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Eigen-concepts in the multiplicative linear algebra context

Autoconceptos en el contexto del álgebra lineal multiplicativa

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Abstract

The concept of eigenvalues is associated with the linearity one, through the structure of vectorial space. The multiplicative linear algebra is a structure in which an expression such as x^3y^2 can be considered a linear combination of variables x and y. This article is reserved to show the corresponding analogues for an Eigenvalue Theory. We exemplify its applications by introducing a connection with the analysis of a nonlinear dynamical system in the standard sense, although a linear recurrence in the multiplicative one.
Keywords . Linearity concept, multiplicative linear algebra, multiplicative linear maps.

Resumen

El concepto de valores propios está asociado al de linealidad, a través de la estructura del espacio vectorial. El álgebra lineal multiplicativa es una estructura en la que una expresión como x^3y^2 puede ser considerada una combinación lineal de las variables x y y. Este artículo está destinado a mostrar los análogos correspondientes para una teoría de valores propios en este contexto. Se ejemplifican sus aplicaciones mediante la introducción de una conexión con el análisis de un sistema dinámico no lineal en el sentido estándar, aunque con una recurrencia lineal en el marco multiplicativo.
Palabras clave. Concepto de linealidad, álgebra lineal multiplicativa, aplicaciones lineales multiplicativas.

1. Introduction. The concept of linearity is not independent of the underlying arithmetic structure, where the expressions make sense. So, in the standard linearity algebraic term, $ax + by$ is a linear combination of the variables x and y with real parameters a and b. Here we observe the concurrence of the addition operation and the multiplication,*i.e.*, we have the action of the complete ordered field $(\mathbb{R}, +, \cdot)$. However, an expression such as $x^a y^b$ could also be considered linear in an appropriate arithmetic context. In fact, it is the case when we consider the complete ordered field $(\mathbb{R}_+, \cdot, *)$, where $\mathbb{R}_+$ is the set of positive real numbers, with the usual product as the first operation and a type of exponentiation $*$ as the second. Specifically, in [1, 2] this numeric field is presented

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in the framework of a non-Newtonian calculus construction, one of multiplicative type in the sense that the respective derivative of a product of functions is the product of the derivative of the factors [3]. This is an isomorphic calculus in relation to the standard one, but with some direct applications [4] that justified its development [5, 6].

In [7], the concepts of m-vectorial space are introduced, m-linear transformations, and the elements of a respective m-matrix theory as its representations, where $(\mathbb{R}_+, \cdot, *)$ is the base numerical field. The idea of a matrix raised to a matrix is another interesting construction, with several applications appearing to have many potentialities. Nevertheless, the basic conceptual elements to introduce a respective spectral theory, with its multiple applications, for example, in the qualitative analysis of dynamic systems, are a debt.

In order to further explore m-linear transformations and advance the m-matrix theory for a wide range of applications, we will introduce the concept of the pair of m-eigenvector and its corresponding m-eigenvalue, and demonstrate its main properties. Moreover, we present specific examples of its determinations and meanings. Although, behind our matrix multiplicative theory, there is the action of an isomorphism with the classic matrices, to install a general operative view or perspective of the linearity concept, we think it necessary to show explicitly the involved objects and their properties.

Intending to facilitate reading and understanding for the reader, it has generally been privileged to work, particularly in the exemplifications, in dimension two, as well as to avoid back-to-detail the most classic and well-known demonstrations of the standard linear algebra. In Section 1, we review one by one the mathematical preliminaries necessary to propose and develop an introduction to the theory of m-eigen-(vector, value). Section 2 is reserved to present the corresponding spectral concepts. Finally, in Section 3, we explore some possibilities for applications, in particular to the theory of dynamical systems. The article concludes with a discussion of this new perspective and its projections.

2. Preliminaries concepts.

2.1. The field of real positives. Just as multiplication shares a conceptual relationship with addition, by repetition (for example, adding three times two is multiplying three by two), we see that raising to a power is related to multiplication. However, in contrast to the product, working with exponentiation has the difficulty of not being a commutative operation. Despite that, the existing difference between 2^3 and 3^2 could be avoided (not without paying a cost) if we introduce some appropriate function, for example the natural logarithm function, so that $2^{\ln(3)} = 3^{\ln(2)}$. So, considering the domain of the natural logarithm function, the positive real numbers $\mathbb{R}_+$, we can consider the triadic $(\mathbb{R}_+, \cdot, *)$ where, to the multiplication operation, we add the commutative exponentiation $a * b = a^{\ln(b)}$. In this way, we obtain the positive real numbers as an isomorphic copy of the field of the real numbers $(\mathbb{R}, +, \cdot)$.

Regarding this field of positive numbers, it is important to remember some notations and properties derived from axiomatic:

- *Concerning the algebraic*: The neutral element for the operation $*$ is the Euler number, e. So, if $a \in \mathbb{R}_+$ and $a \neq 1$, the associated inverse element of a will be denoted $a^{\prec -1 \succ}$ which corresponds to $e^{1/\ln(a)}$. In fact, we have $a * a^{\prec -1 \succ} = a^{1/\ln(a)} = e$. In general, given $a \in \mathbb{R}_+$ and $a \neq 1$, we denote $a * a * \cdots * a$ (n times) and $a^{\prec -1 \succ} * a^{\prec -1 \succ} * \cdots * a^{\prec -1 \succ}$ by $a^{\prec n \succ}$ and $a^{\prec -n \succ}$ respectively. Moreover, $a^{\prec 0 \succ}$ will be another representation of e. Using this notation, we have
$$a^{\prec n \succ} * a^{\prec m \succ} = a^{\prec n+m \succ} \quad \text{and} \quad (a^{\prec n \succ})^{\prec m \succ} = a^{\prec nm \succ},$$
for all $n,\, m \in \mathbb{Z}$.

- *Concerning the order*: If we consider $\mathbb{R}_+ = \mathbb{P}_- \cup \{1\} \cup \mathbb{P}_+$, with $\mathbb{P}_- = (0, 1)$ and $\mathbb{P}_+ = (1, \infty)$, then for $a \in \mathbb{R}_+$ there are three mutually exclusive possibilities: $a \in \mathbb{P}_-$, $a = 1$ or $a \in \mathbb{P}_+$. Moreover, $a \in \mathbb{P}_+$ if only if $a^{-1} \in \mathbb{P}_-$. In the set $\mathbb{P}_+$, both multiplication and exponentiation are closed operations. With this, $\mathbb{P}_+$ is called the set of e-positives numbers. A key function is the relative value of a number $[a]$, which for $a \in \mathbb{R}_+$ is defined by a^{-1} if $a \in \mathbb{P}_-$ and a, if $a \in \mathbb{P}_+ \cup \{1\}$. Additionally, introducing the distance $[a/b]$ in $\mathbb{R}_+$, we have the positive real number set as a relative metric space.

