Journal homepage http://revistas.unitru.edu.pe/index.php/SSMM

SELECCIONES MATEMÁTICAS
Universidad Nacional de Trujillo
ISSN: 2411-1783 (Online)
2025; Vol.12(1):97-122.

Tangential intersection curves of two surfaces in the three-dimensional Lorentz-Minkowski space

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Received, Mar. 10, 2025; *Accepted, Jul. 01, 2025;* *Published, Jul. 27, 2025*

How to cite this article:
Alessio O, Cintra Neto L.A.R. *Tangential intersection curves of two surfaces in the three-dimensional Lorentz-Minkowski space*. Selecciones Matemáticas. 2025;12(1):97–122. http://dx.doi.org/10.17268/sel.mat.2025.01.09

Abstract

We present algorithms for computing the differential geometry properties of tangential intersection curves of two surfaces in the three-dimensional Lorentz-Minkowski space $\mathbb{E}^3_1$. We compute the tangent vector of tangential intersection curves of two surfaces parametric, where the surfaces can be: spacelike, timelike, or lightlike. The first method computed the tangent vector using the equality of the projection of the second derivative vector onto the normal vector and second method computes the tangent vector by applying a rotation to a vector projected onto the tangent space, where the axis of rotation is the normal vector of the surface. In Minkowski space, there are three types of rotations, since the normal vectors can be: spacelike, lightlike, or timelike.

Keywords . Euler Rodrigues formula, Tangential Intersection, Lorentz Minkowski space, Surface-surface intersection.

Resumen

Presentamos algoritmos para calcular las propiedades de la geometría diferencial de las curvas de intersección tangencial de dos superficies en el espacio de Lorentz-Minkowski tridimensional $\mathbb{E}^3_1$. Calculamos el vector tangente de las curvas de intersección tangencial de dos superficies paramétricas, donde las superficies pueden ser: espaciales (spacelike), temporales (timelike) o isotrópicas (lightlike). El primer método calcula el vector tangente utilizando la igualdad de la proyección del vector derivada segunda sobre el vector normal. El segundo método calcula el vector tangente aplicando una rotación a un vector proyectado sobre el espacio tangente, donde el eje de rotación es el vector normal de la superficie. En el espacio de Minkowski, existen tres tipos de rotaciones, ya que los vectores normales pueden ser: espaciales, isotrópicos o temporales.

Palabras clave. Fórmula Euler Rodrigues; Intersección Tangencial; Espacio Lorentz Minkowski; Intersección Superficie-superficie.

1. Introduction. Rotation matrices are very important matrices especially for computer sciences. For this reason, the generation of a rotation matrix becomes one of the most important problems for mathematicians. In differential geometry and mechanics, Euler-Rodrigues formula is a quite useful formula to generate the rotation matrix for a given rotation angle θ around a given rotation axis in Euclidean 3-space. If we take the skew-symmetric matrix as

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$$W = \begin{bmatrix} 0 & w_z & -w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix}.$$

where $\mathbf{w} = (w_x, w_y, w_z)$ is a unit vector, then we get the Rodrigues formula

$$R = I + \sin(\theta)W + (1 - \cos(\theta))W^2$$

.

In Minkowski 3-space, the rotation axis can be spacelike, timelike, or lightlike. Consequently, the Euler–Rodrigues formula in Minkowski 3-space varies depending on the nature of the rotation axis. The Rodrigues rotation formulas in Minkowski 3-space are provided in [1, 2]. Rotation matrices in Minkowski 3-space are generated using unit timelike split quaternions, as shown in [3]. The Rodrigues equation for rotations with a spacelike rotation axis is presented in [4]. Geometric and algebraic interpretations of the Euler–Rodrigues formula in Minkowski 3-space, as well as the derivation of the formula for spacelike, timelike, and lightlike axes, are discussed in [5].

If the Lorentz metric (inner product) of signature $(+,+,-)$, then the semi-skew-symmetric matrix

$$W = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ w_y & -w_x & 0 \end{bmatrix}.$$

where $\mathbf{w} = (w_x, w_y, w_z)$ is rotation axis, then we get the Euler-Rodrigues formula

$$If\ \mathbf{w}\ is\ unit\ spacelike\ vector,\ we\ have\quad \mathcal{R}_{sp}(\theta, \mathbf{w}) = I_3 + sinh(\theta)W + (-1 + cosh(\theta))W^2$$

$$If\ \mathbf{w}\ is\ unit\ timelike\ vector,\ we\ have\quad \mathcal{R}_{tm}(\theta, \mathbf{w}) = I_3 + sin(\theta)W + (1 - cos(\theta))W^2$$

$$If\ \mathbf{w}\ is\ lightlike\ vector, we\ have\quad \mathcal{R}_{lg}(\theta, \mathbf{w}) = I_3 + \theta\, W + \frac{\theta^2}{2}\, W^2.$$

The Rodrigues' rotation formula was introduced into the surface intersection problem by Bahar and Mustafa in [6]. In that paper, the authors proposed new approaches for analyzing both tangential and transversal intersections of two surfaces in Euclidean 3-space using Rodrigues' rotation formula. In [7, 8], the authors employed Rodrigues' rotation formula to derive the geometric properties of the transversal intersection curve of two regular parametric and implicit surfaces in $\mathbb{R}^3$, including the computation of geodesic curvature and geodesic torsion.

The surface–surface intersection (SSI), is a fundamental problem in *computational geometry* and geometric modeling of complex shapes. For general parametric surface intersections, the most commonly used methods include *subdivision* and *marching*. Marching-based algorithms begin by finding a starting point on a intersection curve, and proceed to march along the curve. The marching method which yields linear approximation of the intersection curve is used to generate the sequences of exact points of the intersection curve in a direction obtained by the local differential geometry. To compute the intersection curve with precision, we need higher-order approximation, i.e. higher-order derivative vectors of the intersection curve.

We can find the geometric properties of parametric curves in the classical literature on differential geometry in Euclidean space $\mathbb{E}^3$ [9, 10, 11, 12, 13] and in the contemporary literature on geometric modeling [14, 15, 16, 17]. There is a textbook with a systematic study of curves and surfaces in Lorentz-Minkowski space such as it occurs in the Euclidean space, in the books [18]. Some of the topics of this paper can be found in some books [19, 20, 21] and thesis in Minkowski space [22] and papers [23, 24, 25, 26, 27, 28, 29, 30]. A general reference including many topics in semi-Riemannian geometry is the classical book [21].

Differential geometry of intersection curves of $(n-1)$ hypersurfaces in Euclidean space $\mathbb{E}^n$, $n \geq 3$ can be found in several articles, on the other hand, there is very little or almost nothing literature for differential geometry of transversal intersection curves of $(n-1)$ hypersurfaces in Lorentz-Minkowski space $\mathbb{E}^n_1$, $n \geq 3$ and there is no tangential intersection curve of $(n-1)$ hypersurfaces in Lorentz-Minkowski space $\mathbb{E}^n_1$, $n = 3, 4$.

For transversal intersections in Euclidean spaces $\mathbb{E}^n_1$, $n = 3, 4$, various studies have addressed the computation of differential geometric properties of the intersection curves. Willmore [13] obtained the unit tangent, unit principal normal, and unit binormal vectors, along with the curvatures of the intersection curve of two implicit surfaces in $\mathbb{E}^3$, using the operator $\triangle = \lambda \frac{d}{ds} = \left(h_1 \frac{\partial}{\partial x} + h_2 \frac{\partial}{\partial y} + h_3 \frac{\partial}{\partial z} \right)$, where $h = \nabla f \times \nabla g$. Faux and Pratt [14] provided an expression for the curvature of the intersection curve of two parametric surfaces. Using the Implicit Function Theorem, Hartmann [31] derived formulas for computing the curvature and geodesic curvature of intersection curves for all three types of intersection problems in $\mathbb{E}^3$. Ye and Maekawa [32] expressed the curvature vector as a linear combination of the normal vectors of the intersecting surfaces and represented the third-order derivative vector as a linear combination of the tangent and normal vectors, thereby obtaining the Frenet frame and curvatures of the intersection curve. Goldman [33] derived closed-form expressions for computing the curvatures of the intersection curve of two implicit surfaces in $\mathbb{E}^3$, as well as the first curvature of the intersection curve in $(n+1)$-dimensional space. Aléssio [34] computed the curvature and torsion of the transversal intersection of two implicit surfaces using the Implicit Function Theorem. Building on the method of Ye and Maekawa and the Implicit Function Theorem, Aléssio [35] developed a technique for computing the Frenet apparatus of the transversal intersection curve of three implicit hypersurfaces in $\mathbb{E}^4$. Düldül and Çalışkan [36] computed the geodesic torsion and geodesic curvature of the intersection curve of two regular surfaces defined by parametric-parametric and implicit-implicit equations. Düldül [37] proposed a method to compute the Frenet frame and curvatures of the transversal intersection curve of three parametric hypersurfaces in $\mathbb{E}^4$. Nassar et al. [38], in CAGD, provided a method for computing the Frenet vectors and curvatures of the transversal intersection curve of implicit-parametric-parametric and implicit-implicit-parametric hypersurfaces in $\mathbb{E}^4$. Uyar Düldül and Düldül [39] extended Willmore's method to four-dimensional space. Aléssio [40] computed the normal curvature, the first geodesic curvature, and the first geodesic torsion of the transversal intersection curve of $n-1$ implicit hypersurfaces in $\mathbb{E}^n$. Following the ideas of Willmore, Düldül and Akbaba [41] proposed a new method for analyzing the intersection of two surfaces in three-dimensional space and three hypersurfaces in four-dimensional space, where at least one (hyper)surface is defined parametrically. Finally, Bahar and Mustafa Düldül [6] introduced two new approaches for analyzing the transversal intersection of two surfaces in Euclidean 3-space using Rodrigues' rotation formula.

