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Analysis and numerical simulation of a parabolic equation with non-local terms

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Abstract
In this work, we investigate the existence and uniqueness of global strong solutions, as well as the exponential decay of these solutions in bounded domains, for an initial-boundary value problem associated with parabolic equations involving nonlocal terms. The theoretical results are complemented by numerical simulations obtained using the finite element method for the spatial variable and the finite difference method for the temporal variable.

Keywords . Parabolic equations, Nonlocal nonlinearities, Well-posedness, Energy decay, Numerical Simulation.

1. Introduction. This work addresses the existence, uniqueness, and regularity of solutions, as well as the energy decay and the decay in the $H^3(a,b)$-norm, for a parabolic equation involving nonlocal terms. The results obtained for the decay in the $H^3(a,b)$-norm constitute original contributions of this study. The theoretical analysis is carried out by means of the Faedo–Galerkin method combined with compactness arguments, whereas the numerical analysis employs linear finite elements for the spatial discretization and the Crank–Nicolson scheme for the temporal one, with simulation results implemented using the FreeFem++ software [1].

2. Problem Formulation. Our goal is to find a function $u : (a,b) \times [0,T] \to \mathbb{R}$, solution to the problem

$$\begin{cases} u_t(x,t) - \bar{a}\Big(\displaystyle\int_a^b u_x(x,t)^2 dx\Big) u_{xx} + f(u) = g(x,t) & a < x < b, \quad 0 < t < T, \\ u(0,t) = u(1,t) = 0 & 0 < t < T, \\ u(x,0) = u_0(x), & a < x < b, \end{cases} \tag{2.1}$$

with

H1: Let $\bar{a} \in C_b^1(\mathbb{R})$, satisfying

$$0 < m \leq \bar{a}(s) \leq N, \quad \text{where } \{m, N\} \in \mathbb{R}^+.$$

H2: The function $f : \mathbb{R} \to \mathbb{R}$ belongs to the class $C_b^1(\mathbb{R})$ and satisfies

$$sf(s) \geq 0, \quad \forall s \in \mathbb{R}, \qquad f(0) = 0.$$

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H3: The function $g \in L^2\left((a,\,b)\times(0,T)\right)$;
H4: The initial data $u_0 \in H_0^1(a,b)$.

It is usually said that problem (2.1) contains non-local nonlinearities due to the term $\bar{a}(\int_a^b u_x(x,t)^2\,dx)$. Problems with non-local terms have important applications in physics and biology, for example in the case of migrations of populations such as bacteria in a container, the diffusion coefficient at time t depends on the total population. For topics on well-posedness, decay, and other theoretical properties of models similar to (2.1), see references [2], [3], [4], [5], [6], [7], [8],[9], [10], [11] and [12]. Numerical methods for similar models to (2.1), see references [13], [14], [9], [15], [16], [11], [17], [18][19] and [12]. Topics on mathematical models related to (2.1), see references [20, 21]. Some topics on control of equations similar to (2.1), see references [22], [23], [24], [13], [9], [25][26] and [27]. Now, we pesent the principal results:

Theorem 2.1. ***(i)*** *Under hypotheses* **H1–H4** *given above there exist a unique weak solution of initial-boundary value problem* (2.1)

(ii) *Under hypotheses* **H1**, **H2** *with* $g \in L^2(0,T,H_0^1(a,b))$ *and* $u_0 \in H_0^1(a,b)\cap H^2(a,b)$. *There exists a unique regular solution of* (2.1) *in the class* $u \in L^\infty(0,T;H_0^1(a,b)\cap H^2(a,b)),\ u_t \in L^2(0,T;H_0^1(a,b))$.

Theorem 2.2. *(Energy Decay) Under the conditions of Theorem 2.1* ***(ii)*** *considering* $g=0$ *and the initial data satisfying:*

$$|u_{0x}|^2 + |u_{0xx}|^2 \le \epsilon,$$

for convenient $\epsilon > 0$. Denoting the energy by $E(t) = |u(t)|^2 + |u_x(t)|^2 + |u_{xx}(t)|^2$, then we obtain the exponential decay of energy. That is,

$$E(t) \le c\left(|u_{0x}|, |u_{0xx}|\right) e^{\frac{-mt}{4\tilde{c}_1}}\ ,$$

where $\tilde{c}_1$ is defined below.

Theorem 2.3. *Considering the hypotheses* **H1** and **H2**, $u_0 \in H^3(a,b)$ *there exists a positive constant* ϵ *such that*

$$|u_0| + |u_{0x}| + |u_{0xx}| + |u_{0xxx}| \le \epsilon\,,$$

then the solution of (2.1) *satisfies:*

1. $\|u_t\|_{L^2(a,b)} \le ce^{-mt/8\bar{c}_1}$;
2. $\|u\|_{H^3(a,b)} \le c_7 e^{-\gamma_1 t}$.

3. Exitence, Uniqueness and regularity of solutions.

Proof: **Proof of Theorem 2.1** **(*i*)**: Let (ϕ_i) be the eigenfunctions of a special basis of $H_0^1(a,b)$ that solve the following spectral problem:

$$\begin{cases} -(\phi_i)_{xx} = \lambda_i \phi_i \quad \text{in} \quad (a,b)\,, \\ \phi_i(a) = \phi_i(b) = 0\,, \end{cases} \tag{3.1}$$

and $V_m = [\phi_1, \phi_2, ..., \phi_m]$ the subspace generated by the m first eigenvectors ϕ_i in $H_0^1(a,b)$.

We look for $u_m \in V_m$, that is, $u_m(x,t) = \sum_{i=1}^{m} d_{im}(t)\,\phi_i(x)$, satisfying:

$$\begin{cases} (u_m'(x,t),\, \phi_i(x)) \;+\; \bar{a}\left(\displaystyle\int_a^b |u_{mx}(x,t)|^2\,dx\right)(u_{mx}(x,t),\,\phi_{ix}(x)) \\ \qquad\qquad +\; (f(u_m(x,t)),\,\phi_i(x)) \;=\; (g(x,t),\,\phi_i(x)), \qquad i=1,2,\dots,m, \\ \qquad u_m(0) = u_{0m} \;\longrightarrow\; u_0 \quad \text{in } L^2(a,b). \end{cases} \tag{3.2}$$

Replacing u_m in (3.2) we have:

$$\Big(\sum_{i=1}^{m} d_{im}'(t)\,\phi_i(x),\,\phi_i(x)\Big) \;+\; \bar{a}\left(\int_a^b \left(\sum_{k=1}^{m} d_{km}(t)\,\phi_{ix}(x)\right)^2 dx\right)\Big(\sum_{k=1}^{m} d_{km}(t)\,\phi_{kx}(x),\,\phi_{ix}(x)\Big)$$
$$+\;\left(f\left(\sum_{k=1}^{m} d_{km}(t)\,\phi_i(x)\right),\,\phi_i(x)\right) \;=\; (g(x,t),\,\phi_i(x)), \qquad i=1,2,\dots,m.$$

Let us rewrite the expression as a system of nonlinear ordinary differential equations.