- *Concerning completeness*: Given a bounded subset A of $\mathbb{R}_+$ and $\varepsilon > 1$, there exists $a_\ominus, a_\oplus \in A$, such that $a_\ominus < \varepsilon \cdot \inf(A)$ and $\sup(A)/\varepsilon < a_\oplus$. By A as a bounded set, we understand the existence of $p \in \mathbb{P}_+$ such that $1/p < a < p$ for all $a \in A$.

Other properties can be found in references [1, 2].

2.2. Multiplicative vector spaces. Given a not empty set W, we use the field $\mathbb{R}_+$ to call a pair $(W, (\mathbb{R}_+, \cdot, *))$ a m-vector space, if there are:

- A product in W, that to each pair $(u, v) \in W \times W$ associates some $u \cdot v \in W$, which implies that $(W, \cdot)$ forms an Abelian group. The null element will be denoted $\vec{1}$. Moreover, given $u \in W$, the vector $u^{-1} \in W$ is the associated inverse.

- A scalar product, that to each $(a, v) \in \mathbb{R}_+ \times W$ corresponds $a * v \in W$, such that we have:

  (a) Compatibility of scalar product $*$ with the commutative exponentiation in the field: For scalars $\alpha, \beta \in \mathbb{R}_+$ and a vector $v \in W$, it is satisfied
  $$(\alpha * \beta) * v = \alpha * (\beta * v).$$

  (b) Distributivity of scalar product $*$ regarding vector $\cdot$ product: For a scalar $\alpha \in \mathbb{R}_+$ and vectors v and w in W, we have
  $$\alpha * (v \cdot w) = (\alpha * v) \cdot (\alpha * w).$$

  (c) Distributivity of vector multiplication $\cdot$ regarding scalar product $*$: For scalars $\alpha, \beta \in \mathbb{R}_+$, and a vector $v \in W$, we have
  $$(\alpha \cdot \beta) * v = (\alpha * v) \cdot (\beta * v).$$

  (d) Regularity in the sense that $e * v = v$, for each $v \in W$.

2.3. Multiplicative Euclidean space. An example of multiplicative vector space, that has the obvious extension to higher dimensions, is the multiplicative Euclidean space $\mathbb{R}^2_+ := \mathbb{R}_+ \times \mathbb{R}_+$ defined by the operations:

$$\begin{aligned} (x_1, y_1) \cdot (x_2, y_2) &= (x_1 x_2, y_1 y_2), \quad \text{and} \\ a * (x_1, y_1) &= (a * x_1, a * y_1) = (x_1^{\ln(a)}, y_1^{\ln(a)}), \end{aligned}$$

for any (x_1, y_1), $(x_2, y_2) \in \mathbb{R}^2_+$ and $a \in \mathbb{R}_+$.

Regarding the idea of what we are going to understand by a m-linear combination, let us note that

$$(x, y) = (x * \hat{e}_1) \cdot (y * \hat{e}_2),$$

where $\hat{e}_1 = (e, 1)$ and $\hat{e}_2 = (1, e)$, which are denominated the m-canonical vectors, that form the $\mathcal{B} = \{\hat{e}_1, \hat{e}_2\}$ an m-base, the canonical one. In Figure 2.1, the canonical vectors $\hat{e}_i$, $i \in \{1, 2\}$, defining the coordinate axes, are represented, which intersect in the neutral element $\vec{1} := (1, 1)$ of the product.

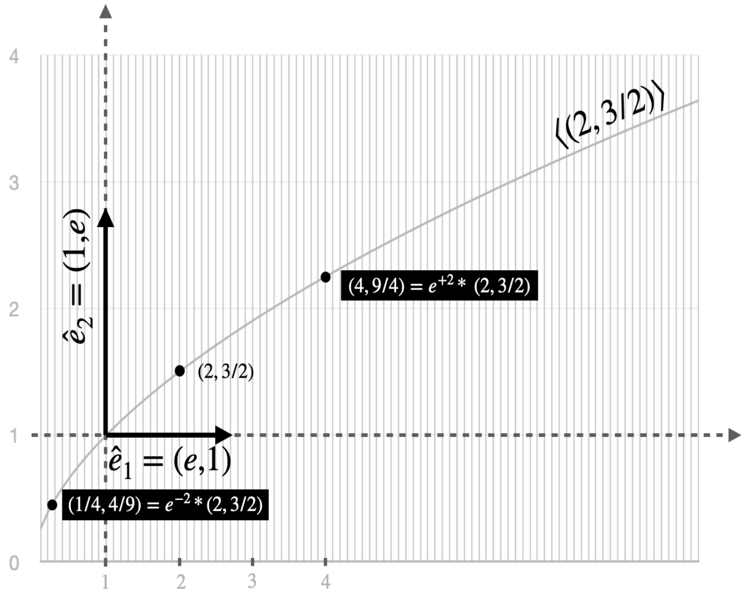

Figure 2.1: Representation of the canonical m-base, $\mathcal{B} = \{\hat{e}_1, \hat{e}_2\}$. In addition, we highlight three elements of the vector space $\langle(2,\ 3/2)\rangle$. Note the convenience of considering the coordinate axes, which are $\langle\hat{e}_1\rangle$ and $\langle\hat{e}_2\rangle$, intersecting at the proper origin $(1,1)$.

It is well known that in the standard Euclidean space $\mathbb{R}^2 := \mathbb{R}\times\mathbb{R}$ the addition of vectors has a geometric interpretation through the parallelogram rule, that is, if two vectors u and v are located having the same initial point at the center of coordinates $(0,0)$, and by parallel translation of them a parallelogram is completed, then $u+v$ can be represented by the diagonal arrow that begins at the point of origin $(0,0)$ and ends at its opposite vertex. Parallelism has to do with the notion of parallel straight lines. In this sense, first, let us agree that a m-straight line in $\mathbb{R}^2_+$ through the origin $(1,1)$ is given by a vector $\omega = (a,b)$, $a,b>0$, and all points $(x,y)\in\mathbb{R}\times\mathbb{R}$ such that $x*\omega$, with $x\in\mathbb{R}_+$. The equation for this m-straight-line is

$$L:\ y = \{b * a^{\prec -1\succ}\} * x = x^{\ln(b)/\ln(a)}, \quad x\in\mathbb{R}_+.$$

The term $b * a^{\prec -1\succ}$ corresponds to the m-slope. Associated with L we have the straight line

$$L':\ y/y_0 = \{b * a^{\prec -1\succ}\} * (x/x_0) = (x/x_0)^{\ln(b)/\ln(a)}, \quad x\in\mathbb{R}_+,$$

which is the m-affine-straight-line that passes through the point (x_0,y_0). Notice that if $(x_0,y_0)\notin L$, then $L\cap L' = \emptyset$.