For tangential intersections in $\mathbb{E}^3$, the available literature is relatively limited. Ye and Maekawa [32] proposed an algorithm for evaluating higher-order derivatives of the tangential intersection curve of two surfaces, considering all three types of surface-surface intersection problems in $\mathbb{E}^3$. Çalışkan and Düldül [42] computed the unit tangent vector and the geodesic torsion of the tangential intersection curve of two surfaces, also addressing all three intersection types. Nassar et al. [43] investigated the differential geometric properties of the Frenet apparatus (t, n, b, κ, τ) of intersection curves of two implicit surfaces in $\mathbb{R}^3$, treating both transversal and tangential cases using the Implicit Function Theorem. Bahar and Mustafa Düldül [6] introduced two new approaches for analyzing the tangential intersection of two surfaces in Euclidean 3-space, utilizing Rodrigues' rotation formula and Willmore's method. For tangential intersections in $\mathbb{E}^4$, the authors of [44] studied the non-transversal intersection of parametric-parametric-parametric hypersurfaces. In [45], they addressed the non-transversal intersection of implicit-implicit-parametric and implicit-parametric-parametric hypersurfaces, and in [46], the non-transversal intersection of implicit-implicit-implicit hypersurfaces in $\mathbb{E}^4$.

The differential geometry of intersection curves arising from transversal intersections in Lorentz-Minkowski spaces $\mathbb{E}^3_1$ and $\mathbb{E}^4_1$ has been studied in several works [47, 48, 49]. Aléssio and Guadalupe [47] investigated the transversal intersection curve of two parametric spacelike surfaces in $\mathbb{E}^3_1$, considering the parametric-parametric intersection problem. Zafer and Yusuf [50] studied the intersection curve of two parametric timelike surfaces in $\mathbb{E}^3_1$. Karaahmetoğlu and Aydemir [51] examined the intersection curves between parametric spacelike and timelike surfaces in $\mathbb{E}^3_1$. Düldül and Çalışkan [48] computed the Frenet vectors and curvatures of the spacelike intersection curve of three spacelike hypersurfaces defined parametrically in four-dimensional Minkowski space $\mathbb{E}^4_1$. Aléssio et al. [52] studied the differential geometry of the transversal intersection curves of two surfaces in Minkowski 3-space $\mathbb{E}^3_1$, where the surfaces may be spacelike, timelike, or lightlike. They considered all combinations of pairs: spacelike-lightlike, timelike-lightlike, and lightlike-lightlike.

In this paper, we compute the tangent vector of tangential intersection curves of two surfaces in the three-dimensional Lorentz-Minkowski space $\mathbb{E}^3_1$, where the combination of the surfaces (spacelike, timelike, or lightlike) can be parametric-parametric. We propose two new approaches for handling the tangential intersection of two surfaces in $\mathbb{E}^3_1$. Our first method differs from the approach of Ye and Maekawa only in the metric used, along with certain adaptations for cases involving lightlike surfaces. Our second method modifies the use of Rodrigues' rotation formula. In the work of Bahar and Mustafa, the rotation was carried out in Euclidean space, whereas in our approach, the rotation is performed in Lorentz-Minkowski space. This method uses the Euler-Rodrigues formula in cases where the rotation axis—given by the normal vector of the surface—is spacelike, timelike, or lightlike in $\mathbb{E}^3_1$.

The remainder of the paper is organized as follows: Section 2 introduces some notation and definitions, and reviews relevant aspects of differential geometry in the Lorentz-Minkowski 3-space $\mathbb{E}^3_1$. Section 3 further develops the notation and revisits the differential geometry of curves and surfaces in $\mathbb{E}^3_1$. Section 4 presents the Euler-Rodrigues formula adapted to Minkowski 3-space. Section 5 focuses on the differential geometry of the tangential intersection curve of two surfaces in $\mathbb{E}^3_1$, where we propose two methods. The first method is analogous to the one by Ye and Maekawa [32] in Euclidean space, while the second is based on the approach of Bahar and Mustafa [6], but adapted using the Euler-Rodrigues formula in Minkowski 3-space. Some numerical results are provided in Section 6, and concluding remarks are given in Section 7.

2. Preliminaries. In this paper we denote by the Lorentz-Minkowski 3-space $\mathbb{E}^3_1$, the pair $(\mathbb{R}^3, \langle,\rangle_1)$ where $\mathbb{R}^3$ is a three dimensional real vector space equipped with a Lorentz metric (inner product) of signature (2,1). That is, if $\mathbf{v} = (x_1, x_2, x_3)$ and $\mathbf{u} = (y_1, y_2, y_3)$, then

$$\langle \mathbf{v}, \mathbf{u}\rangle_L = x_1y_1 + x_2y_2 - x_3y_3.$$

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{E}^3_1$ are said to be *orthogonal* if $\langle \mathbf{u}, \mathbf{v}\rangle_L = 0$. An arbitrary vector $\mathbf{u}$ in $\mathbb{E}^3_1$ which satisfies $\langle \mathbf{u}, \mathbf{u}\rangle_1 = \pm 1$ is called a *unit vector*.

We say that an arbitrary vector $\mathbf{v} \neq 0$ in $\mathbb{E}^3_1$ is called *spacelike***,** *timelike* or *lightlike*(*null*), if respectively holds $\langle \mathbf{v}, \mathbf{v}\rangle_L > 0$, $\langle \mathbf{v}, \mathbf{v}\rangle_L < 0$ or $\langle \mathbf{v}, \mathbf{v}\rangle_L = 0$. In particular, the vector $\mathbf{v} = 0$ is *spacelike*. If $\mathbf{v} = (x_1, x_2, x_3) \in \mathbb{E}^3_1$ we define it norm by

$$\|\mathbf{v}\|_L = |\langle \mathbf{v}, \mathbf{v}\rangle_1|^{\frac{1}{2}} = \sqrt[2]{|x_1x_1 + x_2x_2 - x_3x_3|}.$$

The timelike vectors can be separate in two disjoint sets, with describes the next definition.

Definition 2.1. *Let $\mathcal{F}$ be the set of all timelike vectors in $\mathbb{E}^3_1$. If $\mathbf{u}$ is a timelike vector, the timelike cone of $\mathbf{u}$ is the set*

$$\mathcal{C}(\mathbf{u}) = \{\mathbf{v} \in \mathcal{F} | \langle \mathbf{u}, \mathbf{v}\rangle_L < 0\}.$$

The opposite timelike cone is $\mathcal{C}(-\mathbf{u}) = -C(\mathbf{u}) = \{\mathbf{v} \in \mathcal{F} | \langle \mathbf{u}, \mathbf{v}\rangle_L > 0\}$. Since $\mathbf{u}^\perp$ is spacelike, $\mathcal{F}$ is the disjoint union of these two timelike cones, i.e, $\mathcal{F} = \mathcal{C}(-\mathbf{u}) \bigcup \mathcal{C}(\mathbf{u})$. Furthermore we can conclude of the definition that, two timelike vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{E}^3_1$ are in the same timelike cone if and only if $\langle \mathbf{v}, \mathbf{w}\rangle_L < 0$. The Results the following show the relation of two vectors in $\mathbb{E}^3_1$ with the usual or hyperbolic angle formed between them.

