(t)\}_{t\geq 0}$ is a semigroup of class C_o.*

Let ψ_1 and ψ_2 be close data in H^s_{per}, then we will prove that their corresponding $\mathcal{S}(\cdot)\psi_1$ and $\mathcal{S}(\cdot)\psi_2$ respectively, are also close. Indeed, since $\{\mathcal{S}(t)\}_{t\geq 0}$ is a contraction semigroup, we have

$$\|\mathcal{S}(t)\psi_1 - \mathcal{S}(t)\psi_2\|_s = \|\mathcal{S}(t)(\psi_1-\psi_2)\|_s \leq \|\psi_1-\psi_2\|_s\,.$$

Taking the supremum over $(0,\infty)$ we have

$$\sup_{t\in(0,\infty)} \|\mathcal{S}(t)\psi_1 - \mathcal{S}(t)\psi_2\|_s \leq \|\psi_1-\psi_2\|_s\,. \tag{3.30}$$

From here we have that if $\psi_1 \to \psi_2$ then $\mathcal{S}(\cdot)\psi_1 \to \mathcal{S}(\cdot)\psi_2$.

We will prove: If $\varphi \in H^s_{per}$ then $\partial_t\mathcal{S}(t)\varphi \in H^{s-3}_{per}$ and $\|\partial_t\mathcal{S}(t)\varphi\|_{s-3} \leq \sqrt{\max\{1,a^2\}}\|\varphi\|_s$.

In effect, using $|k^3 i - a|^2 \leq \max\{1,a^2\}(1+k^2)^3$, $\forall k \in Z$, the equality $|e^{i\theta}| = 1$, $\forall\theta \in R$ and $0 < e^{-at} < 1$ for $t>0$, we have

$$\begin{aligned} \|\partial_t\mathcal{S}(t)\varphi\|_{s-3}^2 &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-3} \left| (k^3 i - a) e^{(ik^3-a)t} \widehat{\varphi}(k) \right|^2, \\ &\leq 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-3} |k^3 i - a|^2 \, |\widehat{\varphi}(k)|^2\,, \\ &\leq 2\pi \cdot \max\{1,a^2\} \sum_{k=-\infty}^{+\infty} (1+k^2)^s \, |\widehat{\varphi}(k)|^2 < \infty \\ &= \max\{1,a^2\}\|\varphi\|_s^2\,. \end{aligned}$$

That is, $\|\partial_t \mathcal{S}(t)\varphi\|_{s-3} \le \sqrt{\max\{1,a^2\}}\|\varphi\|_s$. From this inequality, we obtain

$$\|\partial_t \mathcal{S}(t)\psi_1 - \partial_t \mathcal{S}(t)\psi_2\|_{s-3} \le \sqrt{\max\{1,a^2\}}\|\psi_1-\psi_2\|_s\,,$$

with $\psi_j \in H^s_{per}$ for $j \in \{1,2\}$.

So, taking supremum over $(0,\infty)$ we have

$$\sup_{t\in(0,\infty)} \|\partial_t \mathcal{S}(t)\psi_1 - \partial_t \mathcal{S}(t)\psi_2\|_{s-3} \le \sqrt{\max\{1,a^2\}}\|\psi_1-\psi_2\|_s\,.$$

Finally, if $\varphi \in H^s_{per}$ we will demonstrate the continuity of $t \to \partial_t \mathcal{S}(t)\varphi$. That is,

$$\|\partial_t \mathcal{S}(t+h)\varphi - \partial_t \mathcal{S}(t)\varphi\|_{s-3} \longrightarrow 0 \text{ when } h \to 0\,.$$

Indeed, by applying item 3 from the proof of the previous Theorem, we proceed

$$\begin{aligned}
\|\partial_t \mathcal{S}(t+h)\varphi - \partial_t \mathcal{S}(t)\varphi\|^2_{s-3} &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-3}\left|\left(e^{(ik^3-a)(t+h)} - e^{(ik^3-a)t}\right)\cdot(k^3 i - a)\widehat{\varphi}(k)\right|^2,\\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-3}\left|\left(e^{(ik^3-a)h} - 1\right)e^{(ik^3-a)t}\cdot(k^3 i - a)\widehat{\varphi}(k)\right|^2,\\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-3}\left|e^{(ik^3-a)h} - 1\right|^2\left|e^{ik^3t}\right|^2\cdot e^{-2at}|k^3 i - a|^2\,|\widehat{\varphi}(k)|^2,\\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-3}|\beta(h)|^2\cdot e^{-2at}|k^3 i - a|^2\,|\widehat{\varphi}(k)|^2, \qquad (3.31)
\end{aligned}$$

where $\beta(h) := e^{(ik^3-a)h} - 1$.

We observe that $\lim_{h\to 0}\beta(h) = 0$.

To ensure the interchange of limits, we need the uniform convergence of the series (3.31). For this, we will bound the k-th term of the series, that is,

$$\begin{aligned}
I_{k,t,h} &= 2\pi(1+k^2)^{s-3}|\beta(h)|^2\cdot e^{-2at}|k^3 i - a|^2\,|\widehat{\varphi}(k)|^2,\\
&\le 8\pi\cdot\max\{1,a^2\}(1+k^2)^s\,|\widehat{\varphi}(k)|^2\,,
\end{aligned}$$

where we have employed the triangle inequality, $|k^3 i - a|^2 \le \max\{1,a^2\}(1+k^2)^3,\ \forall k \in Z$, the equality $|e^{i\theta}| = 1, \forall\theta \in R$ and $0 < e^{-2at} < 1$ for $t > 0$.

Thus,

$$\sum_{k=-\infty}^{+\infty} I_{k,t,h} \le 4\cdot\max\{1,a^2\}\|\varphi\|^2_s < \infty\,, \qquad (3.32)$$

and using the Weierstrass M-Test, we obtain that the series (3.32) converges uniformly. So, it is possible to interchange of limits and obtain

$$\lim_{h\to 0}\|\partial_t \mathcal{S}(t+h)\varphi - \partial_t \mathcal{S}(t)\varphi\|^2_{s-3} = \sum_{k=-\infty}^{+\infty}\underbrace{\lim_{h\to 0} I_{k,t,h}}_{=0} = 0,$$

hence, we conclude

$$\lim_{h\to 0}\|\partial_t \mathcal{S}(t+h)\varphi - \partial_t \mathcal{S}(t)\varphi\|_{s-3} = 0\,.$$

□

We will provide some additional properties of $\{\mathcal{S}(t)\}_{t\ge 0}$.