Consequently, if $u = (5/2,2)$ and $v = (2,5/2)$, we can associate the m-lines

$$L_1:\ y = \{2*(5/2)^{-1}\}*x \quad L_1':\ y/2 = \{2*(5/2)^{-1}\}*(x/(5/2))$$

and

$$L_2:\ y = \{(5/2)*2^{-1}\}*x \quad L_2':\ y/(5/2) = \{(5/2)*2^{-1}\}*(x/2).$$

We affirm that: $L_i \parallel L_i'$, $i\in\{1,2\}$, in the sense that $L_i\cap L_i' = \emptyset$. Then, we observe $L_1'\cap L_2' = \{(5,5)\}$, and $(5,5) = u\cdot v = (5/2,2)\cdot(2,5/2)$, which gives us the geometric interpretation of the product in $\mathbb{R}^2_+$ as the diagonal of a multiplicative parallelogram, as illustrated by Figure 2.2.

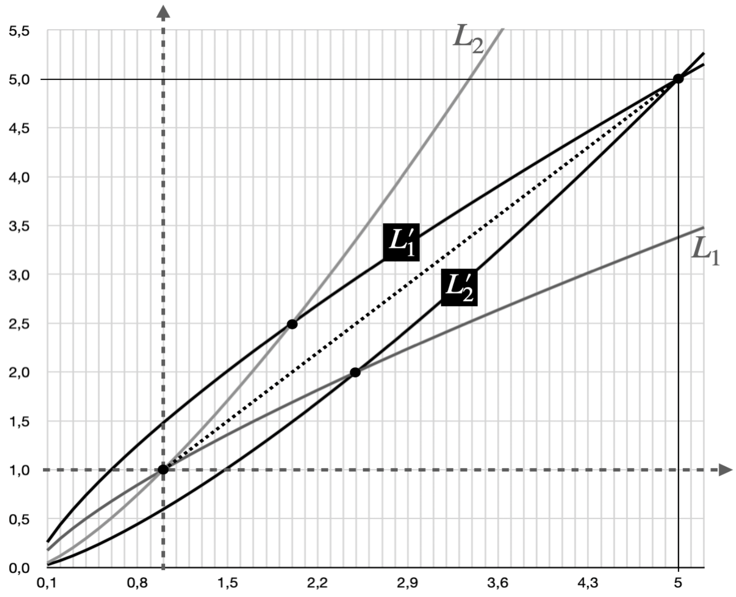

Figure 2.2: Representation of vectors $u = (5/2, 2)$ and $v = (2, 5/2)$ defining m-straight-lines L_1 and L_2 respectively. We have the m-affine-straight-lines L_1' and L_2' as its respective parallels. They form a parallelogram whose diagonal in a dotted line represents the product $u \cdot v$.

2.4. Matrices as a multiplicative algebra. Another example of m-vector space is the set of matrices $M_{n\times m}(\mathbb{R}_+)$, i.e., with positive inputs and the following products operations:

- Given $A, B \in M_{n\times m}(\mathbb{R}_+)$, with $A = (a_{ij})$ y $B = (b_{ij})$, we have $C = A \odot B = (c_{ij})$, defined by $c_{ij} = a_{ij}b_{ij}$. For its inverse operation, we denote $D = A \oplus B = (d_{ij})$, where $d_{ij} = a_{ij}/b_{ij}$.

- Given $A \in M_{n\times m}(\mathbb{R}_+)$, with $A = (a_{ij})$ and $a \in \mathbb{R}_+$, we have $C = a * A = (c_{ij})$, defined by $c_{ij} = a * a_{ij}$.

We note that matrices also form an algebra considering the following operation:

Definition 2.1 ($*$ operation). *Given $A \in M_{n\times p}(\mathbb{R}_+)$ and $B \in M_{p\times m}(\mathbb{R}_+)$, $A = (a_{ij})$ y $B = (b_{ij})$, let us define*

$$A * B = (c_{ij}) \in M_{n\times m}(\mathbb{R}_+) \quad \textit{where} \quad c_{ij} = \Pi_{l=1}^{p}(a_{il} * b_{lj}).$$

This is called the $*$-product and follows the same idea as the usual product, where a row of the first matrix is operated term by term with a column of the second matrix. It is associative, distributive regarding the standard product of the matrices, and has neutral elements given by any square matrix $E = e_I$, where I is the usual identity matrix. This is, if $E = (e_{ij})$, then $e_{ij} = 1$ if $i \neq j$ and $e_{ij} = e$ if $i = j$. Notice that for a square matrix $A \in M_{n\times n}(\mathbb{R}_+)$, we will denote $A^{\prec k \succ}$ the $*$-product $A * A * \cdots * A$ with k factors.

This product of matrices is associative, that is, for any trio of matrices A, B, and C, with compatible dimensions for the $*$ products, satisfy $(A * B) * C = A * (B * C)$. Moreover, we have the following distributive property

$$A * (B \odot C) = (A * B) \odot (A * C) \quad \text{and} \quad (A \odot B) * C = (A * B) \odot (B * C).$$

Moreover, the new product satisfies a bilinearity property or m-bilinearity, in the sense that:

$$\{(\alpha * A) \odot (\beta * B)\} * C = \{\alpha * (A * C)\} \odot \{\beta * (B * C)\}$$

and

$$A * \{(\alpha * B) \odot (\beta * C)\} = \{\alpha * (A * B)\} \odot \{\beta * (A * C)\}.$$

2.5. Multiplicative linear transformations. Let be two m-vector spaces W_1 and W_2, we say that a function $T : W_1 \to W_2$ is a m-linear transformation (or m-linear map) if and only if

$$T((\alpha * u) \cdot (\beta * v)) = (\alpha * T(u)) \cdot (\beta * T(v)),$$

for any $u, v \in W_1$ and $\alpha, \beta \in \mathbb{R}_+$.