Proposition 2.1. *[21] Let $\mathbf{v}$ and $\mathbf{w}$ be timelike vectors in $\mathbb{E}^3_1$. Then*
- $|\langle \mathbf{v}, \mathbf{w}\rangle_L| \geq \|\mathbf{v}\|_L \|\mathbf{w}\|_L$, *with equality if and only if $\mathbf{v}$ and **w** are collinear (reverse inequality Cauchy-Schwarz);*
- *If $\mathbf{v}$ and $\mathbf{w}$ are in the same timelike cone of $\mathbb{E}^3_1$, there is a unique $\theta \geq 0$, called the hyperbolic angle between v and w, such that $\langle \mathbf{v}, \mathbf{w}\rangle_L = -\|\mathbf{v}\|_L \|\mathbf{w}\|_L cosh(\theta)$;*
- *If $\mathbf{v}$ and $\mathbf{w}$ are not in the same timelike cone of $\mathbb{E}^3_1$, there is a unique $\theta \geq 0$, called the hyperbolic angle between v and w, such that $\langle \mathbf{v}, \mathbf{w}\rangle_L = \|\mathbf{v}\|_L \|\mathbf{w}\|_L cosh(\theta)$.*

Definition 2.2. *[25] Let $\boldsymbol{u}, \boldsymbol{v} \in E^3_1$. The Lorentizian vector product of **u** and **v** is to the unique vector denoted by $\boldsymbol{u} \times_L \boldsymbol{v}$ that satisfies*

$$\langle \boldsymbol{u} \times_L \boldsymbol{v}, \boldsymbol{w}\rangle_L = det(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \tag{2.1}$$

*where $det(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ is the determinant of the matrix obtained by replacing by columns the coordinates of the three vectors **u**, **v** and **w**.*

We also define the vector product of $\mathbf{u}$ and $\mathbf{v}$ (in that order) as the unique vector $\mathbf{u} \times_L \mathbf{v} \in \mathbb{E}^3_1$ such that

$$\mathbf{u} \times_L \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & -\mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), -(u_1v_2 - u_2v_1)),$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical basis of $\mathbb{E}^3_1$ and $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. We have $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical basis of $\mathbb{R}^3_1$, which satisfy $\mathbf{e}_1 \times_L \mathbf{e}_2 = -\mathbf{e}_3$, $\mathbf{e}_2 \times_L \mathbf{e}_3 = \mathbf{e}_1$ and $\mathbf{e}_3 \times_L \mathbf{e}_1 = \mathbf{e}_2$.

Corollary 2.1. *[25] (Lagrange's Identities). Let* $\boldsymbol{u}, \boldsymbol{v} \in E_1^3$*. Then*

$$\langle \boldsymbol{u} \times_L \boldsymbol{v}, \boldsymbol{u} \times_L \boldsymbol{v} \rangle_L \quad = \quad -det \begin{bmatrix} \langle \boldsymbol{u}, \boldsymbol{u} \rangle_L & \langle \boldsymbol{u}, \boldsymbol{v} \rangle_L \\ \langle \boldsymbol{v}, \boldsymbol{u} \rangle_L & \langle \boldsymbol{v}, \boldsymbol{v} \rangle_L \end{bmatrix}. \tag{2.2}$$

Remark 2.1. *[25] Let us observe that if* $\mathbf{u}$ *and* $\mathbf{v}$ *are two non-degenerate vectors, then* $B = \{\boldsymbol{u},\ \boldsymbol{v}, \boldsymbol{u} \times_L \boldsymbol{v}\}$ *is basis of* $\in \mathbb{E}_1^3$*. However, and in contrast to the Euclidean space, the causal character of* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *determines if the basis is or is not positively oriented. Exactly, if* $\mathbf{u}$ *and* $\mathbf{v}$ *are spacelike vectors of* $\mathbb{E}_1^3$ *then* $\mathbf{u} \times_L \mathbf{v}$ *is a timelike and* B *is negatively oriented because* $det(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u} \times_L \boldsymbol{v}) = \langle \boldsymbol{u} \times_L \boldsymbol{v}, \boldsymbol{u} \times_L \boldsymbol{v} \rangle_L < 0$*. Is* $\mathbf{u}$ *and* $\mathbf{v}$ *have different causal character, then* B *is positively oriented.*

Proposition 2.2. *[25] Let* $\mathbb{E}_1^3$*, then*

- *Two null (lightlike) vectors are linearly dependent if and only if they are orthogonal;*
- *Two timelike vectors are never orthogonal;*
- *A timelike vector is never orthogonal to a null (lightlike) vector.*

Proposition 2.3. *[25] For* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{E}_1^3$ *we have*

- $\langle \mathbf{u} \times_L \mathbf{v}, \mathbf{u} \rangle_L = 0$ *and* $\langle \mathbf{u} \times_L \mathbf{v}, \mathbf{v} \rangle_L = 0$;
- $\langle \mathbf{u} \times_L \mathbf{v}, \mathbf{u} \times_L \mathbf{v} \rangle_L = \langle \mathbf{u}, \mathbf{v} \rangle_L^2 - \langle \mathbf{u}, \mathbf{u} \rangle_L \langle \mathbf{v}, \mathbf{v} \rangle_L$;
- *Let* $\mathbf{u}$ *be a spacelike vector,* $\mathbf{v}$ *be a null vector, then* $\langle \mathbf{u}, \mathbf{v} \rangle_L \neq 0$ *if and only if* $\mathbf{u} \times_L \mathbf{v}$ *is spacelike. Also* $\langle \mathbf{u}, \mathbf{v} \rangle_L = 0$ *if and only if* $\mathbf{u} \times_L \mathbf{v}$ *is null;*
- *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are null vectors, then* $\mathbf{u} \times_L \mathbf{v}$ *is a spacelike vector;*
- *If* $\mathbf{u}$ *is a timelike vector,* $\mathbf{v}$ *is a null vector, then* $\mathbf{u} \times_L \mathbf{v}$ *is spacelike vector.*

Proposition 2.4. *[25] Let* $\mathbb{E}_1^3$*, then*

- *Two null (lightlike) vectors are linearly dependent if and only if they are orthogonal;*
- *Two timelike vectors are never orthogonal;*
- *A timelike vector is never orthogonal to a null (lightlike) vector.*

Proposition 2.5. *[25] Let* U *be a vector subspace of* $\mathbb{E}_1^3$*. The following statements are equivalent:*

- U *is a lightlike subspace;*
- U *contains a lightlike vector bute not a timelike one;*
- $U \cap \mathcal{C} = \mathcal{L} - \{0\}$*, and* $dim\ \mathcal{L} = 1$*.*

Proposition 2.6. *(Causal Character). Let* $M \subset E_1^3$ *be a regular surface. We say that:*

- M *is spacelike, if for each* $p \in M,\ T_pM$ *is a spacelike plane;*
- M *is timelike, if for each* $p \in M,\ T_pM$ *is a timelike plane;*
- M *is lightlike, if for each* $p \in M,\ T_pM$ *is a lightlike plane.*

Remark 2.2.

- *We will have a situation similar to what happened for curves: by continuity, if* T_pM *is spacelike or timelike for some* $p \in M$*, then* T_qM*will have the same causal type for each* q *in some neighborhood of* $pinM$*. This way,we may possibly restrict our attention to surfaces with constant causal character, when needed.*
- *If* M *has no points* p *for which* T_pM *is lightlike, we'll simply say that M is non-degenerate.*

3. Curves in E_1^3.

The follows definition classifies the curves in $\mathbb{E}_1^3$.

Definition 3.1. *A regular curve* $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ *can locally be a spacelike, timelike or null (lightlike), if all of its velocity vectors* $\alpha'(s)$ *are respectively spacelike, timelike or null (lightlike).*

Definition 3.2. *For any non-lightlike vector* $\vec{v} \neq 0$*, we set its indicator* ϵ_v *to be the sign of* $\langle \vec{v}, \vec{v} \rangle_L$*, that is,* $\epsilon_v = 1$ *if* $\vec{v}$ *is spacelike,* $\epsilon_v = -1$ *if* $\vec{v}$ *is timelike, and* $\epsilon_v = 0$ *if* $\vec{v}$ *is lightlike.*

Definition 3.3. *[18] Let* $\alpha : I \to \mathbb{E}_1^3$ *be a parametrized curve. We'll say that* α *is admissible if:*

- α *is biregular, that is,* $\{\alpha'(t), \alpha''(t)\}$ *is linearly independent for all* $t \in I$*;*
- *both* $\alpha'(t)$ *and span* $\{\alpha'(t), \alpha''(t)\}$ *are not lightlike, for all* $t \in I$*.*

Remark 3.1. *If* $\alpha(s)'$ *and* $\alpha''(s)$ *are orthogonal and span* $\{\alpha'(t), \alpha''(t)\}$ *are not lightlike, then* $\alpha''(s)$ *cannot be lightlike.*

Definition 3.4. *Let* $\alpha : I \to \mathbb{E}_1^3$ *be a unit admissible curve. The indicador of* α *is* $\epsilon_\alpha = \langle \mathbf{t}_\alpha, \mathbf{t}_\alpha \rangle_L$ *and coindicador of* α *is* $\eta_\alpha = \langle \mathbf{n}_\alpha, \mathbf{n}_\alpha \rangle_L$*, where* $\mathbf{t}_\alpha = \alpha'(s)$ *and* $\mathbf{n}_\alpha = \dfrac{\alpha''(s)}{\kappa_\alpha(s)}$*.*