Corollary 3.3. *Based on the hypothesis of the previous Theorem, the following statements hold*

1. *If $\phi \in H^s_{per}$ then $\mathcal{S}(t)\phi \in H^r_{per}$ and $\|\mathcal{S}(t)\phi\|_r \le \|\phi\|_s$, $\forall t \ge 0$, $\forall r < s$. That is, $\mathcal{S}(t) \in L(H^s_{per}, H^r_{per})$, $\forall t \ge 0$, $\forall r < s$.*
2. *If $\phi \in H^s_{per}$ then $\mathcal{S}(\cdot)\phi \in C([0,\infty), H^r_{per})$, $\forall r < s$.*

3. *The application:* $\phi \to \mathcal{S}(\cdot)\phi$ *is continuous and verifies*

$$\begin{aligned} \|\mathcal{S}(t)\phi_1 - \mathcal{S}(t)\phi_2\|_r &\leq \|\phi_1 - \phi_2\|_s, \ \forall t \in [0,\infty), \ \forall r < s\,, \\ \sup_{t\in(0,\infty)} \|\mathcal{S}(t)\phi_1 - \mathcal{S}(t)\phi_2\|_r &\leq \|\phi_1 - \phi_2\|_s, \ \forall r < s, \end{aligned}$$

with $\phi_j \in H^s_{per}$ *for* $j = 1, 2$.

4. *If* $\phi \in H^s_{per}$ *then* $\partial_t \mathcal{S}(t)\phi \in H^{r-3}_{per}$ *and* $\|\partial_t \mathcal{S}(t)\phi\|_{r-3} \leq \sqrt{\max\{1, a^2\}}\|\phi\|_s$, $\forall t > 0$, $\forall r < s$. *That is,* $\partial_t \mathcal{S}(t) \in L(H^s_{per}, H^{r-3}_{per})$, $\forall t > 0$, $\forall r < s$, *where*

$$\partial_t \mathcal{S}(t)\varphi = \left[\left((ik^3 - a)e^{(ik^3-a)t}\widehat{\varphi}(k)\right)_{k\in Z}\right]^{\vee} \in H^{r-3}_{per}\,, \ \forall \varphi \in H^s_{per}\,, \forall r < s\,.$$

5. *If* $\varphi \in H^s_{per}$ *then* $\partial_t \mathcal{S}(\cdot)\varphi \in C((0,\infty), H^{r-3}_{per})$, $\forall r < s$
6. *The application:* $\psi \to \partial_t \mathcal{S}(\cdot)\psi$ *is continuous and verifies:*

$$\|\partial_t \mathcal{S}(t)\psi_1 - \partial_t \mathcal{S}(t)\psi_2\|_{r-3} \leq \sqrt{\max\{1, a^2\}}\|\psi_1 - \psi_2\|_s\,, \forall t \in (0,\infty), \ \forall r < s\,,$$

and

$$\sup_{t\in[0,\infty)} \|\partial_t \mathcal{S}(t)\psi_1 - \partial_t \mathcal{S}(t)\psi_2\|_{r-3} \leq \sqrt{\max\{1, a^2\}}\|\psi_1 - \psi_2\|_s\,, \ \forall r < s,$$

with $\psi_j \in H^s_{per}$ *for* $j \in \{1, 2\}$.

Proof: Its proof is similar to the proof of the second Corollary of Theorem 3.1, in which we use the continuous Sobolev embedding. □

Next, we state Theorem 3.1 in terms of the semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$.

Theorem 3.3. *Let* $s \in R$, $a > 0$ *and* $\{\mathcal{S}(t)\}_{t\geq 0}$ *the semigroup of class* C_o *from Theorem 3.2, then* $\mathcal{S}(\cdot)\psi$ *is the unique solution of*

$$\left|\begin{array}{l} u \in C([0,+\infty), H^s_{per}) \cap C^1((0,+\infty), H^{s-3}_{per}), \\ \partial_t u = Au \ \in H^{s-3}_{per}, \\ u(0) = \psi \in H^s_{per}\,. \end{array}\right.$$

in the sense that

$$\lim_{h\to 0}\left\|\frac{\mathcal{S}(t+h)\psi - \mathcal{S}(t)\psi}{h} - A\mathcal{S}(t)\psi\right\|_{s-3} = 0, \tag{3.33}$$

where $A := -\partial_x^3 - aI$, *and if* $\psi_1 \sim \psi_2$ *then* $\mathcal{S}(\cdot)\psi_1 \sim \mathcal{S}(\cdot)\psi_2$.

In addition, the following regularity holds: if $\psi \in H^s_{per}$ *then* $\mathcal{S}(t)\psi \in H^r_{per}$, $\forall r \leq s$, $\forall t \geq 0$ *and* $\|\mathcal{S}(t)\psi\|_r \leq \|\psi\|_s$, $\forall t \geq 0$, $\forall r \leq s$.

Also, $\|\partial_t \mathcal{S}(t)\psi\|_{r-3} \leq \sqrt{\max\{1, a^2\}}\|\psi\|_s$, $\forall t > 0$, $\forall r \leq s$.

Proof: The proof of (3.33) is similar to item 4 of the proof of Theorem 3.1. And the proof of the remaining statement follows in a similar way to the proof of Theorem 3.1 and as a consequence of Theorem 3.2. □

4. Generalization of results to n-th order equation. In this section, we will generalize the results obtained in the previous section.