Example 2.1. *Consider $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(x) = \sqrt{x}$. It is a m-linear transformation from the m-space $W_1 = \mathbb{R}_+$ to itself. In fact,*

$$\begin{aligned} f((\alpha * u) \cdot (\beta * v)) &= f(u^{\ln(\alpha)} v^{\ln(\beta)}) \\ &= \sqrt{u^{\ln(\alpha)} v^{\ln(\beta)}} \\ &= \sqrt{u^{\ln(\alpha)}} \cdot \sqrt{v^{\ln(\beta)}} \\ &= (\alpha * f(u)) \cdot (\beta * f(v)). \end{aligned}$$

Example 2.2. *Note that $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with $g(x, y) = x^3 y^2$ is a linear transformation between the m-space $W_1 = \mathbb{R}_+ \times \mathbb{R}_+$ and $W_2 = \mathbb{R}_+$ as m-Euclidean space of two and one dimension, respectively. Indeed,*

$$\begin{aligned} g((\alpha * (x_1, y_1)) \cdot (\beta * (x_2, y_2))) &= g(x_1^{\ln(\alpha)} x_2^{\ln(\beta)}, y_1^{\ln(\alpha)} y_2^{\ln(\beta)}) \\ &= x_1^{3\ln(\alpha)} x_2^{3\ln(\beta)} y_1^{2\ln(\alpha)} y_2^{2\ln(\beta)} \\ &= (x_1^3 y_1^2)^{\ln(\alpha)} \cdot (x_2^3 y_2^2)^{\ln(\beta)} \\ &= (\alpha * (x_1^3 y_1^2),\ \beta * (x_2^3 y_2^2)) \\ &= (\alpha * g(x_1, y_1)) \cdot (\beta * g(x_2, y_2)). \end{aligned}$$

3. The eigen-concepts. In this section, we present the definitions, results, and corresponding proofs related to the extension of eigen-concepts.

Definition 3.1 **(m-Eigenvector).** *Given $T : W \to W$ a m-linear transformation. We say that $w \in W$, $w \neq \vec{1}$, is an m-eigenvector if $T(w) = a * w$, for some $a \in \mathbb{R}_+$.*

Definition 3.2 **(m-Eigenvalue).** *In Definition 3.1 the parameter a such that $T(w) = a * w$ is called an m-eigenvalue.*

Definition 3.3 **(m-Eigenspace).** *Given an m-eigenvalue of a m-linear transformation $T : W \to W$, the set $W_a = \{v \in W : T(v) = a * v\}$, is called the m-eigenspace.*

Importantly, W_a is a vector subspace of W since given $u, v \in W_a$ and $\alpha, \beta \in \mathbb{R}_+$, then

$$\begin{aligned} T((\alpha * u) \cdot (\beta * v)) &= (\alpha * T(u)) \cdot (\beta * T(v)) = (\alpha * (a * u)) \cdot (\beta * (a * v)) \\ &= ((\alpha * a) * u) \cdot ((\beta * a) * v) = a * [(\alpha * u) \cdot (\beta * v)], \end{aligned}$$

so that, we have $(\alpha * u) \cdot (\beta * v) \in W$. Therefore, W_a is a subspace of W.

Definition 3.4 **(m-Exponencial function).** *For a matrix $A = (a_{ij}) \in M_{n\times m}(\mathbb{R})$, we define*

$$\exp_m(A) := (e^{a_{ij}}),$$

it is called the m-exponential of A.

Definition 3.5 **(m-Logarithm function).** *For a matrix $A = (a_{ij}) \in M_{n\times m}(\mathbb{R}_+)$, we define*

$$\ln_m(B) := (\ln(b_{ij})),$$

it is called the m-logarithm of A. From this definition, it follows that $\exp_m(I) = E$, $\ln_m(E) = I$,

$\exp_m(A + B) = \exp_m(A) \odot \exp_m(B)$, $\exp_m(A - B) = \exp_m(A) \ominus \exp_m(B)$, $\ln_m(A \odot B) = \ln_m(A) + \ln_m(B)$ and $\ln_m(A \ominus B) = \ln_m(A) - \ln_m(B)$, for A and B matrices in the domain and with sizes where operations and functions are defined.

Theorem 3.1. *Given matrices $A = (a_{ij}) \in M_{n\times p}(\mathbb{K})$ and $B = (b_{ij}) \in M_{p\times m}(\mathbb{K})$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{R}_+$, according to the necessary domain, we have:*

(a) $\exp_m(A \cdot B) = \exp_m(A) * \exp_m(B)$ *and* $\exp_m(A^{-1}) = \exp_m^{\prec -1 \succ}(A)$ *if the inverse* A^{-1} *exists.*

(b) $\ln_m(A * B) = \ln_m(A) \cdot \ln_m(B)$ *and* $\ln_m(A^{\prec -1 \succ}) = \ln_m^{-1}(A)$ *if the inverse* $A^{\prec -1 \succ}$ *exists.*

Proof:

(a) Notice that the isomorphism $\exp : (\mathbb{R}, +, \cdot) \to (\mathbb{R}_+, \cdot, *)$, satisfies

$$\exp(a \cdot b) = \{\exp(a)\}^b = (e^a)^{\ln(e^b)} = e^a * e^b.$$

So, with which, if $C = (c_{ij}) = A \cdot B$, then:

$$\exp(c_{ij}) = \exp\left(\sum_{k=1}^{p} a_{ik} \cdot b_{kj}\right) = \prod_{k=1}^{p} \exp(a_{ik} \cdot b_{kj}) = \prod_{k=1}^{p} \{\exp(a_{ik}) * \exp(b_{kj})\}.$$

In this way, we get

$$\exp_m(C) = (\exp(c_{ij})) = (\exp(a_{ij})) * (\exp(b_{ij})) = \exp_m(A) * \exp_m(B).$$

(b) Let us remember that isomorphism $\ln : (\mathbb{R}_+, \cdot, *) \to (\mathbb{R}, +, \cdot)$, satisfies

$$\ln(a * b) = \ln(a^{\ln(b)}) = \ln(a) \cdot \ln(b),$$

with which, if $C = (c_{ij}) = A * B$, then:

$$\ln(c_{ij}) = \ln\left(\prod_{k=1}^{p} a_{ik} * b_{kj}\right) = \sum_{k=1}^{p} \ln(a_{ik} * b_{kj}) = \sum_{k=1}^{p} \{\ln(a_{ik}) \cdot \ln(b_{kj})\}.$$

Thus, we have $\ln_m(C) = (\ln(c_{ij})) = (\ln(a_{ij})) \cdot (\ln(b_{ij})) = \ln_m(A) \cdot \ln_m(B)$.

$\square$

Its properties are natural extensions of the properties of exponential and logarithmic functions, $\exp : \mathbb{R} \to \mathbb{R}_+$ and $\ln : \mathbb{R}_+ \to \mathbb{R}$, respectively.