3.1. The Frenet-Serret trihedron.

Definition 3.5. *(Frenet-Serret Trihedron for admissible unit speed curve). Let* $\alpha : I \to \mathbb{E}_1^3$ *be an admissible unit speed curve.The tangent vector to* α *at* s *is* $\mathbf{t}_\alpha(s) = \alpha'(s)$*. The curvature of the curve* α *at s is* $\kappa_\alpha(s) = \|\mathbf{t}'(s)\|_L$*. The assumptions on* α *ensure that* $\kappa_\alpha(s) > 0$ *for all* $s \in I$*, so that the unit vector pointing in the same direction as* $\mathbf{t}'(s)$ *is well defined, allowing us to define the normal vector to* α *at s, by the relation* $\mathbf{t}'_\alpha(s) = \kappa_\alpha(s)\mathbf{n}_\alpha(s)$*. Lastly, the binormal vector to* α *at s is defined as the unique vector* $\mathbf{b}(s)$ *making the basis* $\{\mathbf{t}(s), \mathbf{n}_\alpha(s), \mathbf{b}_\alpha(s)\}$ *orthonormal and positive.*

Theorem 1. *[18] (Frenet-Serret Equations admissible curve (If curve is spacelike, the $\alpha(s)''$ cannot be lightlike)). Let $\alpha : I \to \mathbb{E}_1^3$ be a unit speed admissible curve. Then*

$$\begin{cases} \boldsymbol{t}'_\alpha(s) = \kappa_\alpha \boldsymbol{n}_\alpha(s); \\ \boldsymbol{n}'_\alpha(s) = -\epsilon_\alpha \eta_\alpha \kappa_\alpha(s) \boldsymbol{t}_\alpha(s) + \tau_\alpha(s) \boldsymbol{b}_\alpha(s); \\ \boldsymbol{b}'_\alpha(s) = \epsilon_\alpha \tau_\alpha(s) \boldsymbol{n}_\alpha(s). \end{cases} \tag{3.1}$$

Where

$$\kappa_\alpha(s) = \epsilon_\alpha \left\langle \mathbf{t}'_\alpha(s), \mathbf{n}_\alpha(s) \right\rangle_L, \ \tau_\alpha(s) = -\epsilon_\alpha \eta_\alpha \left\langle \mathbf{n}'_\alpha(s), \mathbf{b}_\alpha(s) \right\rangle_L \ \text{and} \ \mathbf{b}_\alpha(s) = -\epsilon_\alpha \eta_\alpha \mathbf{t}_\alpha(s) \times_L \mathbf{n}_\alpha(s).$$

Definition 3.6. *(Frenet-Serret apparatus for admissible curves not necessarily having unit speed). Let $\alpha : I \to \mathbb{E}_1^3$ be an admissible curve and s be an arclength function for α. Write $\alpha(t) = \tilde{\alpha}(s(t))$. The tangent, normal and binormal vectors to α em t are defined by $\mathbf{t}_\alpha(t) = \mathbf{t}_{\tilde{\alpha}}(s(t))$, $\mathbf{n}_\alpha(t) = \mathbf{n}_{\tilde{\alpha}}(s(t))$ and $\mathbf{b}_\alpha(t) = \mathbf{b}_{\tilde{\alpha}}(s(t))$. The curvature of the curve $\kappa_\alpha(t) = \kappa_{\tilde{\alpha}}(s(t))$ and torsion $\tau_\alpha(t) = \tau_{\tilde{\alpha}}(s(t))$*

Proposition 3.1. *Let $\alpha : I \to \mathbb{E}_1^3$ be an admissible curve. Given $t \in I$, the formulas hold:*

$$\kappa_\alpha(t) = \frac{\|\alpha'(t) \times_L \alpha''(t)\|_L}{\|\alpha'(t)\|_L^3};$$

$$\tau_\alpha(t) = \frac{det\left(\alpha'(t), \alpha''(t), \alpha'''(t)\right)}{\|\alpha'(t) \times_L \alpha''(t)\|_L^2}.$$

*These expressions represent, respectively, the **curvature** $\kappa_\alpha(t)$ and the **torsion** $\tau_\alpha(t)$ of the curve α in the three-dimensional Minkowski space $\mathbb{E}_1^3$, where $\times_L$ and $\|\cdot\|_L$ denote the Lorentzian cross product and norm, respectively.*

Definition 3.7. *[18] A unit speed curve $\alpha : I \to \mathbb{E}_1^3$ is called semi-lightlike if span $\{\alpha'(s), \alpha''(s)\}$ is degenerate (and, thus,$\alpha''(s)$ is lightlike) for all $s \in I$*

- *In view of this definition, we'll allow the indicator ϵ_α and the coindicator η_α to be zero. This way, if α is lightlike, we have that $(\epsilon_\alpha, \eta_\alpha) = (0, 1)$, while if a is semi-lightlike, we have $(\epsilon_\alpha, \eta_\alpha) = (1, 0)$. This is done to treat both cases simultaneously*
- *Since the arclenght parameter is denoted by s and arc-photon parameter by ϕ. we will allow ourselves to simply omit the parameter when discussing results for both cases.*

Theorem 2. *[18] (Arc-photon) $\alpha : I \to \mathbb{E}_1^3$ be a lightlike parametrized curve such that $\|\alpha''(t)\| \neq 0$ for all $t \in I$. Then α admits an arc-photon reparametrization, that is, there exists an open interval J and a diffeomorphism $h : J \to I$ such that $\beta = \alpha(h)$ satisfies $\|\alpha''(\phi)\|_L = 1$ for all $\phi \in J$.*

Let $\beta \colon J \subset \mathbb{R} \to \mathbb{R}_1^3$ be a regular curve parametrized by parameter ϕ, with the same trace of the null curve $\alpha(s)$, i.e.

$\beta(\phi) = \alpha(h(\phi))$, $h(\phi) = \int_{\phi_0}^{\phi} \|\alpha''(u)\|_L^{-\frac{1}{2}} du$ and $\dfrac{dh}{d\phi} = \|\alpha''(\phi)\|_L^{-\frac{1}{2}}$. Thus,

Notice that, if $\frac{dh}{d\phi} = 0$ then the curve is a straight-line.

Definition 3.8. *[18] Let $\alpha : I \to \mathbb{E}_1^3$ be lightlike or semi-lightlike. We define the tangent and the normal to the curve by*

$$\mathbf{t}_\alpha \doteq \alpha' \ \text{and} \ \mathbf{n}_\alpha \doteq \alpha''.$$

Proposition 3.2. *[18] Let $\alpha : I \to \mathbb{E}_1^3$ be semi-lightlike, then $\langle \boldsymbol{t}_\alpha, \boldsymbol{t}_\alpha \rangle_L = 1$, $\langle \boldsymbol{n}_\alpha, \boldsymbol{n}_\alpha \rangle_L = 0$, if defined*

$$\boldsymbol{b}_\alpha = \boldsymbol{t}_\alpha \times_E \boldsymbol{n}_\alpha \ (cross\ product\ Euclidean)$$

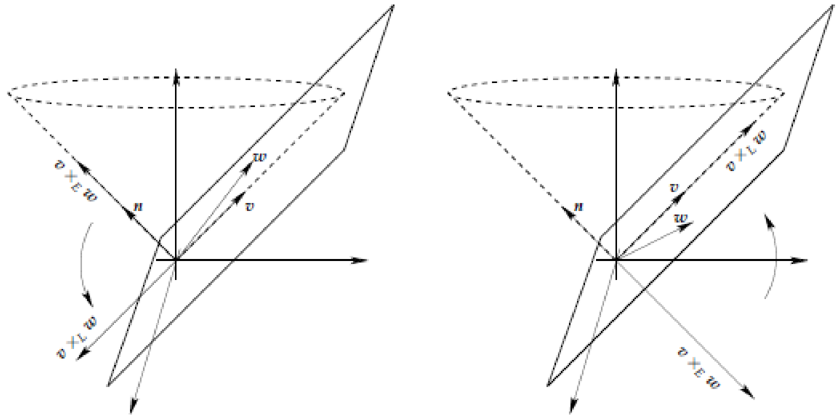
and $\langle \boldsymbol{n}_\alpha, \boldsymbol{b}_\alpha \rangle_L = -1$, and $\langle \boldsymbol{b}_\alpha, \boldsymbol{t}_\alpha \rangle_L = 0$. The triple $\{\mathbf{t}_\alpha, \mathbf{n}_\alpha, \mathbf{b}_\alpha\}$ is a positive basis for $\mathbb{E}_1^3$. Consequently $\langle \boldsymbol{b}_\alpha, \boldsymbol{b}_\alpha \rangle_L = \langle \boldsymbol{n}, \boldsymbol{t} \rangle_L = 0$.