Theorem 4.1. *Let* s *be a fixed real number,* $a > 0$, n *a natural number such that* $n - 1$ *is an even number which is not divisible by four and*

$$(P_\Sigma) \quad \left|\begin{array}{l} u \in C([0,+\infty), H^s_{per}), \\ \partial_t u + \partial_x^n u + au = 0 \ \in H^{s-n}_{per}, \\ u(0) = \psi \in H^s_{per}\,. \end{array}\right.$$

then (P_Σ) *is globally well-posed, that is,* $\exists g \in C([0,\infty), H^s_{per}) \cap C((0,\infty), H^{s-n}_{per})$ *verifying equation* (P_Σ) *so that the application:* $\psi \to g$, *which to every initial data* ψ *assigns the solution* g *of the IVP* (P_Σ), *is continuous.*

Moreover, the solution g *satisfies* $g(t) \in H^r_{per}\,, \ \forall t \geq 0\,, \ \forall r \leq s$ *with* $\|g(t)\|_s \leq \|\psi\|_s$ *and* $\|g(t)\|_r \leq \|\psi\|_s$, $\forall r < s$, $\forall t \geq 0$.

Also, $\|\partial_t g(t)\|_{r-n} \leq \sqrt{\max\{1, a^2\}}\|\psi\|_s\,, \ \forall r \leq s\,, \forall t > 0$.

Proof: Its proof is similar to the proof of Theorem 3.1. □

Consequently, we obtain the following outcome.

Corollary 4.1. *The unique solution of (P_Σ) is*

$$u(t)=\sum_{k=-\infty}^{+\infty} e^{ik^n t}e^{-at}\widehat{\psi}(k)\varphi_k,$$

where $\varphi_k(x)=e^{ikx}$ for $x\in R$.

Next, we define a family of operators which will verify the conditions of being a contraction semigroup of class C_o.

Theorem 4.2. *Let $s\in R$, $a>0$, n a natural number such that $n-1$ is an even number not multiple of four. The application*

$$\begin{array}{rcl}\mathcal{T}_n:[0,\infty) & \to & L(H^s_{per}),\\ t & \to & \mathcal{T}_n(t),\end{array}$$

such that $\mathcal{T}_n(t)=e^{-(\partial_x^n+aI)t}$, that is, applies

$$\mathcal{T}_n(t)\varphi=\left[\left(e^{(ik^n-a)t}\widehat{\varphi}(k)\right)_{k\in Z}\right]^{\vee},\ \varphi\in H^s_{per},$$

then $\{\mathcal{T}_n(t)\}_{t\geq 0}$ is a contraction semigroup of class C_o on H^s_{per}. Thus, $\{\mathcal{T}_n\}_{n\in M}$ is a family of semigroups on H^s_{per} where

$$M:=\{n\in N/n-1 \text{ is an even number not multiple of four }\}.$$

And for simplicity we will denote to $\mathcal{T}_n$ as $\mathcal{T}$.

Additionally, the following statements hold:

1. *If $\varphi\in H^s_{per}$ then $\mathcal{T}(\cdot)\varphi\in C([0,\infty),H^s_{per})$.*
2. *The application: $\varphi\to\mathcal{T}(\cdot)\varphi$ is continuous and verifies:*

$$\|\mathcal{T}(t)\psi_1-\mathcal{T}(t)\psi_2\|_s\leq\|\psi_1-\psi_2\|_s,\forall t\in[0,\infty),$$

and

$$\sup_{t\in(0,\infty)}\|\mathcal{T}(t)\psi_1-\mathcal{T}(t)\psi_2\|_s\leq\|\psi_1-\psi_2\|_s,$$

with $\psi_j\in H^s_{per}$ for $j\in\{1,2\}$.

3. *$\mathcal{T}(t)\in L(H^s_{per})$ and $\|\mathcal{T}(t)\Theta\|_s\leq\|\Theta\|_s$, $\forall\Theta\in H^s_{per}$, $\forall t\in[0,\infty)$.*
4. *If $\varphi\in H^s_{per}$ then $\partial_t\mathcal{T}(t)\varphi\in H^{s-n}_{per}$ and $\|\partial_t\mathcal{T}(t)\varphi\|_{s-n}\leq\sqrt{\max\{1,a^2\}}\|\varphi\|_s$, $\forall t\in[0,\infty)$. That is, $\partial_t\mathcal{T}(t)\in L(H^s_{per},H^{s-n}_{per})$, $\forall t\in(0,\infty)$, where*

$$\partial_t\mathcal{T}(t)\varphi=\left[\left((ik^n-a)e^{(ik^n-a)t}\widehat{\varphi}(k)\right)_{k\in Z}\right]^{\vee}\in H^{s-n}_{per},\ \forall\varphi\in H^s_{per}.$$

5. *If $\varphi\in H^s_{per}$ then $\partial_t\mathcal{T}(\cdot)\varphi\in C((0,\infty),H^{s-n}_{per})$.*
6. *The application: $\psi\to\partial_t\mathcal{T}(\cdot)\psi$ is continuous and verifies:*

$$\|\partial_t\mathcal{T}(t)\psi_1-\partial_t\mathcal{T}(t)\psi_2\|_{s-n}\leq\sqrt{\max\{1,a^2\}}\|\psi_1-\psi_2\|_s,\forall t\in(0,\infty),$$

and

$$\sup_{t\in(0,\infty)}\|\partial_t\mathcal{T}(t)\psi_1-\partial_t\mathcal{T}(t)\psi_2\|_{s-n}\leq\sqrt{\max\{1,a^2\}}\|\psi_1-\psi_2\|_s,$$

with $\psi_j\in H^s_{per}$ for $j\in\{1,2\}$.

Proof: Its proof is similar to the proof of Theorem 3.2 □

We will provide some additional properties of the family $\{\mathcal{T}(t)\}_{t\geq 0}$.

Corollary 4.2. *Based on the hypothesis of the previous Theorem, the following statements hold*

1. *If $\phi\in H^s_{per}$ then $\mathcal{T}(\cdot)\phi\in C([0,\infty),H^r_{per})$, $\forall r<s$.*

2. *The application:* $\phi \to \mathcal{T}(\cdot)\phi$ *is continuous and verifies*

$$\begin{aligned} \|\mathcal{T}(t)\phi_1 - \mathcal{T}(t)\phi_2\|_r &\le \|\phi_1 - \phi_2\|_s,\ \forall t \in [0,\infty),\ \forall r < s\,, \\ \sup_{t\in(0,\infty)} \|\mathcal{T}(t)\phi_1 - \mathcal{T}(t)\phi_2\|_r &\le \|\phi_1 - \phi_2\|_s,\ \forall r < s \end{aligned}$$

with $\phi_j \in H^s_{per}$ *for* $j = 1, 2$.