To this end, introduce the notation

$$d(t) = \big(d_{1m}(t),\, d_{2m}(t),\, \dots,\, d_{mm}(t)\big)^{\tau},$$

where τ denotes the transpose of the row vector

$$(d_{1m}(t),\, d_{2m}(t),\, \dots,\, d_{mm}(t)),$$

so that $d(t)$ is a column vector with m components;

$$C(d(t)) = a\left(\int_0^1 \left(\sum_{k=1}^{m} d_{im}(t)\phi_{ix}(x)\right)^2\right)(\phi_{ik}(x), \phi_{ix}(x)) + (\theta(\sum_{k=1}^{m} d_{km}(t)\phi_k(x)) \cdot \phi_k(x), \phi_i(x))\,;$$

$A = (\phi_k(x), \phi_i(x))$ is a square matrix of order $m \times m$ and $C(d(t))$ is a square matrix of order $m \times m$ that depends on $d(t)$, $G(t) = ((g(x,t), \phi_1), ..., (g(x,t), \phi_m))^{\tau}$, G is a column vector that only depends on t.

Thus, considering the vectors $d(t), G(t)$ and the matrices A and $C(d(t))$ defined above, we have the following non-linear ODE system:

$$Ad'(t) + C(d(t))d(t) = G(t)\,. \tag{3.3}$$

Note that the second term on the left is nonlinear.

Also : $u_m(0) = u_{0m} \to u_0$ in $L^2(0,1)$.

We can consider $u_{0m} = \displaystyle\sum_{i=1}^{m}(u_0, \phi_i)\phi_i$ since

$$u_{0m} = \sum_{i=1}^{m}(u_0, \phi_i)\phi_i \to \sum_{i=1}^{\infty}(u_0, \phi_i)\phi_i = u_0\,,$$

also as $u_m(t) = \displaystyle\sum_{i=1}^{m} d_{im}(t)\phi_i(x)$, therefore

$$u_m(0) = \sum_{i=1}^{m} d_{im}(0)\phi_i(x) = \sum_{i=1}^{m}(u_0, \phi_i)\phi_i = u_{0m}\,,$$

whence $d_{im}(0) = (u_0, \phi_i)$. So we have

$$d(0) = (d_{1m}(0), ..., d_{mm}(0))^{\tau} = ((u_0, \phi_1), ..., (u_0, \phi_m))^{\tau} = d_0\,.$$

Therefore, we have the nonlinear ODE system with initial conditions

$$\begin{cases} Ad'(t) + C(d(t))d(t) = G(t), \\ d(0) = d_0, \end{cases} \tag{3.4}$$

where $d_0 = \big((u_0, \phi_1), \dots, (u_0, \phi_m)\big)^{\tau}$.

For each dimension m, the approximate problem falls into a nonlinear ordinary differential equation. Once m is fixed, the Carathéodory's Theorem is used to obtain the existence of a local solution, u_m, defined on $(0, t_m)$ with $t_m < T$. To extend the local solution to a global one, ODE estimates are necessary.

Estimate 1. Multiplying the approximate equation (3.2) by $d_{im}(t)$ and summing over $i = 1$ to m, we obtain

$$\big(u'_m(t), u_m(t)\big) + \bar{a}\left(\int_a^b |u_{mx}(x,t)|^2\,dx\right)\big(u_{mx}(t), u_m(t)\big) + \big(f(u_m(t)),\, u_m(t)\big) = \big(g(x,t),\, u_m(x,t)\big). \tag{3.5}$$

The following identities are verified:

$$\big(u'_m(t),\, u_m(t)\big) = \frac{1}{2}\,\frac{d}{dt}\,|u_m(t)|^2\,; \tag{3.6}$$

$$\bar{a}\left(\int_a^b |u_{mx}(x,t)|^2\,dx\right)\big(u_{mx}(t),\, u_{mx}(t)\big) = \bar{a}\left(\int_a^b |u_{mx}(x,t)|^2\,dx\right)|u_{mx}(t)|^2\,; \tag{3.7}$$

$$(f(u_m(t)),\, u_m(t)) = \int_a^b f(u_m(x,t))\,u_m(x,t)\,dx \geq 0\,. \tag{3.8}$$

Therefore, using (3.6)–(3.8) in (3.5) and applying the Schwarz inequality, we obtain:

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \bar{a}\left(\int_a^b |u_{mx}(x,t)|^2 dx\right)|u_{mx}(t)|^2 \leq \frac{1}{2}|g(t)|^2 + \frac{1}{2}|u_m(t)|^2\,. \tag{3.9}$$

Also, under assumption **H1**, we have

$$\bar{a}\left(\int_a^b |u_{mx}(x,t)|^2\,dx\right) \geq m.$$

Thus, integrating (3.9) from 0 to t, we obtain:

$$\frac{1}{2}|u_m(t)|^2 + m\int_0^t |u_{mx}(t)|^2 dt \leq \frac{1}{2}|u_0|^2 + \frac{1}{2}\int_0^t |g(t)|^2 ds + \frac{1}{2}\int_0^t |g(t)||u_m(t)|^2\,dt\,. \tag{3.10}$$

Using the Gronwall's inequality, we obtain **Estimate 1**:

$$|u_m(t)|^2 \;+\; 2m\int_0^t |u_{mx}(t)|^2\,dt \;\leq\; \Big(|u_0|^2 + \int_0^T |g(t)|\Big)\exp\Big(\int_0^T |g(t)|\,dt\Big). \tag{3.11}$$

From which it follows that u_m is limited in $L^\infty(0,T,L^2(a,b))$ and u_{mx} is limited in $L^2(0,T,L^2(a,b))$.

Estimate 2. Multiplying the approximate equation (3.2) by $d'_{im}(t)$ and summing from $i = 1$ to m we have:

$$\begin{aligned}(u'_m(x,t),\, u'_m(x,t)) \;+\; \bar{a}\left(\int_a^b |u_{mx}(x,t)|^2\,dx\right)(u_{mx}(x,t),\, u'_{mx}(x,t))\\ +\; \big(f(u_m(x,t)),\, u'_m(x,t)\big) \;=\; \big(g(x,t),\, u'_m(x,t)\big).\end{aligned} \tag{3.12}$$

The following identities are verified:

$$\begin{aligned}&\bar{a}\left(\int_0^1 |u_{mx}(x,t)|^2 dx)(u_m(x,t), u'_m(x,t)\right) =\\ &\frac{1}{2}\bar{a}\left(\int_a^b |u_{mx}(x,t)|^2 dx\right)\left(\frac{d}{dt}|u_{mx}(t)|^2\right) = \frac{1}{2}\frac{d}{dt}\left(\bar{a}(|u_{mx}(t)|^2)\right),\end{aligned} \tag{3.13}$$

where $\overline{a}(s) = \displaystyle\int_0^s a(\lambda)d\lambda$, and

$$\big(f(u_m(x,t)),\, u'_m(x,t)\big) = \int_a^b \frac{d}{dt}\bar{f}(u_m)\,dx, \tag{3.14}$$

where $\bar{f}(s) = \displaystyle\int_0^s f(\lambda)d\lambda \geq 0$.