Definition 3.6 (m-Determinant). *Given* $A \in M_{n\times n}(\mathbb{R}_+)$ *and considering the standard invariant* **det** $: M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ *determinant, we define the* m*-determinant of a positive matrix* A*, by:*

$$\mathbf{det}_m(A) := \exp(\mathbf{det}(\ln_m(A))).$$

Let us consider the following function $M_{n\times n}(\mathbb{R}_+)$ itself related to elementary operations. We denote by $E_{ij}(\cdot)$ the operation that exchanges the row i with row j. The function that $*$-multiply the i-th matrix row by a positive number α, will be denoted F_i^α. We will denote by R_{ji}^α the operation that multiply j $*$-multiplied by α to row i.

Theorem 3.2. *Let us consider matrices* A, $B \in M_{n\times n}(\mathbb{R}_+)$*, then function* $\mathbf{det}_m(\cdot)$ *has the following properties:*

(a) Rows/columns m*-linearity: If* $C = (c_{ij})$ *is such that* $c_{ij} = a_{ij} = b_{ij}$ *for all* j *and* $i \neq k$*, some* $k \in \{1, \ldots, n\}$*, then*

$$\mathbf{det}_m(C) = \{\alpha * \mathbf{det}_m(A)\} \cdot \{\beta * \mathbf{det}_m(B)\},$$

if $c_{kj} = (\alpha * a_{kj}) \cdot (\beta * b_{kj})$*, for all* $j \in \{1, \ldots, n\}$*.*

(b) Permutation of rows / columns: If $B = F_{ij}(A)$*, some par* $i, j \in \{1, \ldots, n\}$*, then* $\mathbf{det}_m(B) = 1/\mathbf{det}_m(A)$*.*

(c) Row/column $*$*-amplified by a scalar: If* $B = F_i^\alpha(A)$*, some* $i \in \{1, \ldots, n\}$*, then* $\mathbf{det}_m(B) = \alpha * \mathbf{det}_m(A)$*.*

(d) m*-triangular-matrix: If* $A = (a_{ij})$ *such that* $a_{ij} = 1$*, for any pair* (i, j) *with* $j > i$ *(i.e., a* m*-triangular matrix), then* $\mathbf{det}_m(A) = a_{11} * \cdots * a_{nn}$*. In particular,* $\mathbf{det}_m(E) = e$*.*

(e) The $*$*-product of matrices:* $\mathbf{det}_m(A * B) = \mathbf{det}_m(A) * \mathbf{det}_m(B)$*. In particular, since* $A * A^{\prec -1 \succ} = E$*, we have* $\mathbf{det}_m(A^{\prec -1 \succ}) = (\mathbf{det}_m(A))^{\prec -1 \succ}$*.*

(f) *Transposed matrix:* $\mathbf{det}_m(A^\top) = \mathbf{det}_m(A)$.

(g) *Matrix with identical rows/columns: If $A = (a_{ij})$ is such that $a_{ij} = a_{kj}$, for all j and some pair $i,\ k$, then* $\mathbf{det}_m(A) = 1$.

(h) *Matrix with unitary row/column: If $A = (a_{ij})$ is such that $a_{kj} = 1$, for some $k \in \{1, \ldots, n\}$ and all j, then* $\mathbf{det}_m(A) = 1$.

Proof: As an illustration, we will only demonstrate some items, since the rest is very similar to the standard case.

(a) Notice that k-th row of $\ln_m(C)$ is given by elements $\ln(c_{kj}) = \ln(\alpha) \cdot \ln(a_{kj}) + \ln(\beta) \cdot \ln(b_{kj})$. Considering all other inputs of $\ln_m(A)$ and $\ln_m(B)$ are equal, we have $\mathbf{det}(\ln_m(C)) = \ln(\alpha) \cdot \mathbf{det}(\ln_m(A)) + \ln(\beta) \cdot \mathbf{det}(\ln_m(B))$. Thus, the proof is achieved by applying the exponential function.

(b) Since the row i and j of $\ln_m(F_{ij}(A))$ are the permutations of $\ln_m(A)$, we have that $\mathbf{det}(\ln_m(F_{ij}(A))) = -\mathbf{det}(\ln_m(A))$ and when applying the exponential function, the required equality immediately follows.

(e) Notice that $\mathbf{det}_m(A * B) = \exp(\mathbf{det}(\ln_m(A * B))) = \exp(\mathbf{det}(\ln_m(A) \cdot \ln_m(B))) = \exp(\mathbf{det}(\ln_m(A)) \cdot \mathbf{det}(\ln_m(B)))$. Since $\exp(a \cdot b) = \exp(a) * \exp(b)$, for all $a,\ b \in \mathbb{R}$, the proof ends.

$\square$

Definition 3.7 (m-Trace). *Given $A \in M_{n\times n}(\mathbb{R}_+)$ and considering the standard invariant*

$$\mathbf{det},\ \mathbf{Tr} : M_{n\times n}(\mathbb{R}) \to \mathbb{R},$$

the standard trace, we define the m-trace of a positive matrix A, by:

$$\mathbf{Tr}_m(A) := \exp(\mathbf{Tr}(\ln_m(A))).$$

Notice that, if $A = (a_{ij})$ then $\mathbf{Tr}_m(A) := \exp(\mathbf{Tr}(\ln_m((a_{ij}))))$, which is $\exp(\mathbf{Tr}(\ln(a_{ij})))$. So,

$$\mathbf{Tr}_m(A) = \exp(\sum_{i=1}^{n} \ln(a_{ii})) = \prod_{i=1}^{n} a_{ii}$$

is obtained.

Theorem 3.3. *Let us consider matrices $A,\ B \in M_{n\times n}(\mathbb{R}_+)$, then function $\mathbf{Tr}_m(\cdot)$ has the following properties:*

(a) *The m-linearity: For any pair α and β in $\mathbb{R}_+$, it is satisfy*

$$\mathbf{Tr}_m(\{\alpha * A\} \odot \{\beta * B\}) = \{\alpha * \mathbf{Tr}_m(A)\} \cdot \{\beta * \mathbf{Tr}_m(B)\}.$$

(b) *Invariance under transposition:* $\mathbf{Tr}_m(A^\top) = \mathbf{Tr}_m(A)$.

(c) *Invariance under change of basis: If $B = P^{\prec -1 \succ} * A * P$, then* $\mathbf{Tr}_m(B) = \mathbf{Tr}_m(A)$.

(d) *The $*$-product of matrices:* $\mathbf{Tr}_m(A * B) = \mathbf{Tr}_m(B * A)$.

(e) *The $\odot$-product of matrices:* $\mathbf{Tr}_m(A \odot B) = \mathbf{Tr}_m(A) \cdot \mathbf{Tr}_m(B)$.

Proof: As an illustration, we will only demonstrate some items, since the rest is very similar to the standard case.