3. $\mathcal{T}(t) \in L(H^s_{per}, H^r_{per})$ *and* $\|\mathcal{T}(t)\theta\|_r \le \|\theta\|_s$, $\forall r < s$, $\forall t \ge 0\,, \forall \theta \in H^s_{per}$.
4. *If* $\phi \in H^s_{per}$ *then* $\partial_t\mathcal{T}(t)\phi \in H^{r-n}_{per}$ *and* $\|\partial_t\mathcal{T}(t)\phi\|_{r-n} \le \sqrt{\max\{1,a^2\}}\|\phi\|_s$, $\forall t > 0$, $\forall r < s$. *That is,* $\partial_t\mathcal{T}(t) \in L(H^s_{per}, H^{r-n}_{per})$, $\forall t > 0$, $\forall r < s$ *where*

$$\partial_t\mathcal{T}(t)\varphi = \left[\left((ik^n - a)e^{(ik^n-a)t}\widehat{\varphi}(k)\right)_{k\in Z}\right]^{\vee} \in H^{r-n}_{per}\,,\ \forall\varphi \in H^s_{per}\,, \forall r < s\,.$$

5. *If* $\varphi \in H^s_{per}$ *then* $\partial_t\mathcal{T}(\cdot)\varphi \in C((0,\infty), H^{r-n}_{per})$, $\forall r < s$
6. *The application:* $\psi \to \partial_t\mathcal{T}(\cdot)\psi$ *is continuous and satisfies:*

$$\|\partial_t\mathcal{T}(t)\psi_1 - \partial_t\mathcal{T}(t)\psi_2\|_{r-n} \le \sqrt{\max\{1,a^2\}}\|\psi_1 - \psi_2\|_s\,, \forall t \in (0,\infty),\ \forall r < s\,,$$

and

$$\sup_{t\in(0,\infty)} \|\partial_t\mathcal{T}(t)\psi_1 - \partial_t\mathcal{T}(t)\psi_2\|_{r-n} \le \sqrt{\max\{1,a^2\}}\|\psi_1 - \psi_2\|_s\,,,\ \forall r < s,$$

with $\psi_j \in H^s_{per}$ *for* $j \in \{1, 2\}$.

Proof: Its proof is similar to the proof of Corollary 3.3. □

Now, we state another version of Theorem 4.1 in terms of the semigroup $\{\mathcal{T}(t)\}_{t\ge0}$.

Theorem 4.3. *Let* $s \in R$, $a > 0$, n *a natural number such that* $n - 1$ *is an even number not multiple of four and* $\{\mathcal{T}(t)\}_{t\ge0}$ *the semigroup of class* C_o *from Theorem 3.2, then* $\mathcal{T}(\cdot)\psi$ *is the unique solution of*

$$\left|\begin{array}{l} u \in C([0,+\infty), H^s_{per}) \cap C^1((0,+\infty), H^{s-n}_{per}), \\ \partial_t u = A_\Sigma u \ \in H^{s-n}_{per}, \\ u(0) = \psi \in H^s_{per}\,. \end{array}\right.$$

in the sense that

$$\lim_{h\to0}\left\|\frac{\mathcal{T}(t+h)\psi - \mathcal{T}(t)\psi}{h} - A_\Sigma\mathcal{T}(t)\psi\right\|_{s-n} = 0, \tag{4.1}$$

where $A_\Sigma := -\partial^n_x - aI$, *and if* $\psi_1 \sim \psi_2$ *then* $\mathcal{S}(\cdot)\psi_1 \sim \mathcal{S}(\cdot)\psi_2$.

In addition, the following regularity holds: if $\psi \in H^s_{per}$ *then* $\mathcal{T}(t)\psi \in H^r_{per}$, $\forall r \le s$, $\forall t \ge 0$ *and* $\|\mathcal{T}(t)\psi\|_r \le \|\psi\|_s$, $\forall t \ge 0$, $\forall r \le s$.

Also, if $\psi \in H^s_{per}$ *then* $\partial_t\mathcal{T}(t)\psi \in H^{r-n}_{per}\,,\ \forall r \le s\,,\ \forall t \in (0,\infty)$ *and* $\|\partial_t\mathcal{T}(t)\psi\|_{r-n} \le \sqrt{\max\{1,a^2\}}\|\psi\|_s$, $\forall t > 0$, $\forall r \le s$.

Proof: Its proof is similar to the proof of Theorem 3.3. □

Below we state some additional results that can be obtained.

Remark 4.1. *Analogous results to Theorem 4.1 are obtained when* n *is a natural number such that* $n - 1$ *is a multiple of four, where the solution would be*

$$v(t) = \left[\left(e^{-(ik^n+a)t}\widehat{\varphi}(k)\right)_{k\in Z}\right]^{\vee},$$

for the initial data $\varphi \in H^s_{per}$.

Remark 4.2. *Similar results to Theorem 4.2 and Corollary 4.2 are obtained when* n *is a natural number such that* $n - 1$ *is a multiple of four, where the family of operators introduced is*

$$T(t)\varphi = \left[\left(e^{-(ik^n+a)t}\widehat{\varphi}(k)\right)_{k\in Z}\right]^{\vee}\,,\ \forall\varphi \in H^s_{per}\,.$$

Therefore, the analogous version to Theorem 4.3 is valid.

Remark 4.3. *When n is a number multiple of four, the problem (P_Σ) has a solution and the associated family of operators*

$$\Gamma(t)\varphi = \left[\left(e^{-(k^n+a)t}\widehat{\varphi}(k)\right)_{k\in Z}\right]^{\vee}\ ,\ \forall\varphi\in H^s_{per},$$

forms a contraction semigroup of class C_o.

Finally,

Remark 4.4. *When n is an even number which is not divisible by four, the problem (P_Σ) has no solution.*

5. Dissipative property of (P_1) and (P_Σ). The properties that will be obtained in this section do not depend on the explicit form of the solution.