So from (3.12) and (3.13) in (3.11), we have:

$$\frac{1}{2}|u'_m(t)|^2 + \frac{1}{2}\frac{d}{dt}\bar{a}\left(|u_{mx}|^2\right) \leq C|g|^2 + C_\epsilon|u_m|^2 + \frac{d}{dt}\int_a^b \bar{f}(u_m) \leq \frac{1}{2}|g|^2. \tag{3.15}$$

Integrating from 0 to t above we have:

$$\begin{aligned}\int_0^t |u'_m(t)|^2 + \bar{a}\left(|u_{mx}|^2\right) + \int_a^b \bar{f}(u_m)\,dx &\leq C\int_0^t |g|^2 + \bar{a}(|u_{0x}|^2)\\ &\leq C\left(|u_{0x}|^2 + \int_0^T |g|^2\right).\end{aligned} \tag{3.16}$$

By assumption **H1**, we have $\bar{a}(s) \geq m$, therefore $\bar{a}(s) \geq ms$, as follows:

$$\bar{a}\left(|u_{mx}|^2\right) \geq m|u_{mx}|^2. \tag{3.17}$$

Using the previous results, we have:

$$\int_0^t |u_m'(s)|^2\, ds + m|u_{mx}|^2 + \int_a^b \bar{f}(u_m(t))\, dx \leq c\left(|u_{0x}|^2 + \int_0^T |g|^2\right). \tag{3.18}$$

From which it can be concluded that: u_m' is bounded in $L^2(0,T,L^2(a,b))$, u_{mx} is limited in $L^\infty(0,T,L^2(a,b))$.

Estimate 3. Multiplying the approximate equation (3.2) by $\lambda_i d_{im}(t)$ and adding from $i = 1,\dots,m$ we have:

$$(u_m'(x,t), -u_{mxx}(x,t)) + \bar{a}(\int_a^b |u_{mx}(x,t)|^2)(u_{mx}, (-u_{mxx}(x,t))_x) \tag{3.19}$$

$$+ (f(u_m(x,t)), -u_{mxx}(x,t)) = (g(x,t), -u_{mxx}). \tag{3.20}$$

Analogously to **Estimate 2** we have:

$$|(f(u_m), -u_{mxx})| \leq \int_a^b |f'(u_m)||u_{mx}|^2\, dx \leq c\int_a^b |u_{mx}|^2\, dx,$$

$$|(g(x,t), -u_{mxx}| \leq \frac{1}{2m}|g(x,t)|^2 + \frac{m}{2}|u_{mxx}|^2.$$

Thus, we have

$$\begin{aligned} &\frac{1}{2}|u_{mx}(t)|^2 + m\int_0^T |u_{mxx}(t)|^2 dt + \int_0^T\int_a^b |f'(u_m)||u_{mx}|^2 dx \\ &\leq \frac{1}{2}|u_{0x}|^2 + \frac{1}{2m}\int_0^T |g(t)|^2 + \frac{c}{2m}\Big(|u_0|^2 + \int_0^T |g(t)|\Big) exp\Big(\int_0^T g(t)\Big) \leq c_1(u_{0x},\, f,\, g). \end{aligned} \tag{3.21}$$

From which it can be concluded that:

- u_{mxx} is limited in $L^2(0,T,L^2(a,b))$;
- u_{mx} is limited in $L^\infty(0,T,L^2(a,b))$.

Passing to the limit. From the **Estimates 1, 2 and 3** we have that:

- (u_m) is limited in $L^\infty(0,T,H_0^1(a,b)) \cap L^2(0,T,H_0^1(a,b) \cap H^2(a,b))$;
- (u_m') is limited to $L^2(0,T,L^2(a,b))$.

Since $H^2(a,\, b) \cap H_0^1(a,b)$ is compactly embedded in $H_0^1(a,b)$, using the Aubin-Lions-Simon compactness theorem, we have

$$W = \{Z \in L^2(0,T,H_0^1(a,b) \cap H^2(a,b)) \text{ such that } Z' \in L^2(0,T,L^2(a,b))\},$$

is compactly immersed in $L^2(0,T,H_0^1(a,b))$.

We can thus obtain a subsequence of u_m such that:

$$\begin{cases} u_m \to u & \text{weak in} \quad L^2(0,T,H_0^1(a,b) \cap H^2(a,b)), \\ u_m' \rightharpoonup u' & \text{weak}^* \text{ in} \quad L^\infty(0,T,H_0^1(a,b)), \\ u_m' \to u' & \text{weak in} \quad L^2(0,T,L^2(a,b)), \\ u_m \to u & \text{strong in} \quad L^2(0,T,H_0^1(a,b)). \end{cases} \tag{3.22}$$

Multiplying (3.5) by $\theta_i(t) \in L^2(0,T)$ and adding from $i = 1$ to m, we have

$$\begin{aligned} &\int_0^T (u_m', \varphi) + \int_0^T \bar{a}\left(\int_a^b |u_{mx}(x,t)|^2 dx\right)(u_{mx}, \varphi_x) \\ &+ \int_0^T (f(u_m), \varphi) = \int_0^T (g, \varphi), \quad \forall \varphi = \sum_{i=1}^m \theta_i(t)\phi_i(x). \end{aligned} \tag{3.23}$$

Also from (3.22), $u_m' \to u'$ weak in $L^2(0,T,L^2(a,b))$ and since $\varphi \in L^2(0,T,H_0^1(a,b))$ which is included in $L^2(0,T,L^2(0,1))$, has It is clear from the definition of weak convergence that

$$\int_0^T (u_m', \varphi) \to \int_0^T (u', \varphi). \tag{3.24}$$

Also using the notation:

$$\|u_m\|^2 = \int_a^b |u_{mx}(x,t)|^2 dx\,.$$

We will prove that:

$$\int_0^T \overline{a}(\|u_m\|^2)(u_{mxx},\varphi) \to \int_0^T a(\|u\|^2)(u_{xx},\varphi). \tag{3.25}$$

In effect, using Green's Theorem:

$$\begin{cases} |\displaystyle\int_0^T \overline{a}(\|u_m\|^2)(u_{mxx},\varphi) - \int_0^T \overline{a}(\|u\|^2)(u_{xx},\varphi)| \\ = |\displaystyle\int_0^T \overline{a}(\|u_m\|^2)(u_{mx},\varphi_x) - \int_0^T \overline{a}(\|u\|^2)(u_x,\varphi_x)| \\ + \displaystyle\int_0^T \overline{a}(\|u\|^2(u_{mx}-u_x,\varphi_x) = I_1 + I_2\,. \end{cases} \tag{3.26}$$

We have the Mean Value Theorem:

$$\begin{cases} |\overline{a}(\|u_m\|^2) - \overline{a}(\|u\|^2)| \leq \overline{a'}(\theta)(\|u_m\|^2) - (\|u\|^2) \leq c(\|u_m\|^2 - \|u\|^2) \\ \leq c(\|u_m - u\|)(\|u_m\| + \|u\|). \end{cases} \tag{3.27}$$

Remark 1: From (3.22) we have that $u_m \to u$ strong in $L^2(0,T,H_0^1(a,b))$, therefore $u_m - u \to 0$ strong in $L^2(0,T,H_0^1(a,b))$, whence $\|u_m - u\| \to 0$ strong in $L^2(0,T)$.