Proposition 3.3. *[18] If $\alpha : I \to \mathbb{E}_1^3$ be lightlike, then $\langle \boldsymbol{t}_\alpha, \boldsymbol{t}_\alpha \rangle_L = 0$ and $\langle \boldsymbol{n}_\alpha, \boldsymbol{n}_\alpha \rangle_L = 1$, if defined*

$$\boldsymbol{b}_\alpha = \boldsymbol{t}_\alpha \times_E \boldsymbol{n}_\alpha \ (cross\ product\ Euclidean)$$

and $\langle \boldsymbol{n}_\alpha, \boldsymbol{b}_\alpha \rangle_L = 0$, and $\langle \boldsymbol{b}_\alpha, \boldsymbol{t}_\alpha \rangle_L = -1$. The triple $\{\mathbf{t}_\alpha, \mathbf{n}_\alpha, \mathbf{b}_\alpha\}$ is a positive basis for $\mathbb{E}_1^3$. Consequently $\langle \boldsymbol{b}_\alpha, \boldsymbol{b}_\alpha \rangle_L = \langle \boldsymbol{t}_\alpha, \boldsymbol{n}_\alpha \rangle_L = 0$.

Orientations for a lightlike plane [18].



(a) (v, w) positive $(\lambda < 0)$ (b) (v, w) negative $(\lambda > 0)$

Figure 3.1: Orientations for a lightlike plane.

Definition 3.9. *[18] Let $\alpha : I \to \mathbb{E}^3_1$ be lightlike or semi-lightlike. The pseudo-torsion of α is given by*

$$\tau_\alpha \doteq -\langle \mathbf{n}'_\alpha, \mathbf{b}_\alpha \rangle_L .$$

Remark 3.2. *In the literature, this pseudo-torsion is also called the Cartan curvature of α.*

Theorem 3. *[18] (Frenet-Serret Equations -lightlike or semi-lightlike curves) Let $\alpha : I \to \mathbb{E}^3_1$. Then we have*

$$\begin{cases} \boldsymbol{t}'_\alpha = \mathbf{n}; \\ \boldsymbol{n}'_\alpha = \eta_\alpha\, \tau_\alpha\, \mathbf{t}_\alpha + \epsilon_\alpha\, \tau_\alpha\, \mathbf{n}_\alpha; \\ \boldsymbol{b}'_\alpha = \epsilon_\alpha \mathbf{t}_\alpha + \eta_\alpha\, \tau_\alpha\, \mathbf{n}_\alpha - \epsilon_\alpha\, \tau_\alpha\, \mathbf{b}_\alpha. \end{cases} \tag{3.2}$$

3.2. Surfaces in E^3_1.

Definition 3.10. *[18] [Regular Parametrized Surface] A smooth map $S : U \subset \mathbb{R}^2 \longrightarrow \mathbb{E}^3_1$ is called a regular parametrized surface if $D_S(u, v)$ has full rank for all $(u, v) \in U$.*

3.2.1. Normal vector of parametric Spacelike, Timelike and Ligthlike Surfaces in $\mathbb{E}^3_1$.

*The **unit normal** vector field $\mathbf{N}$ of a parametric spacelike (or a timelike) surface $\mathbf{M}$*

. The **unit normal** vector is given by

$$\mathbf{N} = \frac{\mathbf{S}_u \times_L \mathbf{S}_v}{\|\mathbf{S}_u \times_L \mathbf{S}_v\|_L}. \tag{3.3}$$

*The **unit normal** vector field $\mathbf{N}$ of a parametric lightlike surface $\mathbf{M}$*

. The **unit normal** vector is given by

$$\mathbf{N} = \mathbf{S}_u \times_L \mathbf{S}_v. \tag{3.4}$$

Remark 3.3. *Since $\boldsymbol{S}_u \times_L \boldsymbol{S}_v$ is lightlike, it follows from Proposition (2.3) that the vectors $\boldsymbol{S}_u$ and $\boldsymbol{S}_v$ can be either lightlike or spacelike. If $\langle \boldsymbol{S}_u, \boldsymbol{S}_v \rangle_L = 0$, then one element of the sett $\{\boldsymbol{S}_u, \boldsymbol{S}_v\}$ is lightlike and the other is spacelike. If both elements of the set $\{\boldsymbol{S}_u, \boldsymbol{S}_v\}$ are spacelike, then $\langle \boldsymbol{S}_u, \boldsymbol{S}_u \rangle_L \langle \boldsymbol{S}_v, \boldsymbol{S}_v \rangle_L = \left(\langle \boldsymbol{S}_u, \boldsymbol{S}_v \rangle_L\right)^2$. Both vectors cannot be lightlike, as we would have $\boldsymbol{S}_u \times_L \boldsymbol{S}_v$ being spacelike.*

3.3. Curves and Surface in E^3_1.

3.3.1. Curves in Parametric Surface in E^3_1. Consider an parametric surface represented by $S : U \subset \mathbb{R}^2 \longrightarrow V \cap S \subset \mathbb{E}^3_1$ and let $\alpha(s)$ a curve in the surface defined by $\alpha(s) = \mathbf{S}(u(s), v(s))$. The $\alpha'(s)$, $\alpha''(s)$ and $\alpha'''(s)$ is

$$\alpha'(s) = \mathbf{S}_u u' + \mathbf{S}_v v', \tag{3.5}$$

$$\alpha''(s) = \mathbf{S}_u u'' + \mathbf{S}_v v'' + \mathbf{S}_{uu}(u')^2 + 2\mathbf{S}_{uv}u'v' + \mathbf{S}_{vv}(v')^2. \tag{3.6}$$

Therefore, the projection of the vectors $\alpha'(s)$, $\alpha''(s)$ and $\alpha'''(s)$ onto the unit normal vector field ($\mathbf{N}$) of the surface $S(u,v)$ are given respectively by

$$\langle \boldsymbol{\alpha}', \boldsymbol{N} \rangle_L = 0, \tag{3.7}$$

$$\langle \boldsymbol{\alpha}'', \boldsymbol{N} \rangle_L = \langle \mathbf{S}_{uu}, \boldsymbol{N} \rangle_L (u')^2 + 2 \langle \mathbf{S}_{uv}, \boldsymbol{N} \rangle_L u'v' + \langle \mathbf{S}_{vv}, \boldsymbol{N} \rangle_L (v')^2. \tag{3.8}$$

4. Euler-Rodrigues formula in Minkowski 3-space ($\mathbb{E}_1^3$). In [5, 4], the Lorentz metric (inner product) of signature $(-,+,+)$, the semi-skew-symmetric matrix W obtained from the vector $\mathbf{w} = (w_x, w_y, w_z)$ **axis with unit length** be

$$W = \begin{bmatrix} 0 & w_z & -w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix}.$$

In this work, the Lorentz metric (inner product) of signature $(+,+,-)$, then the semi-skew-symmetric matrix W obtained from the vector $\mathbf{w} = (w_x, w_y, w_z)$ **axis with unit length** be

$$W = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ w_y & -w_x & 0 \end{bmatrix}.$$

Case 4. *[5, 4] Assume that $\mathbf{w} = (w_x, w_y, w_z)$ is unit spacelike vector, then we get*

$$\mathcal{R}_{sp}(\theta, \mathbf{w}) = I_3 + sinh(\theta)W + (-1 + cosh(\theta))W^2, \tag{4.1}$$

where in this work, the Lorentz metric (inner product) of signature $(+,+,-)$, then we have

$$\mathcal{R}_{sp} = \begin{bmatrix} 1 + (-1 + cosh(\theta)(w_y^2 - w_z^2) & -sinh(\theta)w_z - (-1 + cosh(\theta))w_x w_y & sinh(\theta)w_y + (-1 + cosh(\theta))w_x w_z \\ sinh(\theta)w_z - (-1 + cosh(\theta))w_x w_y & 1 + (-1 + cosh(\theta))(w_x^2 - w_z^2) & -sinh(\theta)w_x + (-1 + cosh(\theta))w_y w_z \\ sinh(\theta)w_y - (-1 + cosh(\theta))w_x w_z & -sinh(\theta)w_x - (-1 + cosh(\theta))w_y w_z & 1 + (-1 + cosh(\theta))(w_x^2 + w_y^2) \end{bmatrix}.$$

Case 5. *[5, 4] Assume that $\mathbf{w} = (w_x, w_y, w_z)$ is unit timelike axis, then we get*

$$\mathcal{R}_{tm}(\theta, \mathbf{w}) = I_3 + sin(\theta)W + (1 - cos(\theta))W^2, \tag{4.2}$$

where in this work, the Lorentz metric (inner product) of signature $(+,+,-)$, then we have

$$\mathcal{R}_{tm} = \begin{bmatrix} 1 + (1 - cos(\theta)(w_y^2 - w_z^2) & -sin(\theta)w_z - (1 - cos(\theta))w_x w_y & sin(\theta)w_y + (1 - cos(\theta))w_x w_z \\ sin(\theta)w_z - (1 - cos(\theta))w_x w_y & 1 + (1 - cos(\theta))(w_x^2 - w_z^2) & -sin(\theta)w_x + (1 - cos(\theta))w_y w_z \\ sin(\theta)w_y - (1 - cos(\theta))w_x w_z & -sin(\theta)w_x - (1 - cos(\theta))w_y w_z & 1 + (1 - cos(\theta))(w_x^2 + w_y^2) \end{bmatrix}.$$