5.1. Dissipative property of (P_1). Let s be a fixed real number, $a > 0$ and the problem

$$(P_1)\quad \left|\begin{array}{l} w\in C([0,+\infty),H^s_{per})\cap C^1((0,\infty),H^{s-3}_{per}),\\ \partial_t w+\partial_x^3 w+aw=0\ \in H^{s-3}_{per},\\ w(0)=\psi\in H^s_{per}\,. \end{array}\right.$$

Theorem 5.1. *Let w the solution of (P_1) with initial data $\psi\in H^s_{per}$ then we get the following results:*

1. $\partial_t\|w(t)\|^2_{s-3} = -2a\|w(t)\|^2_{s-3}\leq 0.$
2. $\|w(t)\|_{s-3} = e^{-at}\|\psi\|_{s-3}\leq e^{-at}\|\psi\|_s\leq\|\psi\|_s\,,\ t\geq 0.$
3. $\lim\limits_{t\to+\infty}\|w(t)\|_{s-3}=0\,.$

Proof: As $H^s_{per}\subset H^{s-3}_{per}$ then the following expressions: $<\partial_t w, w>_{s-3}$ and $<w,\partial_t w>_{s-3}$ are well-defined.
So we have

$$\begin{aligned}\partial_t\|w(t)\|^2_{s-3} &= \partial_t<w(t),w(t)>_{s-3}\\ &= <\partial_t w(t),w(t)>_{s-3}+<w(t),\partial_t w(t)>_{s-3}\\ &= 2Re<\partial_t w(t),w(t)>_{s-3}\ . \end{aligned}\tag{5.1}$$

Also, we obtain

$$\begin{aligned}<\partial_x^3 w,w>_{s-3} &= 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{s-3}\widehat{\partial_x^3 w}(k)\cdot\overline{\widehat{w}(k)}\\ &= 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{s-3}(ik)^3\widehat{w}(k)\cdot\overline{\widehat{w}(k)}\\ &= -i2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{s-3}k^3\widehat{w}(k)\cdot\overline{\widehat{w}(k)}\\ &= -i\,\underbrace{2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{s-3}k^3|\widehat{w}(k)|^2}_{\delta:=}\ . \end{aligned}\tag{5.2}$$

At this point, we will prove the convergence of the series (5.2). Indeed, using the inequality: $|k|^3\leq |k|^6=(|k|^2)^3\leq(1+k^2)^3$, $\forall k\in\mathbb{Z}$ and $w(t)\in H^s_{per}$, we obtain

$$\begin{aligned}\left|\sum_{k=-\infty}^{+\infty}(1+k^2)^{s-3}k^3|\widehat{w}(k)|^2\right| &\leq \sum_{k=-\infty}^{+\infty}(1+k^2)^{s-3}|k|^3|\widehat{w}(k)|^2\\ &\leq \sum_{k=-\infty}^{+\infty}(1+k^2)^{s-3}(1+|k|^2)^3|\widehat{w}(k)|^2\\ &= \sum_{k=-\infty}^{+\infty}(1+k^2)^{s}|\widehat{w}(k)|^2=\frac{1}{2\pi}\|w(t)\|^2_s<\infty\,. \end{aligned}$$

Then the series (5.2) is convergent, that is,

$$< \partial_x^s w(t), w(t) >_{s-3} = -i\delta \text{ with } \delta \in R\,. \tag{5.3}$$

From (5.1), using $\partial_t w = -\partial_x^3 w - aw$ and the equality (5.3) we get

$$\begin{aligned}
\partial_t \|w(t)\|_{s-3}^2 &= 2Re < \partial_t w(t), w(t) >_{s-3} \\
&= 2Re\{< -\partial_x^3 w(t), w(t) >_{s-3} - a < w(t), w(t) >_{s-3}\} \\
&= -2\underbrace{Re < \partial_x^3 w(t), w(t) >_{s-3}}_{=0} - 2a\|w(t)\|_{s-3}^2 \\
&= -2a\|w(t)\|_{s-3}^2 \leq 0\,.
\end{aligned}$$

Therefore, $\|w(t)\|_{s-3}^2$ is not incressing. Then $\|w(t)\|_{s-3}^2 \leq \|w(0)\|_{s-3}^2\,, \forall t \geq 0$.
As

$$(\|w(t)\|_{s-3} - \|w(0)\|_{s-3})(\|w(t)\|_{s-3} + \|w(0)\|_{s-3}) \leq 0\,,$$

we have

$$\|w(t)\|_{s-3} \leq \|w(0)\|_{s-3} \leq \|w(0)\|_s\ ,\ \forall t \geq 0\,.$$

That is,

$$\|w(t)\|_{s-3} \leq \|\psi\|_{s-3} \leq \|\psi\|_s\ ,\ \forall t \geq 0\,.$$

To be exact, solving the equation we obtain $\|w(t)\|_{s-3}^2 = e^{-2at}\|w(0)\|_{s-3}^2$.
That is

$$\|w(t)\|_{s-3} = e^{-at}\|w(0)\|_{s-3} \leq e^{-at}\|w(0)\|_s \leq \|w(0)\|_s\,,\ \forall t \geq 0.$$

Taking limit to $\|w(t)\|_{s-3} = e^{-at}\|w(0)\|_{s-3}$ when $t \to +\infty$, we obtain $\lim\limits_{t\to+\infty} \|w(t)\|_{s-3} = 0$. □

Corollary 5.1 (Continuous dependence of the solution of (P_1)). *Let u and v be solutions of (P_1) with initial data ψ_1 and ψ_2 in H^s_{per}, respectively. Then the following statements hold*

$$\partial_t\|u(t) - v(t)\|_{s-3}^2 = -2a\|u(t) - v(t)\|_{s-3}^2 \leq 0,$$

and

$$\|u(t) - v(t)\|_{s-3} = e^{-at}\|\psi_1 - \psi_2\|_{s-3} \leq \|\psi_1 - \psi_2\|_s\,, \quad t \geq 0\,. \tag{5.4}$$