Also as u_m is limited in $L^\infty(0,T,H_0^1(a,b))$, therefore $\|u_m(t)\|$ is limited. Thus, we have

$$\|u_m(t)\| + \|u\| \leq c_2\,.$$

From (3.27) and Remark 1 above:

$$\left|\overline{a}(\|u_m\|^2) - \overline{a}(\|u\|^2)\right| \to 0\,,$$

strong in $L^2(0,T)$. There is also

$$\begin{aligned} \int_0^T |(u_{mx},\varphi_x)|^2 &\leq \int_0^T |u_{mx}|^2|\varphi_x|^2 \leq \|u_{mx}\|_\infty^2 \|\varphi_x\|^2_{L^2(0,T,L^2(a,b))} \\ &\leq \|u_{mx}\|^2_{L^\infty(0,T,H_0^1(a,b)))}\|\varphi\|_{L^2(0,T,H_0^1(a,b))} \leq c. \end{aligned} \tag{3.28}$$

We will prove that I_1 and I_2 converge to zero:

$$I_1 = \int_0^T \overline{a}(\|u_m\|^2) - \overline{a}(\|u\|^2))(u_{mx},\varphi_x) \leq (\int_0^T |\overline{a}(\|u_m\|^2) - \overline{a}(\|u\|^2)|^2)^{\frac{1}{2}}(\int_0^T (u_{mx},\varphi_x)^2)^{\frac{1}{2}}$$

$$\leq c(\int_0^T |\overline{a}(\|u_m\|^2) - \overline{a}(\|u\|^2)|^2)^{\frac{1}{2}} \mapsto 0.$$

Also by (3.22) $|u_m - u|_{L^2(0,T,H_0^1(0,1)} \mapsto 0$ therefore:

$$I_2 = \int_0^T \overline{a}(\|u\|^2)|(u_{mx} - u_x,\varphi_x)| \leq \|u_m - u\|_{L^2(0,T,H_0^1(a,b))}\|\varphi_x\|_{L^2(0,T,H_0^1(a,b))} \mapsto 0.$$

Let us now prove that:

$$\int_0^T (f(u_m),\varphi) \to \int_0^T (f(u),\varphi). \tag{3.29}$$

With effect from (3.22), we have a subsequence of u_m, also denoted by u_m, such that:

$$u_m \to u \; q.s \text{ in } (0,1)\times(0,T)\,.$$

As f is Lipschitzian, therefore f is a continuous function, so we have:

$$f(u_m) \to f(u) \text{ q.s } (a,b)\times(0,T)\,. \tag{3.30}$$

Also as f satisfies $|f(s)| \le M|s|$ then from **Estimate 1** we have:

$$\int_0^T\int_a^b |f(u_m)|^2 \le \int_0^T\int_0^1 M|u_m|^2dx \le c\int_0^T\int_a^b |u_{mx}|^2 \le c\,. \tag{3.31}$$

Thus, using the Lions lemma of (3.30) and (3.31), we have: $f(u_m)\to f(u)$ weak in $L^2(0,T,L^2(a,\,b))$, therefore, by the definition of weak convergence we have:

$$\int_0^T (f(u_m),\varphi) \to \int_0^T (f(u),\varphi). \tag{3.32}$$

So from (3.22), (3.25) we can go to the limit in (3.23) and we have:

$$\int_0^T (u',\varphi) + \int_0^T a(\|u\|^2)(u_{xx},\varphi) + \int_0^T (f(u),\varphi) = \int_0^T (g,\varphi)\,, \quad \forall\ \varphi(x,t) = \sum_{i=1}^m \theta_i(t)\phi_i(x). \tag{3.33}$$

With $\theta_i(t)\in L^2(0,T)$ and ϕ_i element of the base of $H_0^1(0,1)$ given in (3.24), as $\{\sum_{i=1}^m \theta_i(t)\phi_i(x)\}$ are dense in $L^2(0,T,H_0^1(0,1))$, we have for all $\varphi\in L^2(0,T,H_0^1(0,1))$

$$\int_0^T (u',\varphi) + \int_0^T a(\|u\|^2)(u_{xx},\varphi) + \int_0^T (f(u),\varphi) = \int_0^T (g,\varphi). \tag{3.34}$$

Thus, we demonstrate the existence of the weak solution, that is, we conclude the proof of (i) of Theorem 2.1. □

Proof: **Proof Theorem 2.1 - (*ii*) :**

To prove that the weak solution of Theorem 2.1-*(i)* is more regularity. We need additional hypotheses about g and u_0, that is, $g\in L^2(0,T,H_0^1(a,b))$ and $u_0\in H_0^1(a,b)\cap H^2(a,b)$ to obtain the following Estimate:

Multiplying the approximate equation (3.5) by $\lambda_i d'_{im}(t)$ and adding from $i=1$ to m, we have:

$$\begin{aligned}(u'_m(x,t),-u'_{mxx}(x,t)) + \overline{a}(\int_0^1 |u_{mx}(x,t)|^2)(u_{mx}(x,t),(-u'_{mxx}(x,t))_x)\\ +(f(u_m(x,t)),-u'_{mxx}(x,t)) = (g(x,t),-u'_{mxx}(x,t)).\end{aligned} \tag{3.35}$$

Analogously to previous estimates, we have:

$$\begin{aligned}\frac{1}{2}|u'_{mx}(t)|^2 + \frac{1}{2}\frac{d}{dt}\left(\overline{a}\left(\int_a^b |u_{mx}(x,t)|^2dx\right)|u_{mxx}(t)|^2\right)\\ \le C|g_x(t)|^2 + C_1(T,g,u_0) + C(1+|u_{mxx}|^2)(|u_{mxx}|^2).\end{aligned} \tag{3.36}$$

Integrating from 0 to t, and using $\overline{a}\left(\int_0^1 |u_{mx}(x,t)|^2dx\right)\ge m$, we have:

$$\int_0^t |u'_{mx}(t)|^2dt + m|u_{mxx}(t)|^2 \le a(|u_{0x}|^2)|u_{0xx}|^2 + \int_0^t |g_x(t)|^2 + CT + C\int_0^t (1+|u_{mxx}|^2)(|u_{mxx}|^2)dt\,.$$

According to the hypothesis $a(s)\ge m$ and by **Estimate 3**

$$\int_0^T |u_{mxx}|^2 \le C(T,g,u_0),$$

from the previous expression using Gronwall's inequality has:

$$\int_0^T |u'_{mx}(t)|^2 + m|u_{mxx}(t)|^2 \le C\left(|u_{0xx}|^2 + \int_0^T |g_x(t)|^2 + T\right)\exp(\int_0^T |u_{mxx}|^2). \tag{3.37}$$

and so we finally have **Estimate 4**:

$$\int_0^T |u'_{mx}(t)|^2 + m|u_{mxx}(t)|^2 \leq C_2(T, g, u_0)\left(|u_{0xx}|^2 + \int_0^T |g_x(t)|^2 + T\right). \tag{3.38}$$