(a) If $A = (a_{ij})$ and $B = (b_{ij})$, then $C = (\alpha * A) \odot (\beta * B) = ((\alpha * a_{ij}) \cdot (\beta * b_{ij}))$. Thus, $\ln_m(C) = (\ln(\alpha * a_{ij}) + \ln(\beta * b_{ij}))$ and

$$\begin{aligned} \mathbf{Tr}(\ln_m(C)) &= \sum_{k=1}^{n} \{\ln(\alpha * a_{ii}) + \ln(\beta * b_{ii})\}, \\ \ln(\alpha) \sum_{k=1}^{n} \ln(a_{ii}) + \ln(\beta) \sum_{k=1}^{n} \ln(b_{ii}) &= \ln(\alpha)\mathbf{Tr}(\ln_m(A)) + \ln(\beta)\mathbf{Tr}(\ln_m(B)). \end{aligned}$$

Therefore,

$$\exp(\mathbf{Tr}(\ln_m(C))) = \exp(\ln(\alpha)\mathbf{Tr}(\ln_m(A))) \cdot \exp(\ln(\beta)\mathbf{Tr}(\ln_m(B))).$$

Since $\exp(\ln(a) \cdot \ln(b)) = a * b$, finally

$$\mathbf{Tr}_m(C) = \{\alpha * \mathbf{Tr}_m(A)) \cdot \{\beta * \mathbf{Tr}_m(B)\}.$$

(d) We need to prove that

$$\mathbf{Tr}(\ln_m((\prod_{l=1}^{n} a_{il} * b_{lj}))) = \mathbf{Tr}(\ln_m((\prod_{l=1}^{n} b_{il} * a_{lj}))).$$

This is,

$$\mathbf{Tr}((\sum_{l=1}^{n} \ln\{a_{il} * b_{lj}\})) = \mathbf{Tr}((\sum_{l=1}^{n} \ln\{b_{il} * a_{lj}\})).$$

Then

$$\sum_{i=1}^{n}\sum_{l=1}^{n} \ln\{a_{il} * b_{li}\} = \sum_{i=1}^{n}\sum_{l=1}^{n} \ln\{a_{il}\} \cdot \ln\{b_{li}\}$$

has to be equal to

$$\sum_{i=1}^{n}\sum_{l=1}^{n} \ln\{b_{il} * a_{li}\} = \sum_{i=1}^{n}\sum_{l=1}^{n} \ln\{b_{il}\} \cdot \ln\{a_{li}\},$$

which is clear.

(e) Notice that $\mathbf{Tr}(\ln_m(A \odot B)) = \mathbf{Tr}((\ln(a_{ij}b_{ij}))) = \mathbf{Tr}((\ln(a_{ij})) + (\ln(b_{ij})))$. It is equal to

$$\mathbf{Tr}((\ln(a_{ij}))) + \mathbf{Tr}((\ln(b_{ij}))) = \mathbf{Tr}(\ln_m(A)) + \mathbf{Tr}(\ln_m(B)).$$

Thus, all that remains is to apply the exponential function.

□

Definition 3.8 (m-Characteristic polynomial). *Given $A \in M_{n\times n}(\mathbb{R}_+)$ the characteristic polynomial $p_A(\lambda)$ of A is defined by*

$$p_A(\lambda) := \mathbf{det}_m(A \ominus (\lambda * E)).$$

Since $\ln_m(A \ominus (\lambda * E)) = \ln_m(A) - \ln_m(\lambda * E) = \ln_m(A) - \ln(\lambda)\cdot I$, we have $\mathbf{det}(\ln_m(A/(\lambda * E))) = \mathbf{det}(\ln_m(A) - \ln(\lambda)\cdot I)$. Then, it is equal to $\mathbf{q}_{_B}(\ln(\lambda))$, where $\mathbf{q}_{_B}(x)$ is the characteristic polynomial of the matrix $B := \ln_m(A)$. Finally,

$$p(\lambda) = \exp(\mathbf{q}_{_{\ln(A)}}(\ln(\lambda))).$$

Definition 3.9. *Two matrices A and B in $M_{n\times n}(\mathbb{R}_+)$ are m-similar if there exists a matrix m-invertible P such that $B = P^{\prec -1 \succ} * A * P$.*

This similarity relation implies that A and B represent the same m-linear transformation, but on a basis different. Two m-similar matrices have the same eigenvalues. In fact, if a is an eigenvalue of A, then there exists w such that $A * w = a * w$. Therefore, $(P * B * P^{\prec -1 \succ}) * w = a * w$, and $B * (P^{\prec -1 \succ} * w) = a * (P^{\prec -1 \succ} * w)$ is obtained. So, a is an eigenvalue of B. Similarly, an eigenvalue of B is the eigenvalue of A.

Moreover, $\mathbf{det}_m(P^{\prec -1 \succ} * A * P) = \mathbf{det}_m(P^{\prec -1 \succ}) * \mathbf{det}_m(A) * \mathbf{det}_m(P)$. Since, $\mathbf{det}_m(P^{\prec -1 \succ}) = (\mathbf{det}_m(P))^{\prec -1 \succ}$, we have $\mathbf{det}_m(B) = \mathbf{det}_m(A)$. In addition, by (c) of Theorem 3, we have $\mathbf{Tr}_m(B) = \mathbf{Tr}_m(A)$.

4. The 2×2 matrices. If $A = (a_{ij}) \in M_{2\times 2}(\mathbb{R}_+)$, then we have the m-linear transformation $T_A : \mathbb{R}_+^2 \to \mathbb{R}_+^2$, given by

$$T_A(y, y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (a_{11} * x)\cdot(a_{12} * y) \\ (a_{21} * x)\cdot(a_{22} * y) \end{pmatrix} = \begin{pmatrix} x^{\ln(a_{11})}\, y^{\ln(a_{12})} \\ x^{\ln(a_{21})}\, y^{\ln(a_{22})} \end{pmatrix}.$$

Then, the m-linear map $T : \mathbb{R}_+^2 \to \mathbb{R}_+^2$, defined by the formula

$$T(x, y) = (x^2y^3, x^3y^2) = (x^{\ln(e^2)}y^{\ln(e^3)}, x^{\ln(e^3)}y^{\ln(e^2)})$$

is a $T_{\exp_m(B)}$ m-linear transformation with

$$B = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

Finding an eigenvector $w = (x, y)$ of T_A associated with an eigenvalue of λ implies the relation $T_A(w) = \lambda * w$, this is,

$$\begin{pmatrix} x^{\ln(a_{11})}\, y^{\ln(a_{12})} \\ x^{\ln(a_{21})}\, y^{\ln(a_{22})} \end{pmatrix} = \lambda * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^{\ln(\lambda)} \\ y^{\ln(\lambda)} \end{pmatrix},$$

expression that can be rewritten as the following system

$$\begin{cases} x^{\ln(a_{11})-\ln(\lambda)} y^{\ln(a_{12})} & = & 1, \\ x^{\ln(a_{21})} y^{\ln(a_{22})-\ln(\lambda)} & = & 1. \end{cases}$$