Proof: Define $w := u - v$ then w satisfies:

$$\left|\begin{array}{l}
\partial_t w + \partial_x^3 w + aw = 0, \\
w(0) = \psi_1 - \psi_2\,.
\end{array}\right.$$

We conclude using Theorem 5.1. □

Corollary 5.2 (Uniqueness of solution of (P_1)). *The (P_1) problem has a unique solution.*

Proof: Indeed, let u and v be solutions of (P_1) with the same initial data, that is, $\psi_1 = \psi_2 = \psi$.
From (5.4) we obtain $\|u(t) - v(t)\|_{s-3} \leq \|0\|_s = 0$. Then $\|u(t) - v(t)\|_{s-3} = 0$. So, $u(t) = v(t), \forall t \geq 0$, that is, $u = v$. □

5.2. Dissipative property of (P_Σ). Let s be a fixed real number, $a > 0$ and the problem

$$(P_\Sigma)\quad \left|\begin{array}{l}
w \in C([0, +\infty), H^s_{per}) \cap C^1((0, \infty), H^{s-n}_{per}), \\
\partial_t w + \partial_x^n w + aw = 0\ \in H^{s-n}_{per}, \\
w(0) = \varphi \in H^s_{per}\,.
\end{array}\right.$$

Theorem 5.2. *Let n be a natural number such that $n - 1$ is an even number not multiple of four and w the solution of (P_Σ) with initial data $\varphi \in H^s_{per}$ then we obtain the following results:*

1. $\partial_t\|w(t)\|_{s-n}^2 = -2a\|w(t)\|_{s-n}^2 \leq 0$.
2. $\|w(t)\|_{s-n} = e^{-at}\|\varphi\|_{s-n} \leq e^{-at}\|\varphi\|_s \leq \|\varphi\|_s\,,\ t \geq 0$.

3. $\lim_{t\to+\infty} \|w(t)\|_{s-n} = 0$.

Proof: It is analogous to the proof of Theorem 5.1, noting $(ik)^n = -ik^n$ when n is a natural number such that $n-1$ is an even number not multiple of four. □

Corollary 5.3 (Continuous dependence of the solution of (P_Σ)). *Let u and v be solutions of (P_Σ) with initial data ψ_1 and ψ_2 in H^s_{per}, respectively. Then the following statements hold*

$$\partial_t \|u(t) - v(t)\|^2_{s-n} = -2a\|u(t) - v(t)\|^2_{s-n} \leq 0$$

and

$$\|u(t) - v(t)\|_{s-n} = e^{-at}\|\psi_1 - \psi_2\|_{s-n} \leq \|\psi_1 - \psi_2\|_s \,, \quad t \geq 0\,. \tag{5.5}$$

Proof: Define $w := u - v$ then w satisfies:

$$\left| \begin{array}{l} \partial_t w + \partial_x^n w + aw = 0, \\ w(0) = \psi_1 - \psi_2\,. \end{array} \right.$$

From Theorem 5.2 we conclude the proof. □

Corollary 5.4 (Uniqueness of solution of (P_Σ)). *The Cauchy problem (P_Σ) possesses uniqueness of solution.*

Proof: Indeed, let u and v be solutions of (P_Σ) with the same initial data, that is, $\psi_1 = \psi_2 = \psi$. From (5.5) we obtain $\|u(t) - v(t)\|_{s-n} \leq \|0\|_s = 0$. Then $\|u(t) - v(t)\|_{s-n} = 0$. So, $u(t) = v(t)$, $\forall t \geq 0$, that is, $u = v$. □

Below we state some additional results that can be obtained.

Remark 5.1. *Similar results to Theorem 5.2, Corollaries 5.3 and 5.4 are also obtained when n is a natural number such that $n-1$ is a number multiple of four, in this case, note that $(ik)^n = ik^n$.*

Remark 5.2. *The dissipative property of (P_Σ) is also obtained when n is multiple of four, in this case, note that $(ik)^n = k^n$.*

6. Differentiability analysis versus initial data of (P_1) and (P_Σ). In order to deepen and enrich our study, we will investigate the infinite-dimensional space in which differentiability occurs and its connection to the initial data.

6.1. Differentiability analysis versus initial data of (P_1). In this subsection, we will analyze the solution of (P_1).

Theorem 6.1. *Let $s \in R$ and $a > 0$. If $t > 0$ and u is the solution of (P_1) with initial data $\psi \in H^s_{per}$ then for all $r \leq s-3$ it holds*

$$\lim_{h\to 0} \left\| \frac{u(t+h) - u(t)}{h} + \partial_x^3 u(t) + au(t) \right\|_r = 0.$$

This means that differentiability occurs in H^r_{per}, $\forall r \leq s-3$. That is, $\partial_t u(t) = -\partial_x^3 u(t) - au(t)$ in H^r_{per}, $\forall r \leq s-3$, $\forall t > 0$.

Proof: Let $t > 0$, $t + h > 0$ and $\psi \in H^s_{per}$

$$\left\| \frac{u(t+h) - u(t)}{h} + \partial_x^3 u(t) + au(t) \right\|_r^2$$

$$= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^r e^{-2at} \left| \underbrace{\frac{e^{(ik^3-a)h} - 1}{h} - ik^3 + a}_{\mathcal{M}(h):=} \right|^2 \left|\widehat{\psi}(k)\right|^2. \tag{6.1}$$

Using L'Hôpital's rule we have $\mathcal{M}(h) \to 0$ when $h \to 0$.
To ensure the interchange of limits, we need the uniform convergence of the series (6.1). For this, we will bound the k-th term of the series. First, for $h > 0$, from (3.6) we obtain

$$\left| \frac{e^{(ik^3-a)h} - 1}{h} \right| \leq (k^6 + a^2)^{\frac{1}{2}} \leq \sqrt{\max\{1, a^2\}}(k^6+1)^{\frac{1}{2}}\,. \tag{6.2}$$