From the above estimates, it follows that

$$u'_{mx} \text{ is bounded in } L^2(0,T;L^2(a,\,b)) \quad \text{and} \quad u_{mxx} \text{ is bounded in } L^\infty(0,T;L^2(a,\,b)).$$

Consequently, the weak solution, obtained as the limit of the sequence of approximate solutions, exhibits additional regularity; namely,

$$u' \in L^2(0,T;H_0^1(a,\,b)) \quad \text{and} \quad u_{xx} \in L^\infty(0,T;L^2(a,\,b)).$$

□

4. Exponential decay of solutions.

Proof: **Proof Theorem 2.2** By the Banach Steinhaus Theorem we have $E(t) \leq \lim inf\ E_m(t)$ where

$$E_m(t) = |u_m(t)|^2 + |u_{mx}(t)|^2 + |u_{mxx}(t)|^2 \,.$$

Thus, it will be enough to prove the exponential decay for $E_m(t)$, which for simplicity we will denote by $E(t)$. Following as in **Estimate 3**, with $g = 0$ we have

$$\frac{1}{2}\frac{d}{dt}|u_{mx}(t)|^2 + \frac{m}{2}|u_{mxx}(t)|^2 \leq \bar{c}|u_{mx}(t)|^2 \,, \tag{4.1}$$

where $\bar{c} = \dfrac{c_0 M}{m}$. Also analogous to **Estimate 4** with $g = 0$, we obtain from (3.36):

$$\frac{1}{2}|u'_{mx}(t)|^2 + \frac{1}{2}\frac{d}{dt}|\bar{a}\Big(\int_a^b |u_{mx}(x,t)|^2 dx\Big)||u_{mxx}(t)|^2 \leq c_1|u_{mx}(t)|^3|u_{mxx}(t)|^3. \tag{4.2}$$

From **Estimate 1**, (3.9) with $g = 0$ we have

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + m|u_{mx}(t)|^2 \leq 0\,. \tag{4.3}$$

Multiplying (4.1) by $\dfrac{m}{2\bar{c}}$ and adding (4.2) and (4.3) we have:

$$\begin{aligned}&\frac{d}{dt}\left(\frac{m}{4\bar{c}}|u_{mx}(t)|^2 + \frac{1}{2}\bar{a}(\int_a^b |u_{mx}|^2 dx)|u_{mxx}(t)|^2 + \frac{1}{2}|u_m(t)|^2\right)\\ &+\frac{m}{4}\Big(|u_{mx}(t)|^2 + \frac{m}{\bar{c}}|u_{mxx}(t)|^2\Big) + |u_{mxx}(t)|^2\left(\frac{m^2}{4\bar{c}} - c_1|u_{mx}(t)|^2|u_{mxx}(t)|\right) \leq 0\,.\end{aligned} \tag{4.4}$$

Assertion $c_1|u_{mx}(t)|^2|u_{mxx}(t)| \leq \dfrac{m^2}{4\bar{c}}, \forall\, t \geq 0.$

The proof of the statement is done by contradiction using $|u_{0x}|^2 + |u_{0xx}|^2 \leq \epsilon$.

So we have:

$$\begin{aligned}&\frac{d}{dt}\left(\frac{m}{4\bar{c}}|u_{mx}(t)|^2 + \frac{1}{2}|\bar{a}(\int_0^1 |u_{mx}(xt)|^2 dx)|u_{mxx}(t)|^2 + \frac{1}{2}|u_m(t)|^2\right)\\ &+\frac{m}{4}\left(|u_{mx}(t)|^2 + \frac{m}{\bar{c}}|u_{mxx}(t)|^2\right) \leq 0.\end{aligned} \tag{4.5}$$

Note that

$$J(t) = \frac{m}{4\bar{c}}|u_{mx}(t)|^2 + \frac{1}{2}\,\bar{a}\bigg(\int_0^1 |u_{mx}(x,t)|^2\,dx\bigg)|u_{mxx}(t)|^2 + \frac{1}{2}|u_m(t)|^2.$$

Then, there exist positive constants $\hat{c}$ and $\tilde{c}$ such that

$$\hat{c}\,E(t) \leq J(t) \leq \tilde{c}\,E(t).$$

Also from the Poincaré inequality ($|u(t)| \leq c_0|u_x(t)|$) we have

$$J(t) \leq \tilde{c}E(t) \leq \tilde{c}\Big((1+c_0)|u_{mx}(t)|^2 + \frac{m}{\bar{c}}|u_{mxx}(t)|^2\Big) \leq \tilde{c_1}\Big(|u_{mx}(t)|^2 + \frac{m}{\bar{c}}|u_{mxx}(t)|^2\Big), \tag{4.6}$$

where $\tilde{c_1} = \tilde{c}(1+c_0)$. Thus, we have:

$$\frac{d}{dt}J(t) + \frac{m}{4\tilde{c_1}}J(t) \leq 0.$$

Using the integrating factor $e^{\frac{m.t}{4\tilde{c_1}}}$ it must be:

$$J(t) \leq J(0)e^{\frac{-m.t}{4\tilde{c_1}}}.$$

Finally, from the previous observation, we have:

$$\hat{c}E(t) \leq J(t) \leq J(0)e^{\frac{-m.t}{4\tilde{c_1}}},$$

and so

$$E(t) \leq c(|u_{0x}|, |u_{0xx}|)e^{\frac{-m.t}{4\tilde{c_1}}}.$$

This is where the exponential decay of energy comes from. □

Proof: **Proof of Theorem 2.3:**
From (2.1) with $g = 0$ we have:

$$u_t(x,t) = \overline{a}(\|u(t)\|^2)u_{xx} - f(u).$$

According to hypothesis **H1**,

$$0 < m \leq \overline{a}(s) \leq N,$$

for all $s \in \mathbb{R}$, and by **H2**, $|f(s)| \leq M|s|$, we obtain:

$$|u_t(x,t)| \leq N|u_{xx}| + M|u|,$$

from where

$$|u_t|^2_{L^2(0,1)} \leq 2\max\{N, M\}\Big(|u_{xx}|^2_{L^2(0,1)} + |u|^2_{L^2(0,1)}\Big) \leq cE(t).$$

Thus, by Theorem 2.2, we have

$$|u_t|_{L^2(0,1)} \leq cE(t) \leq ce^{\frac{-m}{4\tilde{c_1}}t},$$

whence is proved (i).
Now, we proof (ii).
From (2.1) we have

$$u_{xx}(x,t) = \frac{1}{\overline{a}(\|u(t)\|^2)}(u_t(x,t) + f(u)) = \frac{u_t(x,t)}{\overline{a}(\|u(t)\|^2)} + \frac{f(u)}{\overline{a}\|u(t)\|^2)} = I_1 + I_2 = \theta.$$