When we apply the natural logarithm function on both sides of the equalities, assuming the change of variables $u = \ln(x)$ and $v = \ln(y)$, we get

$$\begin{cases} \ln(a_{11}/\lambda)\, u + \ln(a_{12}) v & = & 0 \\ \ln(a_{21}) u + \ln(a_{22}/\lambda)\, v & = & 0, \end{cases}$$

a system that supports the matrix expression given by

$$\ln_m \begin{pmatrix} a_{11}/\lambda & a_{12} \\ a_{21} & a_{22}/\lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \ln(a_{11}/\lambda) & \ln(a_{12}) \\ \ln(a_{21}) & \ln(a_{22}/\lambda) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and equivalent to $\{\ln_m(A \oplus (\lambda * E))\}(u, v)^\top = (0, 0)$. Then, to have non-null solutions (u, v), we need

$$\mathbf{det}\{\ln_m(A \oplus (\lambda * E))\} = 0, \quad \text{i.e.,} \quad p_A(\lambda) = \mathbf{det}_m\{(A \oplus (\lambda * E))\} = 1,$$

so that

$$p_A(\lambda) = (\frac{a_{11}}{\lambda} * \frac{a_{22}}{\lambda})/(a_{12} * a_{21}), \quad \mathbf{det}_m(A) = p_A(1) \quad \text{and} \quad \mathbf{T}_m(A) = a_{11} \cdot a_{22}$$

are obtained. Importantly, the condition $p_A(\lambda) = 1$, implies

$$(a_{11} \cdot \lambda^{-1}) * (a_{22} \cdot \lambda^{-1}) = a_{12} * a_{21},$$

a condition that can be written

$$\lambda^{\prec 2 \succ} \cdot (\mathbf{Tr}_m(A) * \lambda^{-1}) \cdot \mathbf{det}_m(A) = 1,$$

an equivalent expression to

$$\{\lambda^{-1} \cdot \mathbf{Tr}_m(A)\}^{\prec 2 \succ} = \Delta,$$

with $\Delta = \sqrt{\mathbf{Tr}_m(A)}^{\prec 2 \succ} / \mathbf{det}_m(A) = (\mathbf{Tr}_m(A)^{\prec 2 \succ})^{1/4} / \mathbf{det}_m(A)$. Since $a^{\prec 2 \succ} \geq 1$, for any $a \in \mathbb{R}_+$, a condition for the existence of $\lambda \in \mathbb{R}_+$ is $\Delta \geq 1$, i.e.,

$$\mathbf{Tr}_m(A)^{\prec 2 \succ} \geq \mathbf{det}_m^4(A).$$

Notice that for the $T_{\exp(B)}$ map, we have

$$p_{\exp(B)}(\lambda) = (e^2 \lambda^{-1} * e^2 \lambda^{-1})/(e^3 * e^3) = 1,$$

that implies $e^2 \lambda^{-1} * e^2 \lambda^{-1} = e^3 * e^3$, i.e., $(e^2/\lambda)^{\prec 2 \succ} = (e^3)^{\prec 2 \succ}$. Since $\ln(a^{\prec 2 \succ}) = \ln^2(a)$, we get $|2 - \ln(\lambda)| = 3$. With $z = \ln(\lambda)$, it can be expressed as $z^2 - 4z - 5 = (z-5)(z+1) = 0$, which is the standard characteristic polynomial of the matrix B. So, the roots of the polynomial are the eigenvalues of B, these are $z_1 = +5$ and $z_2 = -1$. Then, we have the pair of m-eigenvalues $\lambda_1 = e^{+5}$ and $\lambda_2 = e^{-1}$.

Taking $\lambda_1 = e^{+5}$, the system for have (u, v) is:

$$[2 - \ln(a_1)]u + 3v = -3u + 3v = -3(u - v) = 0.$$

That determines the subspace: $W(e^5)$ generated by the vector $(u, v) = (1, 1)$, this is, the m-eigenvector $w_1 = (e, e)$. Now, with $\lambda_2 = e^{-1}$, the system for have (u, v) is: $3u + 3v = 3(u+v) = 0$. That determines the vector $(u, v) = (1, -1)$, then the m-eigenvector $w_2 = (e, e^{-1})$. Indeed,

$$T(e, e) = (e^2 e^3, e^3 e^2) = (e^5, e^5) = (e^5 * e, e^5 * e) = e^5 * (e, e),$$

and

$$T(e, e^{-1}) = (e^2 e^{-3}, e^3 e^{-2}) = (e^{-1}, e^{+1}) = e^{-1} * (e, e^{-1})$$

are obtained.

Looking for α and β such that $(x, y) = [\alpha * (e, e)] \cdot [\beta * (e, e^{-1})] = (\alpha\beta, \alpha/\beta)$, we get the decomposition

$$(x, y) = [\sqrt{xy} * (e, e)] \cdot [\sqrt{y/x} * (e, e^{-1})].$$

Using the linearity, we recuperate the transformation formula by:

$$\begin{aligned} T(x,y) &= [\sqrt{xy} * T(e,e)] \cdot [\sqrt{y/x} * T(e,e^{-1})] \\ &= [\sqrt{xy} * e^5 * (e,e)] \cdot [\sqrt{y/x} * e^{-1} * (e,e^{-1})] \\ &= [(x^{5/2}y^{5/2}) * (e,e)] \cdot [x^{1/2}y^{-1/2}) * (e,e^{-1})] \\ &= [(x^{5/2}y^{5/2}, x^{5/2}y^{5/2})] \cdot [x^{1/2}y^{-1/2}, x^{-1/2}y^{1/2})] \\ &= (x^{6/2}y^{4/2}, x^{4/2}y^{6/2}) \\ &= (x^3y^2, x^2y^3). \end{aligned}$$

Now regarding the diagonalization, let'us note that

$$\exp(B) * (w_1|w_2) = \begin{pmatrix} e^2 & e^3 \\ e^3 & e^2 \end{pmatrix} * \begin{pmatrix} e & e \\ e & e^{-1} \end{pmatrix} = \begin{pmatrix} e^5 & e^{-1} \\ e^5 & e^{+1} \end{pmatrix}$$

and

$$(w_1|w_2) * \text{diag}_m(\lambda_1, \lambda_2) = \begin{pmatrix} e & e \\ e & e^{-1} \end{pmatrix} * \begin{pmatrix} e^5 & 1 \\ 1 & e^{-1} \end{pmatrix} = \begin{pmatrix} e^5 & e^{-1} \\ e^5 & e^{+1} \end{pmatrix}.$$