Using the inequality (6.2), we are going to bound $|\mathcal{M}(h)|^2$ in the following way:

$$|\mathcal{M}(h)|^2 \leq 8\max\{1, a^2\}(1+k^2)^3\,. \tag{6.3}$$

For $h < 0$ such that $0 < t + h$, from (3.8) we have that there exists $\xi \in (h, 0)$ such that

$$\left| \frac{e^{(ik^3-a)t} - e^{(ik^3-a)(t+h)}}{h} \right| \le \sqrt{\max\{1, a^2\}}(k^6 + 1)^{\frac{1}{2}} . \tag{6.4}$$

Using the inequality (6.4), we are going to bound $|\mathcal{M}(h)|^2$ in the following way:

$$e^{-2at}|\mathcal{M}(h)|^2 \le 4\max\{1, a^2\}(1 + k^6) \le 8\max\{1, a^2\}(1 + k^2)^3 . \tag{6.5}$$

Therefore, for $h \in R - \{0\}$ and $t + h > 0$,

$$e^{-2at}|\mathcal{M}(h)|^2 \le 8\max\{1, a^2\}(1 + k^2)^3 . \tag{6.6}$$

Thus, we bound the k-th term of the series, where we use inequality (6.6)

$$\begin{aligned} \mathcal{J}_{k,t,r} &:= (1 + k^2)^r e^{-2at}|\mathcal{M}(h)|^2|\widehat{\psi}(k)|^2 \\ &\le \underbrace{8\max\{1, a^2\}}_{C_a:=}|\widehat{\psi}(k)|^2(1 + k^2)^{r+3} . \end{aligned}$$

On the other hand, we know that the series converges

$$2\pi \sum_{k=-\infty}^{+\infty} (1 + k^2)^s|\widehat{\psi}(k)|^2 = \|\psi\|_s^2 < \infty,$$

since $\psi \in H^s_{per}$.

Then, using continuous immersion in periodic Sobolev spaces, that is $H^s_{per} \subset H^{r+3}_{per}$ for $r \le s - 3$, we obtain

$$2\pi \cdot C_a \sum_{k=-\infty}^{+\infty} (1 + k^2)^{r+3}|\widehat{\psi}(k)|^2 = C_a\|\psi\|^2_{r+3} \le C_a\|\psi\|_s^2 < \infty\,,\ \forall r \le s - 3\,.$$

Thus, using the Weierstrass M-Test, we have that the series (6.1) converges uniformly and consequently it is possible to exchange limits and obtain

$$\left\| \frac{u(t + h) - u(t)}{h} + \partial_x^3 u(t) + au(t) \right\|_r^2 \to 0,$$

when $h \to 0$. □

Theorem 6.2. *Let $s \in R$ and $a > 0$. If u is the solution of (P_1) with initial data $\psi \in H^s_{per}$ then for all $r \le s - 3$ it holds*

$$\lim_{h\to 0^+} \left\| \frac{u(h) - \psi}{h} + \partial_x^3 u(0) + au(0) \right\|_r = 0\,.$$

This means that differentiability occurs in H^r_{per}, $\forall r \le s - 3$. That is, $\partial_{t^+} u(0) = -\partial_x^3 u(0) - au(0)$ in H^r_{per}, $\forall r \le s - 3$.

Proof: Let $h > 0$,

$$\begin{aligned} &\left\| \frac{u(h) - \psi}{h} + \partial_x^3 u(0) + au(0) \right\|_r^2 \\ &= 2\pi \sum_{k=-\infty}^{+\infty} (1 + k^2)^r \left| \underbrace{\frac{e^{(ik^3-a)h} - 1}{h} - ik^3 + a}_{\mathcal{M}(h):=} \right|^2 \left|\widehat{\psi}(k)\right|^2 . \end{aligned} \tag{6.7}$$

Using L'Hôpital's rule, we have $\mathcal{M}(h) \to 0$ when $h \to 0$.

To ensure the interchange of limits, we need the uniform convergence of the series (6.7). For this, we will bound the k-th term of the series. Thus, using inequality (6.3) we obtain

$$\mathcal{L}_{k,r} = (1 + k^2)^r|\mathcal{M}(h)|^2|\widehat{\psi}(k)|^2 \le 8\max\{1, a^2\}(1 + k^2)^{r+3}|\widehat{\psi}(k)|^2,$$

On the other hand, we know that the series converges

$$2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^s|\widehat{\psi}(k)|^2=\|\psi\|_s^2<\infty,$$

since $\psi \in H^s_{per}$.

Then, using continuous immersion in periodic Sobolev spaces, that is, $H^s_{per}\subset H^{r+3}_{per}$ for $r\leq s-3$, we obtain

$$16\pi\cdot\max\{1,a^2\}\sum_{k=-\infty}^{+\infty}(1+k^2)^{r+3}|\widehat{\psi}(k)|^2=8\max\{1,a^2\}\|\psi\|_{r+3}^2\leq 8\max\{1,a^2\}\|\psi\|_s^2<\infty\,,\ \forall r\leq s-3\,.$$

Thus, using the Weierstrass M-Test, we have that the series (6.7) converges uniformly and consequently it is possible to exchange limits and obtain

$$\left\|\frac{u(h)-u(0)}{h}+\partial_x^3u(0)+au(0)\right\|_r^2\longrightarrow 0,$$

when $h\to 0^+$. □

Theorem 6.3. *Let $s \in R$, $a>0$ and $\psi\in H^s_{per}$ then the following statements are equivalent*

1. *There exists* $\lim\limits_{h\to 0^+}\left(\frac{\mathcal{S}(h)-I}{h}\right)\psi$ *in* $\left(H^s_{per},\|\cdot\|_s\right)$.
2. $\psi\in H^{s+3}_{per}$.