From hypothesis **H1**:

$$|I_1| \leq \frac{|u_t|}{|\overline{a}(\|u\|^2)|} \leq \frac{1}{m}|u_t|,$$

whence, by item (i):

$$|I_1|_{L^2(0,1)} \leq c_1 e^{\frac{-m}{8c_1}t}.$$

Similarly:

$$|I_2| \leq \frac{|f(u)|}{\overline{a}(\|u\|^2)} \leq \frac{1}{m}M|u|$$

whence, by Theorem 2.2:

$$|I_2| \leq ce^{\frac{-mt}{8c1}}.$$

Therefore $|\theta|_{L^2(0,1)} \leq |I_1| + |I_2| \leq ce^{\frac{-mt}{8c1}}$. Also,

$$\theta_x = I_{1x} + I_{2x}. \tag{4.7}$$

Deriving the approximate problem (3.2), we have

$$\left(u_m'' - \overline{a}(\|u_m\|^2)(u_{mx}, u_{mx}')u_{mxx} - \overline{a}(\|u_m\|^2)u_{mxx}' + f'(u_m)u_m', \phi_i(x)\right) = 0.$$

Multiplying the above equality by $\lambda_i d_{im}'(t)$ and adding from $i = 1, \ldots, m$ we obtain

$$(u_m'', -u_{mxx}') + \bar{a}(\|u_m\|^2)\|u_{mxx}'\|^2 = -2\bar{a}(\|u_m\|^2)(u_{mx}, u_{mx}')(u_{mxx}, u_{mxx}') + (f'(u_m)u_m', u_{mxx}')\,.$$

From the above and following the arguments in the proof of Theorem 2.2, together with

$$|u_0|+|u_{0x}|+|u_{0xx}|+|u_{0xxx}|\leq \epsilon\,,$$

we deduce that

$$(|u_{mx}'(t)|^2 + |u_{mxx}(t)|^2 + |u_{mx}(t)|^2 + |u_m(t)|^2) \leq c_1 e^{-\gamma t}. \tag{4.8}$$

Thus, from the previous expression and hypothesis **H1**, we have:

$$\begin{aligned} |I_{1x}| &= \left| \frac{u_{xt}(x,t)}{\overline{a}(||u(t)||^2)} \right| \leq \frac{1}{m} c_1 e^{-\gamma t}, \\ |I_{2x}| &= \left| \frac{f'(u)u_x}{\overline{a}(||u(t)||^2)} \right| \leq \frac{M|u_x|}{m} \leq \frac{M}{m} c_1 e^{-\gamma t}. \end{aligned} \tag{4.9}$$

Thus, we have:

$$|\theta_x|_{L^2(0,1)} \leq c_5 e^{-\gamma t}.$$

It is:

$$\gamma_1 = \min\left\{\frac{m}{8\bar{c_1}}, \gamma\right\},$$

we also have:

$$|\theta|_{H^1(a,\,b)} \leq |\theta|_{L^2(a,\,b)} + |\theta_x|_{L^2(a,\,b)} \leq c_6 e^{-\gamma_1 t}. \tag{4.10}$$

Just as $u_{xx} = \theta$ therefore $u_{xx} + u = \theta + u$, whence by the elliptic regularity theorem (Theorem 9.25 - Brézis [2].)

Elliptic Regularity Theorem. Let $u \in H_0^1(a,\,b)$ be a weak solution of $u_{xx} + u = f$ if $f \in H^m(a,\,b)$ then $u \in H^{m+2}(a,\,b)$ and

$$|u|_{H^{m+2}(a,\,b)} \leq c|f|_{H^m(a,\,b)}.$$

Using the Theorem above for $m = 1$ and $f = \theta + u$, we have:

$$u \in H^3(a,\,b) \quad \text{and} \quad |u|_{H^3(a,\,b)} \leq |\theta + u|_{H^1(a,\,b)} \leq |\theta|_{H^1(a,\,b)} + |u|_{H^1(a,\,b)} \leq c_6 e^{-\gamma_1 t}. \tag{4.11}$$

□

5. Numerical Approximation Scheme. To complement the proposed problem, we perform numerical simulations that both accompany and validate the theoretical development. To this end, we adopt an approach that combines the Crank–Nicolson method for temporal discretization with the finite element method for spatial discretization. This hybrid method provides the essential accuracy and stability required for solving complex problems (see [8], [11] and [15]).

5.1. Spatial Discretization. Initially, we will adapt the notation for approximate solutions and spaces, considering h as an index indicative of finite element discretization, considering the Galerkin approximation of problem (2.1) for each $t > 0$. Thus, we seek to obtain $u_h(t) \in V_h \subset V$ such that

$$\frac{\partial u_h(t)}{\partial t} - a\left(\int_0^1 \left(\frac{\partial u_h(t)}{\partial x}\right)^2\right)\frac{\partial^2 u_h(t)}{\partial x^2} + f(u_h(t)) = g(x,t), \tag{5.1}$$

with $u_h(0) = u_{0h}$, where $V_h \subset V$ is a suitable finite-dimensional space and u_{0h} is an approximation of u_0 in V_h. Such problem is called semi-discretization of (2.1), in space, as presented in [1].

To provide an algebraic interpretation of (5.1), we introduce a basis $\{\phi_j\}$ for V_h (as (3.2)), and observe that it suffices for the basis functions in order to be satisfied by all functions in the subspace. Moreover, since for each $t > 0$, the solution to the Galerkin problem also belongs to the subspace, we have

$$u_h(x,t) = \sum_{j=1}^{N_h} u_j(t)\phi_j(x),$$

where the coefficients $\{u_j(t)\}$ represent the unknowns of problem (5.1).

5.2. Time Discretization. To discretize the time evolution, we employ the Crank-Nicolson method, which approximates the temporal derivative by averaging over the time interval:

$$\frac{\partial u_h}{\partial t} \approx \frac{u_h^{n+1} - u_h^n}{\Delta t},$$

where u_h^n and u_h^{n+1} are numerical approximations of $u_h(t)$ at times t_n and t_{n+1}, respectively, and Δt is the time step. The time intervals are defined as $t_n = n\Delta t$ for $n = 0, 1, \ldots, N$, spanning the interval $[0, T]$. Substituting the time discretization into the original differential equation yields:

$$\frac{u_h^{n+1} - u_h^n}{\Delta t} - a\left(\int_0^1 \left(\frac{\partial u_h^{n+1}}{\partial x}\right)^2\right)\frac{\partial^2 u_h^{n+1}}{\partial x^2} + f(u_h^{n+1}) = g(x, t_{n+1}),$$

where u_h^{n+1} represents the numerical solution at the next time step t_{n+1}.