Then $\exp(B) * (w_1|w_2) = (w_1|w_2) * \text{diag}_m(\lambda_1, \lambda_2)$. Thus

$$(w_1|w_2)^{\prec -1 \succ} * \exp(B) * (w_1|w_2) = \text{diag}_m(\lambda_1, \lambda_2).$$

5. A non-linear but m-linear dynamical system. Let us consider the non-linear discrete dynamical system with state space $\mathbb{R}^2_+$ and transition function defined by:

$$\begin{cases} x_{k+1} = x_k^{\alpha}\, y_k^{\beta} \\ y_{k+1} = x_k^{\gamma}\, y_k^{\delta}, \end{cases} \tag{5.1}$$

where α, β, γ, and δ are real numbers. We know that for mediation of the logarithm function, it can be seen and studied as a standard linear system. Nevertheless, (5.1) is a linear system in a multiplicative sense (m -linear) because it is defined by the m-liner map:

$$T(x, y) = (x^{\alpha}\, y^{\beta}, x^{\gamma}\, y^{\delta}).$$

As m-linear system, (5.1) admits the following matrix representation:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \exp(A) * \begin{pmatrix} x_k \\ y_k \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

In order, from an initial condition $(x_0, y_0)^{\top}$, to get its positive orbit

$$\Theta((x_0, y_0)^{\top}) = \{(x_k, y_k)^{\top} \,:\, k \geq 0\},$$

we define

$$\Phi(k, v^{\top}) = \exp(A)^{\prec k \succ} * v^{\top},$$

where $v \in \mathbb{R}^2_+$, which satisfies the flux property:

$$\Phi(k_1 + k_0, v^{\top}) = \Phi(k_1, \Phi(k_0, v^{\top})), \quad \text{for all} \quad k_1,\, k_0 \geq 0.$$

Then, returning to system (5.1) the positive orbit of an initial state $(x_0, y_0)^{\top}$ is

$$\Theta((x_0, y_0)^{\top}) = \{\Phi(k, (x_0, y_0)^{\top}) \,:\, k \geq 0\} = \{\exp(A)^{\prec k \succ} * (x_0, y_0)^{\top} \,:\, k \geq 0\}.$$

Equilibria:. It is clear that $(1,1)$ is always a fixed point. Now if there exists another $(x,y) \neq (1,1)$, it is clear that the matrix $\exp(A)$ must have the number one as an m-eigenvalue and $W(1,1)$, the set of respective m-eigenvectors, as the set of all fixed points. Now, if $(x,y)^\top \neq (1,1)$ is a fixed point, then $\exp(A) * (x,y)^\top = (x,y)^\top$. This is, $(\exp(A) \ominus E) * (x,y)^\top = (1,1)$. So, we have the condition $\mathbf{det}_m(\exp(A) \ominus E) = 1$, which is equivalent to $e^{\alpha-1} * e^{\delta-1} = e^{\beta} * e^{\gamma}$, i.e., $(\alpha-1)(\delta-1) = \beta\gamma$. Therefore, $\mathbf{det}(A) = 0$.

Periodic orbits:. Since $\mathbf{det}_m(B * C) = \mathbf{det}_m(B) * \mathbf{det}_m(C)$. We have, that the condition for the existence of a n-periodic orbit $\exp(A)^{\prec n \succ}) * (x,y)^\top = (x,y)^\top$, implying $\{\exp(A)^{\prec n \succ} \ominus E\} * (x,y)^\top = 1$, in terms of the determinant is $\mathbf{det}_m(\exp(A)^n) \ominus E) = \mathbf{det}_m(\exp(A^n) \ominus E) = 1$, Therefore, $\mathbf{det}(A^n) = \mathbf{det}^n(A) = 0$, i.e., $\mathbf{det}(A) = 0$.

Now, if we take $\ln_m(\cdot)$ to both sides of the original system $(x_{k+1}, y_{k+1})^\top = \exp(A) * (x_k, y_k)^\top$, we get $(u_{k+1}, v_{k+1})^\top = A \cdot (u_k, v_k)^\top$. Thus, they are equivalent because (in the sense that) the orbits are bijectively related.

We note that delving into the dynamics in terms of m-eigenvalues and characterization of periodic orbits (e.g., with rotation matrices) involves introducing us to the field of complex numbers. In this regard, it only remains to say that these efforts are outside the objectives of this article. We only give here the required number field structure: $\mathbb{C}_+ := \{u \cdot (v * \kappa) \ : \ u, v \in \mathbb{R}_+\}$, where an element $u_1 \cdot (v_1 * \kappa)$ is equal to the other $u_2 \cdot (v_2 * \kappa)$ if $u_1 = u_2$ and $v_1 = v_2$. Now, introducing in $\mathbb{C}_+$ the operations (i) Product standard: $\{a \cdot (b * \kappa)\} \cdot \{c \cdot (d * \kappa)\} = (ac) \cdot \{(bd) * \kappa\}$ and (ii) Exponentiation: $\{a \cdot (b * \kappa)\} * \{c \cdot (d * \kappa)\} = \{(a * c)/(b * d)\} \cdot \{(a * d)(b * c)\} * \kappa$, we have that $(\mathbb{C}_+, \cdot, *)$ is a field, where $\kappa^{\prec 2 \succ} = 1/e$.

6. Conclusion. Just as the isomorphism between the bodies $(\mathbb{R}, +, \cdot)$ and $(\mathbb{R}_+, \cdot, *)$ implies the existence of isomorphic *calculi*, e.g., the one existing between the Newtonian calculus and multiplicative calculus, this also implies an extension of the isomorphy to the realm of linear algebras. This is the case, which we have tried to highlight, between the matrices with real entries and those with positive entries (isomorphic algebras), and some of their spectral concepts.

Just to let you know, from a purely abstract perspective, the novelty of what was presented could be considered by citing only a nominal exercise. However, in contrast, we must note that the gain is that isomorphic structures are not a reality parallel to the standard one but rather they are inside it and contain it. We mean that the real matrices and their linear structures contain the matrices of positive entries with the presented structure, and so on. Defining a cascade of isomorphic structures within each other. Thus, before each true proposition of the theory, there are infinite true propositions (of similar architectures) within the same theory. For example, it is enough to remember that the geometric mean in real numbers, between positive numbers, is the equivalent, or the isomorphic image, of the arithmetic mean [8, 9]. However, we cannot deny that proper understanding of the geometric mean is a technical necessity and we would not be willing to go back and forth through isomorphism every time we want to calculate it.

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