Proof: Suppose that item 1 holds, then $A=-\partial_x^3-aI$ is the infinitesimal generator of the contraction semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$. Thus, $A\psi=-\partial_x^3\psi-a\psi\in H^s_{per}$, from which we obtain

$$(1+i\partial_x^3)\psi=\psi+i\partial_x^3\psi\in H^s_{per}\,. \tag{6.8}$$

Applying the Fourier transform to (6.8) we have

$$\Big((1+k^3)\widehat{\psi}(k)\Big)_{k\in Z}\in l^2_s\,. \tag{6.9}$$

Using that the $|\cdot|_p$ p-norms, with $p\in[1,\infty]$, are equivalent in R^2, we have $|(1,k)|_2\leq\sqrt{2}|(1,k)|_3$, $\forall k\in Z$; then

$$|(1,k)|_2^6\leq 8|(1,k)|_3^6,\ \forall k\in Z\,.$$

That is,

$$(1+k^2)^3\leq 8(1+k^3)^2\,,\ \forall k\in Z\,. \tag{6.10}$$

Using (6.10) we get

$$\begin{aligned}
\|\psi\|_{s+3}^2 &= 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{s+3}\left|\widehat{\psi}(k)\right|^2\\
&= 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{s}(1+k^2)^3\left|\widehat{\psi}(k)\right|^2\\
&\leq 16\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{s}(1+k^3)^2\left|\widehat{\psi}(k)\right|^2\\
&= 16\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{s}\left|(1+k^3)\widehat{\psi}(k)\right|^2\\
&= 8\cdot\|(1+i\partial_x^3)\psi\|_s^2<\infty\,.
\end{aligned}$$

Then $\psi\in H^{s+3}_{per}$.

Reciprocally, if $\psi\in H^{s+3}_{per}$,

$$\left\|\frac{\mathcal{S}(h)\psi-\psi}{h}+\partial_x^3\psi+a\psi\right\|_s^2 = 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^s\left|\underbrace{\frac{e^{(ik^3-a)h}-1}{h}-ik^3+a}_{\mathcal{M}(h)=}\right|^2\left|\widehat{\psi}(k)\right|^2\,. \tag{6.11}$$

Using L'Hôpital's rule we have $\mathcal{M}(h) \to 0$ when $h \to 0$.

To ensure the interchange of limits, we need the uniform convergence of the series (6.11). For this, we will bound the k-th term of the series. Thus, using inequality (6.3) we obtain

$$\mathcal{L}_{k,s} = (1+k^2)^s \left|\mathcal{M}(h)\right|^2 \left|\widehat{\psi}(k)\right|^2 \leq 8\max\{1,a^2\}(1+k^2)^{s+3}\left|\widehat{\psi}(k)\right|^2.$$

On the other hand, we know that the series converges

$$2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s+3}\left|\widehat{\psi}(k)\right|^2 = \|\psi\|_{s+3}^2 < \infty,$$

since $\psi \in H^{s+3}_{per}$.

Thus, using the Weierstrass M-Test, we have that the series (6.11) converges uniformly and consequently it is possible to exchange limits and obtain

$$\left\| \frac{\mathcal{S}(h)\psi - \psi}{h} + \partial_x^3\psi + a\psi \right\|_s^2 \longrightarrow 0,$$

when $h \to 0^+$. □

6.2. Differentiability analysis versus initial data of (P_Σ). In this subsection, we will analyze the solution of (P_Σ).

Theorem 6.4. *Let n be a natural number such that $n-1$ is an even number not multiple of four, $a > 0$ and $s \in R$. If $t > 0$ and u is the solution of (P_Σ) with initial data $\psi \in H^s_{per}$ then for all $r \leq s-n$ it holds*

$$\lim_{h \to 0} \left\| \frac{u(t+h) - u(t)}{h} + \partial_x^n u(t) + au(t) \right\|_r = 0.$$

This means that differentiability occurs in H^r_{per}, $\forall r \leq s-n$. That is, $\partial_t u(t) = -\partial_x^n u(t) - au(t)$ in H^r_{per}, $\forall r \leq s-n$, $\forall t > 0$.

Proof: Its proof is analogous to the proof of Theorem 6.1. □

Theorem 6.5. *Let n be a natural number such that $n-1$ is an even number not multiple of four, $a > 0$ and $s \in R$. If u is the solution of (P_Σ) with initial data $\psi \in H^s_{per}$ then for all $r \leq s-n$ it holds*

$$\lim_{h \to 0^+} \left\| \frac{u(h) - \psi}{h} + \partial_x^n u(0) + au(0) \right\|_r = 0.$$

This means that differentiability occurs in H^r_{per}, $\forall r \leq s-n$. That is, $\partial_{t^+} u(0) = -\partial_x^n u(0) - au(0)$ in H^r_{per}, $\forall r \leq s-n$.

Proof: Its proof is analogous to the proof of Theorem 6.2. □

Theorem 6.6. *Let n be a natural number such that $n-1$ is an even number not multiple of four, $a > 0$, $s \in R$, and $\psi \in H^s_{per}$ then the following statements are equivalent*

1. *There exists $\lim\limits_{h \to 0^+} \left(\frac{\mathcal{T}(h) - I}{h}\right)\psi$ in $\left(H^s_{per}, \|\cdot\|_s\right)$.*
2. *$\psi \in H^{s+n}_{per}$.*

Proof: Its proof is analogous to the proof of Theorem 6.3. □

Below we state some additional results that can be obtained.

Remark 6.1. *Similar results to Theorems 6.4, 6.5 and 6.6 are obtained when n is a natural number such that $n-1$ is a number multiple of four.*

Remark 6.2. *Analogous results to Theorems 6.4, 6.5 and 6.6 are obtained when n is a number multiple of four.*

7. Conclusions. Based on Fourier theory, we rigorously demonstrated the existence and uniqueness of solution to the (P_1) model, along with the continuous dependence of the solution with respect to the initial data. Subsequently, we introduced the semigroup theory to rewrite the solution of the (P_1) problem using this theory, rendering it much more fine. We used semigroups theory and got important results of existence and approximation. We generalised the results for the n-th order equation when n is a natural number such that $n-1$ is an even number not multiple of four. And we have analyzed the other cases of n.

We also proved the dissipative property of (P_1), which enabled us to deduce the continuous dependence (with respect to the initial data) and uniqueness solution of (P_1), without using the explicit form of the solution. We obtained results analogous to Theorem 5.1, Corollary 5.1 and Corollary 5.2 for the n-th order equation when n is a natural number such that $n-1$ is multiple of four. Also, we gave some remarks about the n-th order equation when n is a multiple of four. Finally, to deepen and enrich our study, we analized the infinite-dimensional space in which differentiability occurs and its connection to the initial data.

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