Case 6. *[5] If $\mathbf{w} = (w_x, w_y, w_z)$ is lightlike axis, then we get*

$$\mathcal{R}_{lg}(\theta, \mathbf{w}) = I_3 + \theta\, W + \frac{\theta^2}{2}\, W^2, \tag{4.3}$$

where in this work, the Lorentz metric (inner product) of signature $(+,+,-)$, then we have

$$\mathcal{R}_{lg} = \begin{bmatrix} 1+(\frac{\theta^2}{2})(w_y^2-w_z^2) & -\theta w_z-(\frac{\theta^2}{2})w_xw_y & \theta w_y+(\frac{\theta^2}{2})w_xw_z \\ \theta w_z-(\frac{\theta^2}{2})w_xw_y & 1+(\frac{\theta^2}{2})(w_x^2-w_z^2) & -\theta w_x+(\frac{\theta^2}{2})w_yw_z \\ \theta w_y-(\frac{\theta^2}{2})w_xw_z & -\theta w_x-(\frac{\theta^2}{2})w_yw_z & 1+(\frac{\theta^2}{2})(w_x^2+w_y^2) \end{bmatrix}.$$

Where

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4.1. Operator $\mathcal{D}_L$. In the paper [?], the authors defined the operator $\mathcal{D}(\mathbf{w}) = \mu \times_E \mathbf{w}$, where $\times_E$ cross product Euclidean space. In this subsection, we defined $\mathcal{D}_L$. Let $\mathbf{w}$ be a nonzero vector in $\mathbb{E}^3_1$. We define $\mathcal{D}_L$ as

$$\mathcal{D}_L(\mathbf{w}) = \mu \times_L \mathbf{w}. \tag{4.4}$$

Where $\times_L$ is cross product Lorentzian spaces.

Proposition 4.1. *If the vector μ is chosen arbitrary such that it is linearly independent with $\mathbf{w}$, then $\mathcal{D}_L(\mathbf{w})$ yields never a zero vector and also we can see that*

i)

$$\langle \mathcal{D}_L(\mathbf{w}), \mathcal{D}_L(\mathbf{w})\rangle_L = \langle \mu \times_L \mathbf{w}, \mu \times_L \mathbf{w}\rangle_L = -det \begin{vmatrix} \langle \mu,\mu\rangle_L & \langle \mu,\mathbf{w}\rangle_L \\ \langle \mathbf{w},\mu\rangle_L & \langle \mathbf{w},\mathbf{w}\rangle_L \end{vmatrix} = \lambda^2 - \varepsilon_u\varepsilon_w,$$

where $\langle \mu, \mathbf{w}\rangle_L = \lambda$. $\varepsilon_u = \langle \mathbf{u},\mathbf{u}\rangle_L$ and $\varepsilon_w = \langle \mathbf{w},\mathbf{w}\rangle_L$.

ii) If μ and $\mathbf{w}$ are spacelike vectors, we have $\mathcal{D}_L(\mathbf{w})$ is spacelike or $\mathcal{D}_L(\mathbf{w})$ is timelike or $\mathcal{D}_L(\mathbf{w})$ is lightlike ;

iii) If μ and $\mathbf{w}$ are timelike vectors, we have $\mathcal{D}_L(\mathbf{w})$ is spacelike;

iv) If μ and $\mathbf{w}$ are lightlike (linearly independent), them $\mathcal{D}_L(w)$ is spacelike vector;

v) If μ is timelike(spacelike) and $\mathbf{w}$ is spacelike (timelike), them $\mathcal{D}_L(w)$ is spacelike vector;

vi) If μ is lightlike (spacelike) and $\mathbf{w}$ is spacelike (lightlike) with $\langle \mu, \mathbf{w}\rangle_L = 0$ them $\mathcal{D}_L(w)$ is lightlike vector;

vii) If μ is lightlike (spacelike) and $\mathbf{w}$ is spacelike (lightlike) with $\langle \mu, \mathbf{w}\rangle_L \neq 0$ them $\mathcal{D}_L(w)$ is spacelike vector;

viii) If μ is lightlike (timelike) and $\mathbf{w}$ is timelike (lightlike), them $\mathcal{D}_L(w)$ spacelike vector, $\lambda \neq 0$.

The operator $\mathcal{D}_L$ will play the main role together with the **Euler-Rodrigues formula in Minkowski 3-space** for finding the tangent vector at the tangential intersection point of the intersection curve of two surfaces.

Remark 4.1. *The rotation $\mathcal{R}_{tm}(\theta,\mathbf{w}) = I_3 + sin(\theta)W + (1-cos(\theta))W^2$ of vector $\frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}$ still keeps it unitary, but the rotation $\mathcal{R}_{sp}(\theta,\mathbf{w}) = I_3 + sinh(\theta)W + (1-cosh(\theta))W^2$ does not maintain its unit length. See the figure below.*

Remark 4.2. *The $\mathcal{R}_{sp}(\theta,\mathbf{N})\dfrac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}$, transform the timelike vectors to timelike vectors, the spacelike vectors to spacelike vectors and the lightlike vectors to lightlike vectors.*

Theorem 7. *The $\mathcal{R}_{lg}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N})$ transform the lightlike vectors to lightlike vectors.*

Proof: Since the axis of the rotation is lightlike, the matrix of rotation is

$$\mathcal{R}_{lg}(\theta,\mathbf{N}) = \begin{bmatrix} 1+(\frac{\theta^2}{2})(w_y^2-w_z^2) & -\theta w_z-(\frac{\theta^2}{2})w_xw_y & \theta w_y+(\frac{\theta^2}{2})w_xw_z \\ \theta w_z-(\frac{\theta^2}{2})w_xw_y & 1+(\frac{\theta^2}{2})(w_x^2-w_z^2) & -\theta w_x+(\frac{\theta^2}{2})w_yw_z \\ \theta w_y-(\frac{\theta^2}{2})w_xw_z & -\theta w_x-(\frac{\theta^2}{2})w_yw_z & 1+(\frac{\theta^2}{2})(w_x^2+w_y^2) \end{bmatrix}.$$

Let $\mathcal{D}_L(\mathbf{N}) = [a,b,c]$ lightlike vector, the multiplication $\mathcal{R}_{lg}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N})$ is

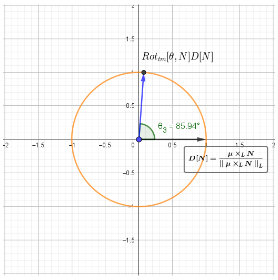


Figure 4.1: $\mathcal{R}_{tm}(\theta,\mathbf{N})\frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}$.

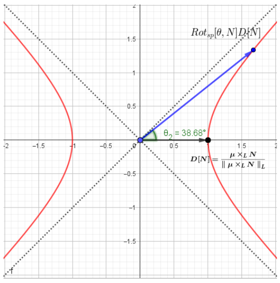


Figure 4.2: $\mathcal{R}_{sp}(\theta,\mathbf{N})\frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}$.

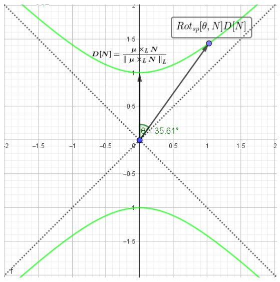


Figure 4.3: $\mathcal{R}_{sp}(\theta,\mathbf{N})\frac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}$.