5.2.1. Crank-Nicolson Discretization. We utilize the Crank-Nicholson method, which is a weighted average of the approximations at time steps: n and $n+1$:

$$\frac{u_h^{n+1} - u_h^n}{\Delta t} = \frac{1}{2}\left(\mathcal{L}u_h^{n+1} + \mathcal{L}u_h^n\right) + \frac{1}{2}\left(f(u_h^{n+1}) + f(u_h^n)\right) + \frac{1}{2}\left(g(x,t^{n+1}) + g(x,t^n)\right), \tag{5.2}$$

rearranging, we obtain:

$$u_h^{n+1} - \frac{\Delta t}{2}\mathcal{L}u_h^{n+1} = u_h^n + \frac{\Delta t}{2}\mathcal{L}u_h^n + \frac{\Delta t}{2}\left(f(u_h^{n+1}) + f(u_h^n)\right) + \frac{\Delta t}{2}\left(g(x,t^{n+1}) + g(x,t^n)\right), \tag{5.3}$$

where

- u_h^n and u_h^{n+1} represent the solution at spatial point x_i and times t_n and t_{n+1}, respectively ;
- $\mathcal{L}u_h^n = -a\left(\int_0^1 \left(\frac{\partial u_h^n}{\partial x}\right)^2 dx\right)\frac{\partial^2 u_h^n}{\partial x^2}$;
- $\mathcal{L}(u_h^n)$ and $\mathcal{L}(u_h^{n+1})$ evaluate $\mathcal{L}(u_h)$ at u_h^n and u_h^{n+1}, respectively ;
- $f(u_h^n)$ and $f(u_h^{n+1})$ evaluate f at u_h^n and u_h^{n+1}, respectively ;
- $g(x_i, t_n)$ and $g(x_i, t_{n+1})$ are the source terms in (x_i, t_n) and (x_i, t_{n+1}), respectively ;
- Δt and Δx are the time and spatial step sizes, respectively.

In the context of the discretized problem described in (5.3), the presence of the non-local term $\mathcal{L}(u)$ precludes the use of a direct method. Therefore, we will opt to use the fixed-point method, which allows us to linearize the problem, enabling the iterative resolution of a linear problem at each step of the algorithm.

5.2.2. Fixed-Point Method. To solve this nonlinear eproblem using the fixed-point method, it is necessary to follow a structured process. The equation is transformed into an iterative form, allowing the computation of successive approximations until achieving the desired convergence. Each iteration is designed to progressively approach the approximate solution of the problem, outlined below:

1. Consider fixed the initial data u_h^n.
2. For each iteration $k \in \{0,1,2,\dots,M\}$, compute the time iterates in $n \in \{0,1,2,\dots N\}$:

$$u_h^{n+1} = u_h^n + \frac{\Delta t}{2}\mathcal{L}(u_h^n) + \frac{\Delta t}{2}\left(f(u_h^{n+1}) + f(u_h^n)\right) + \frac{\Delta t}{2}\left(g(x,t^{n+1}) + g(x,t^n)\right).$$

3. Update u_h^{n+1} using:

$$u_h^{n+1} = u_h^n - \frac{\Delta t}{2}\mathcal{L}(u_h^n),$$

   and return $u_h^n \equiv u_h^{n+1}$.
4. Repeat the steps until convergence criterion is met.

5.3. Iterative Algorithm. Now, to connect the theoretical data with the discretization process performed, we will present the iterative algorithm that was used for data computation. For this, we use the fixed-point algorithm:

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ALGORITHM (Fixed-Point Iteration for the Nonlinearity)
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`Step 1.` Fix $\varepsilon > 0$, choose $u_h^0 \in V_h$.

`Step 2.` For each iteration $k = 0,1,2,\dots,M-1$:

`Step 2.2` Compute the state, using

$$u_h^{n+1,k+1} = \frac{u_h^n + \frac{\Delta t}{2}\mathcal{L}u_h^n + \frac{\Delta t}{2}\left(f(u_h^{n+1,k}) + f(u_h^n)\right) + \frac{\Delta t}{2}\left(g(x,t^{n+1}) + g(x,t^n)\right)}{1 - \frac{\Delta t}{2}\mathcal{L}u_h^n}. \tag{5.4}$$

`Step 2.3` Compute the error ε_h, using:

$$\|u_h^{n+1,k} - u_h^{n,k}\| = \varepsilon_h\,. \tag{5.5}$$

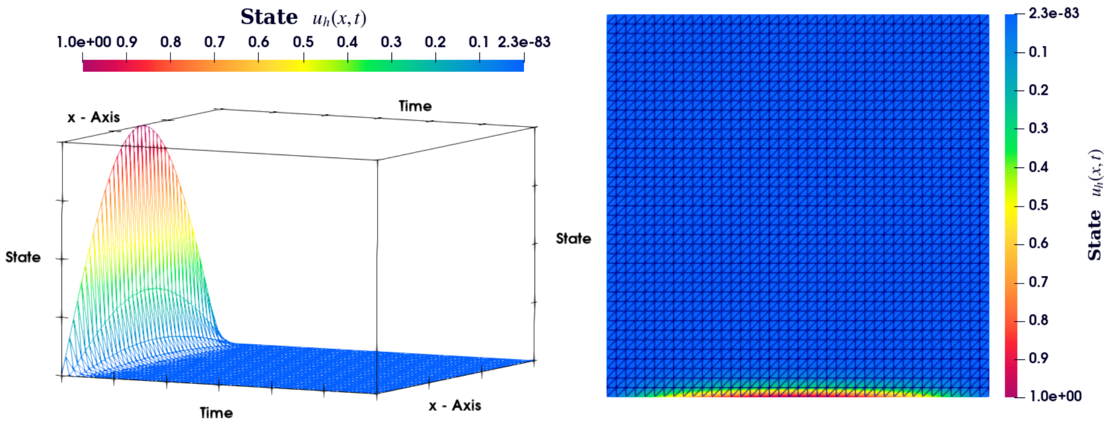
If $\varepsilon_h < \varepsilon$, set $u = u_h^{n+1,k}$ and stop. Otherwise, repeat Step 2 with $k = k+1$.

`Step 3:` After the final time step N, the solution u_h^{n+1} represents the approximate solution at $t = T$.

The combination of the Crank-Nicholson method for time discretization and the finite element method for spatial discretization, along with a fixed point algorithm, provides a robust and accurate approach for solving nonlinear parabolic partial differential equations. This method not only ensures stability and accuracy but also is efficient for solving complex problems with coupled nonlinearities (see [13]).

5.4. Simulations. For the numerical simulations of the linearized problem, we employed the capabilities of the open-source software FreeFem++ (version 4.15) for the implementation, post-processing, and execution of the computational experiments. The space–time discretization of the problem was programmed by adapting several iterative procedures, guided by the approximation and full-system discretization methodologies presented in [22], as well as by mesh-adaptation techniques applied during the iterative process, as described in [27]. These capabilities were complemented with post-processing tools from the open-source software Paraview, enhancing the visual presentation of simulation results.

In the experimental data, we consider $f(s) = 0$, $g(x,t) = 0$, and $a(s) = 1 + s$. The initial data was given by $u_0 \equiv \sin(\pi x)$. The evolution process of the approximate problem is presented in Figure 5.1 below.



a) State evolution in 3D view

b) State evolution in 2D view

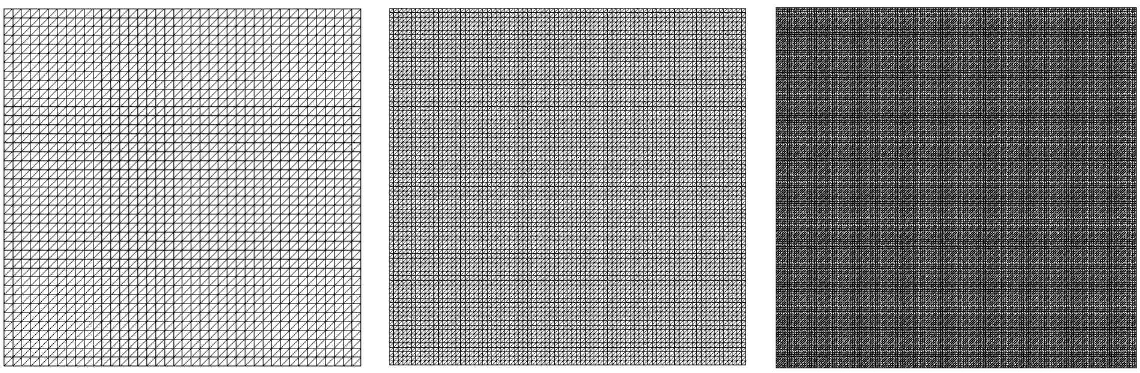
Figure 5.1: To avoid graphical redundancy, Figures a) and b) above display the evolution of the state $u_h(x,t)$ over the interval $[0,1]$, using $\Delta x = \Delta t \equiv 1/80$.

We will include a complementary convergence analysis to indicate the efficiency of the method and the implemented algorithm. For this purpose, we present results regarding the convergence order and the convergence criterion error of the proposed algorithm. The assessment of the algorithm's convergence order involved implementing an isometric mesh refinement process in Figure 5.2, with results detailed in Table (5.1). Specifically, to evaluate the algorithm's convergence order, we computed:

$$p_i = \frac{\log\left(\frac{E(h_i)}{E(h_{i+1})}\right)}{\log\left(\frac{h_i}{h_{i+1}}\right)}, \tag{5.6}$$

with

$$\|E(h_i)\|_{(L^2 \ or \ H^3)} = \|u_{h_i}^{n+1} - u_{h_i}^{n}\|_{(L^2 \ or \ H^3)} \quad \text{and} \quad \|E(h_{i+1})\|_{(L^2 \ or \ H^3)} = \|u_{h_{i+1}}^{n+1} - u_{h_{i+1}}^{n}\|_{(L^2 \ or \ H^3)}. \tag{5.7}$$



a) $\Delta x = \Delta t \equiv 1/40$

b) $\Delta x = \Delta t \equiv 1/80$

c) $\Delta x = \Delta t \equiv 1/160$

Figure 5.2: The discretizations described above were carried out on the spatial domain $[0,1]$ and the temporal interval $[0,1]$, with $\Delta x = \Delta t$. These meshes were used to define the spatial and temporal discretizations via the finite element method. Numerical simulations and comparative analyses of the resulting data were subsequently performed within this framework.

Norms	**Mesh 1**	**Mesh 2**	**Mesh 3**	**Order of Convergence**
$\|E(h_i)\|_{L^2}$	8.66588e-18	2.02503e-18	5.0091e-19	$p_1 \approx 2.098$ $p_2 \approx 2.015$
$\|E(h_i)\|_{H^3}$	1.72512e-16	1.86031-16	2.26643e-16	$p_1 \approx 1.13021$ $p_2 \approx 1.05193$

Table 5.1: Here we are considering the evaluation of convergence results for different mesh refinements, where p_1 denotes the order between the initial discretization and the refined discretization, and p_2 represents the order evaluated between the first and second refinements.

Finally, some results obtained from comparative analyses of norms with mesh refinement are presented in Figure 5.3.

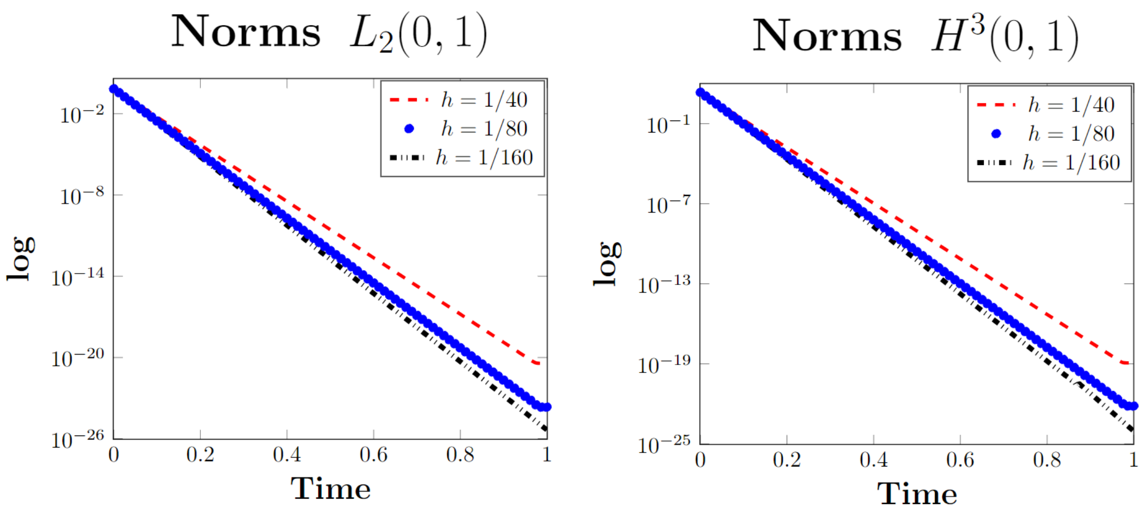


Figure 5.3: In the graphs above, the results of $L_2(0,T)$ and $H^3(0,T)$ norms, presented on a semi-logarithmic scale for clearer perception, were computed using **Mesh 1**, **Mesh 2**, and **Mesh 3**.

Conclusions. In Theorem 2.3, we proved the existence and uniqueness of weak solutions of equation (2.1) for given initial data $u_0 \in H_0^1(a,\, b)$, $g \in L^2((a,b) \times (0,T))$. Moreover, we also established the existence of more regular solutions under additional assumptions on the initial data, $u_0 \in H_0^1(a,\, b) \cap H^2(a,\, b)$, $g \in L^2(0,T,H_0^1(a,b))$. In Theorem 2.2, for small initial data $u_0 \in H_0^1(a,\, b) \cap H^2(a,\, b)$ and $g = 0$, we proved the exponential decay of the system's energy. In Theorem 2.3, for small initial data $u_0 \in H^3(a,\, b)$ and $g = 0$, we proved the exponential decay of solutions in the H^3-norm.

The compactness theorems of Aubin–Lions–Simon and the elliptic regularity theorem are the main analytical tools in the proofs of these three results. We also developed numerical simulations using the Crank–Nicholson scheme for the time discretization and finite element methods for the spatial discretization.

Interesting problems for further study include null controllability, approximate controllability, and trajectory control associated with equation (2.1), as well as the optimal control problem for (2.1) with a suitable cost functional. For these topics, Theorems 2.2 and 2.3 are particularly important in establishing control over large time horizons.

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