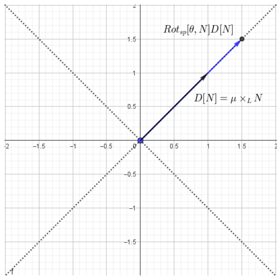


Figure 4.4: $\mathcal{R}_{sp}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N})$.

$$\mathcal{R}_{lg}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N})=\left[\begin{array}{c} a\left(\frac{\theta^2(w_2^2-w_3^2)}{2}+1\right)+b\left(-\frac{\theta^2w_1w_2}{2}-\theta w_3\right)+c\left(\frac{\theta^2w_1w_3}{2}+\theta w_2\right)\\ a\left(-\frac{\theta^2w_1w_2}{2}+\theta w_3\right)+b\left(\frac{\theta^2(w_1^2-w_3^2)}{2}+1\right)+c\left(\frac{\theta^2w_2w_3}{2}-\theta w_1\right)\\ a\left(-\frac{\theta^2w_1w_3}{2}+\theta w_2\right)+b\left(-\frac{\theta^2w_2w_3}{2}-\theta w_1\right)+c\left(\frac{\theta^2(w_1^2+w_2^2)}{2}+1\right)\end{array}\right].$$

The norma is

$$\begin{aligned}\|\mathcal{R}_{lg}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N})\|_L= &\left[a\left(\frac{\theta^2(w_2^2-w_3^2)}{2}+1\right)+b\left(-\frac{\theta^2w_1w_2}{2}-\theta w_3\right)+c\left(\frac{\theta^2w_1w_3}{2}+\theta w_2\right)\right]^2\\ + &\left[a\left(-\frac{\theta^2w_1w_2}{2}+\theta w_3\right)+b\left(\frac{\theta^2(w_1^2-w_3^2)}{2}+1\right)+c\left(\frac{\theta^2w_2w_3}{2}-\theta w_1\right)\right]^2\\ - &\left[a\left(-\frac{\theta^2w_1w_3}{2}+\theta w_2\right)+b\left(-\frac{\theta^2w_2w_3}{2}-\theta w_1\right)+c\left(\frac{\theta^2(w_1^2+w_2^2)}{2}+1\right)\right]^2.\end{aligned}\tag{4.5}$$

Since $w=(w1,w2,w3)$ and $\mathcal{D}_L(\mathbf{N})=[a,b,c]$ are lightlike, we have $w_3^2=w_1^2+w_2^2$ and $a^2=b^2+c^2$. Replacing $w_3^2=w_1^2+w_2^2$ and $a^2=b^2+c^2$ in the equation above, we have the equation below

$$\begin{aligned}\|\mathcal{R}_{lg}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N})\|_L=&\frac{a^2\theta^4w_1^2w_2^2}{4}-\frac{a^2\theta^4w_1^2w_3^2}{4}+\frac{a^2\theta^4w_2^4}{4}-\frac{a^2\theta^4w_2^2w_3^2}{2}+\frac{a^2\theta^4w_3^4}{4}+a^2\\ &-\frac{ab\theta^4w_1^3w_2}{2}-\frac{ab\theta^4w_1w_2^3}{2}+\frac{ab\theta^4w_1w_2w_3^2}{2}+\frac{ac\theta^4w_1^3w_3}{2}+\frac{ac\theta^4w_1w_2^2w_3}{2}\\ &-\frac{ac\theta^4w_1w_3^3}{2}+\frac{b^2\theta^4w_1^4}{4}+\frac{b^2\theta^4w_1^2w_2^2}{4}-\frac{b^2\theta^4w_1^2w_3^2}{2}-\frac{b^2\theta^4w_2^2w_3^2}{4}+\frac{b^2\theta^4w_3^4}{4}+b^2\\ &+\frac{bc\theta^4w_1^2w_2w_3}{2}+\frac{bc\theta^4w_2^3w_3}{2}-\frac{bc\theta^4w_2w_3^3}{2}-\frac{c^2\theta^4w_1^4}{4}-\frac{c^2\theta^4w_1^2w_2^2}{2}+\frac{c^2\theta^4w_1^2w_3^2}{4}\\ &-\frac{c^2\theta^4w_2^4}{4}+\frac{c^2\theta^4w_2^2w_3^2}{4}-c^2,\end{aligned}\tag{4.6}$$

simplifying

$$\|\mathcal{R}_{lg}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N})\|_L=\frac{c\theta^4w_3}{2}\left(aw_1^3+aw_1w_2^2-aw_1w_3^2+bw_1^2w_2+bw_2^3-bw_2w_3^2\right)$$

simplifying

$$\|\mathcal{R}_{lg}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N})\|_L=\frac{c\theta^4w_3}{2}\left(aw_1(w_1^2+w_2^2)-aw_1w_3^2+bw_2(w_1^2+w_2^2)-bw_2w_3^2\right)$$

and replacing again $w_3^2=w_1^2+w_2^2$, we have

$$\|\mathcal{R}_{lg}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N})\|_L=0.$$

$\square$

5. Differential Geometry of Tangential Intersection Curve of Two Surfaces in $\mathbb{E}_1^3$. Now, let us assume the two surfaces S^A and S^B intersect tangentially at a point $P_0 = S^A(u_0, v_0) = S^B(p_0, q_0)$ on the intersection curve $\alpha(s)$, i.e., $N^A(u_0, v_0) \parallel N^B(p_0, q_0)$ at P_0. If the surfaces are spacelike or timelike, by orienting the surfaces properly we can assume that $N^A(u_0, v_0) = N^B(p_0, q_0) = N$. If the surfaces are lightlike, $N^A(u_0, v_0) = \lambda N^B(p_0, q_0)$, for some real $\lambda \in \mathcal{R}$.

5.1. First method.

5.1.1. Parametric-parametric surfaces Spacelike or Timelike

. Let S^A and S^B be two regular surfaces spacelike or timelike given by the parametric equations $S^A(u, v) = (x(u, v), y(u, v), z(u, v))$ and $S^B(p, q) = (x(p, q), y(p, q), z(p, q))$. The vector $\alpha'(s_0)$ of the tangential intersection curve $\alpha(s) = S^A(u(s), v(s)) = S^B(p(s), q(s))$, i.e.

$$\mathbf{t} = S_u^A u' + S_v^A v' = S_p^B p' + S_q^B q' \tag{5.1}$$

and the projection of the vector $\alpha''(s_0)$ onto $\mathbf{N}^A(u_0, v_0)$ and $\mathbf{N}^B(p_0, q_0)$ in the point $P_0 = S^A(u_0, v_0) = S^B(p_0, q_0)$ where surfaces S^A and S^B intersect tangentially

$$\left\langle \mathbf{N}^A(u_0, v_0), \alpha''(s_0) \right\rangle_L = \left\langle \mathbf{N}^B(p_0, q_0), \alpha''(s_0) \right\rangle_L , \tag{5.2}$$

produces the equation in terms of the coefficients of the second fundamental form it becomes

$$e^A(u')^2 + 2f^A u'v' + g^A(v')^2 = e^B(p')^2 + 2f^B p'q' + g^B(q')^2 \ . \tag{5.3}$$

Since the equation (5.1) consists of four variables u', v', p' *and* q', we can write p' *and* q' in terms of u' and v'.

$$\begin{aligned} p' &= a_{11}u' + a_{12}v', \\ q' &= a_{21}u' + a_{22}v', \end{aligned} \tag{5.4}$$

where

$$a_{11} = \frac{\left\langle S_u^A \times_L S_q^B, N \right\rangle_L}{\left\langle S_p^B \times_L S_q^B, N \right\rangle_L}; \quad a_{12} = \frac{\left\langle S_v^A \times_L S_q^B, N \right\rangle_L}{\left\langle S_p^B \times_L S_q^B, N \right\rangle_L}, \tag{5.5}$$

$$a_{21} = \frac{\left\langle S_u^A \times_L S_p^B, N \right\rangle_L}{\left\langle S_q^B \times_L S_p^B, N \right\rangle_L}; \quad a_{22} = \frac{\left\langle S_v^A \times_L S_p^B, N \right\rangle_L}{\left\langle S_q^B \times_L S_p^B, N \right\rangle_L}. \tag{5.6}$$

Substituting (5.4) into (5.3), we have

$$b_{11}(u')^2 + 2b_{12}u'v' + b_{22}(v')^2 = 0, \tag{5.7}$$

where

$$\begin{aligned} b_{11} &= a_{11}^2 e^B + 2a_{11}a_{12}f^B + a_{21}^2 g^B - e^A; \\ b_{12} &= a_{11}a_{12}e^B + 2(a_{11}a_{22} + a_{21}a_{12})f^B + a_{21}a_{22}g^B - f^A; \\ b_{22} &= a_{12}^2 e^B + 2a_{12}a_{22}f^B + a_{22}^2 g^B - g^A. \end{aligned}$$

If $\omega = \dfrac{u'}{v'}$ when $d_{11} \neq 0$ or $\mu = \dfrac{v'}{u'}$ when $d_{22} \neq 0$, we have

$$b_{11}\omega^2 + 2b_{12}\omega + b_{22} = 0, \tag{5.8}$$

$$\mathbf{t} = \frac{\omega S_{u_A}^A + S_{v_A}^A}{\|\omega S_{u_A}^A + S_{v_A}^A\|},$$

or

$$b_{22}\mu^2 + 2b_{12}\mu + b_{11} = 0, \tag{5.9}$$

$$\mathbf{t} = \frac{S^A_{u_A} + \mu S^A_{v_A}}{\|S^A_{u_A} + \mu S^A_{v_A}\|}.$$

5.1.2. Parametric-parametric surfaces Lightlike. Let S^A and S^B be two regular surfaces lightlike given by the parametric equations $S^A(u,v) = (x(u,v), y(u,v), z(u,v))$ and $S^B(p,q) = (x(p,q), y(p,q), z(p,q))$. The vector $\alpha'(s_0)$ of the tangential intersection curve $\alpha(s) = S^A(u(s), v(s)) = S^B(p(s), q(s))$, i.e.

$$\mathbf{t} = S^A_u u' + S^A_v v' = S^B_p p' + S^B_q q' \tag{5.10}$$

and the projection of the vector $\alpha''(s_0)$ onto $\mathbf{N}^A(u_0, v_0)$ and $\mathbf{N}^B(p_0, q_0)$ in the point $P_0 = S^A(u_0, v_0) = S^B(p_0, q_0)$ where surfaces S^A and S^B intersect tangentially

$$\left\langle \mathbf{N}^A(u_0, v_0), \alpha''(s_0) \right\rangle_L = \left\langle \mathbf{N}^B(p_0, q_0), \alpha''(s_0) \right\rangle_L, \tag{5.11}$$

produces the equation in terms of the coefficients it becomes

$$\bar{e}^A(u')^2 + 2\bar{f}^A u'v' + \bar{g}^A(v')^2 = \bar{e}^B(p')^2 + 2\bar{f}^B p'q' + \bar{g}^B(q')^2 \ , \tag{5.12}$$

$$\bar{e}^i = \left\langle \mathbf{N}^i, S^i_{uu} \right\rangle_L ; \bar{f}^i = \left\langle \mathbf{N}^i, S^i_{uv} \right\rangle_L ; \bar{g}^i = \left\langle \mathbf{N}^i, S^i_{vv} \right\rangle_L, \ i \in \{A, B\}.$$

Remark 5.1. *Since* $\mathbf{N}(u_0, v_0) = \boldsymbol{S}^A_v(u_0, v_0) \times_L \boldsymbol{S}^A_v(u_0, v_0) = \lambda \boldsymbol{S}^B_p(p_0, q_0) \times_L \boldsymbol{S}^B_q(p_0, q_0)$ *is lightlike, we have* $\left\langle S^A_u(u_0, v_0) \times_L S^A_v(u_0, v_0), \mathbf{N} \right\rangle_L = 0$ *and* $\left\langle S^B_p(p_0, q_0) \times_L S^B_q(p_0, q_0), \mathbf{N} \right\rangle_L = 0$*, but* $\left\langle S^B_p(p_0, q_0) \times_E S^B_q(p_0, q_0), \mathbf{N} \right\rangle_L \neq 0$.

In fact, let $S^B_p(p_0, q_0) \times_L S^B_q(p_0, q_0) = (a, b, c)$*, then* $S^B_p(p_0, q_0) \times_E S^B_q(p_0, q_0) = (a, b, -c)$*, therefore* $S^B_p(p_0, q_0) \times_E S^B_q(p_0, q_0)$ *is lightlike,*
if the product $\left\langle S^B_p(p_0, q_0) \times_E S^B_q(p_0, q_0), S^B_p(p_0, q_0) \times_L S^B_q(p_0, q_0) \right\rangle_L = 0$ *by Proposition (2.4) we have* $S^B_p(p_0, q_0) \times_L S^B_q(p_0, q_0) \parallel \left(S^B_p(p_0, q_0) \times_E S^B_q(p_0, q_0) \right)$. *Which is absurd.*

Since the equation (5.10) consists of four variables u', v', p' *and* q', p' *and* q' it can be written in terms of u' and v'.

$$\begin{aligned} p' &= a_{11}u' + a_{12}v', \\ q' &= a_{21}u' + a_{22}v', \end{aligned} \tag{5.13}$$

where

$$a_{11} = \frac{\left\langle S^A_u \times_E S^B_q, \mathbf{N} \right\rangle_L}{\left\langle S^B_p \times_E S^B_q, \mathbf{N} \right\rangle_L}; \quad a_{12} = \frac{\left\langle S^A_v \times_E S^B_q, \mathbf{N} \right\rangle_L}{\left\langle S^B_p \times_E S^B_q, \mathbf{N} \right\rangle_L}; \tag{5.14}$$

$$a_{21} = \frac{\left\langle S^A_u \times_E S^B_p, \mathbf{N} \right\rangle_L}{\left\langle S^B_q \times_E S^B_p, \mathbf{N} \right\rangle_L}; \quad a_{22} = \frac{\left\langle S^A_v \times_E S^B_p, \mathbf{N} \right\rangle_L}{\left\langle S^B_q \times_E S^B_p, \mathbf{N} \right\rangle_L}. \tag{5.15}$$

Substituting (5.4) into (5.12), we have

$$b_{11}(u')^2 + 2b_{12}u'v' + b_{22}(v')^2 = 0, \tag{5.16}$$

where

$$\begin{aligned} b_{11} &= a_{11}^2 \bar{e}^B + 2a_{11}a_{12}\bar{f}^B + a_{21}^2 \bar{g}^B - \bar{e}^A; \\ b_{12} &= a_{11}a_{12}\bar{e}^B + 2(a_{11}a_{22} + a_{21}a_{12})\bar{f}^B + a_{21}a_{22}\bar{g}^B - \bar{f}^A; \\ b_{22} &= a_{12}^2 \bar{e}^B + 2a_{12}a_{22}\bar{f}^B + a_{22}^2 \bar{g}^B - \bar{g}^A. \end{aligned}$$

If $\omega = \dfrac{u'}{v'}$ when $d_{11} \neq 0$ or $\mu = \dfrac{v'}{u'}$ when $d_{22} \neq 0$, we have

$$b_{11}\omega^2 + 2b_{12}\omega + b_{22} = 0. \tag{5.17}$$

$$\mathbf{t} = \frac{\omega S^A_{u_A} + S^A_{v_A}}{\|\omega S^A_{u_A} + S^A_{v_A}\|},\ if\ \mathbf{t}\ is\ spacelike.$$

$$\mathbf{t} = \omega S^A_{u_A} + S^A_{v_A},\ if\ \mathbf{t}\ is\ lightlike.$$

or

$$b_{22}\mu^2 + 2b_{12}\mu + b_{11} = 0. \tag{5.18}$$

$$\mathbf{t} = \frac{S^A_{u_A} + \mu S^A_{v_A}}{\|S^A_{u_A} + \mu S^A_{v_A}\|},\ if\ \mathbf{t}\ is\ spacelike.$$

$$\mathbf{t} = S^A_{u_A} + \mu S^A_{v_A},\ if\ \mathbf{t}\ is\ lightlike.$$

5.1.3. Solution of the Equations. There are four distinct cases to the solution of the equations (5.7, 5.16) depending upon the discriminant $\Delta = b_{12}^2 - 4b_{11}b_{22}$.

1. Isolated tangential contact point: If $\Delta < 0$ then the equations does not have any solution. Thus, P is an isolated contact point of S^A and S^B.
2. Tangential intersection curve: If $\Delta = 0$ and $b_{11}^2 + b_{12}^2 + b_{22}^2 \neq 0$ then the equations has double roots and $\mathbf{t}$ is unique. Thus, S^A and S^B intersect at P and at its neighborhood.
3. Branch point: If $\Delta > 0$ then the equations has distinct roots. Thus, P is a branch point of the intersection curve $\alpha(s)$: there is another intersection branch crossing $\alpha(s)$ at P.
4. Higher-order contact point: Is $b_{11} = b_{12} = b_{22} = 0$ then the equations are vanishes for any values of u' and v'. Thus, S^A and S^B has contact of at least second order at P.

5.2. Second method. This method will be called the Euler-Rodrigues formula in Minkowski 3-space ($\mathbb{E}^3_1$).

The rotation with rotation angle θ around the axis in the direction of $\mathbf{N}$ is $\mathcal{R}_{type}(\theta, \mathbf{N})$, where $type = \{sp, tm, lg\}$, depending on whether $\mathbf{N}$ is spacelike or timelike or lightlike, respectively.

By the definition of $\mathcal{D}$, the vector $\mathcal{D}(\mathbf{N})$ lies in the common tangent plane of the surfaces S^A and S^B. Thus, after a suitable rotation with rotation angle θ around the axis in the direction of $\mathbf{N}$, the vector of $\mathcal{D}(\mathbf{N})$ is multiple of the unit tangent vector $\mathbf{t}$ of the intersection curve at P ($0 \leq \theta < \pi$).

The normal vector of the surface $\mathbf{N}$ can be spacelike or timelike or lightlike. The rotation $\mathcal{R}_{sp}(\theta, \mathbf{N})\dfrac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}$ around axis $\mathbf{N}$ may not preserve the length of vector $\dfrac{\mathcal{D}_L(\mathbf{N})}{\|\mathcal{D}_L(\mathbf{N})\|_L}$ and furthermore the vector $\mathcal{D}_L(\mathbf{N})$ cannot be unitary, as it can be lightlike. Thus, the rotation will be $\mathcal{R}_{type}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N})$, and the vector $\mathcal{D}_L(\mathbf{N})$ can not be unitary. Therefore instead of $\alpha'(s) = \lambda \mathcal{R}_{type}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N})$ we will use $\alpha'(t) = \mathcal{R}_{type}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N})$.

5.2.1. Parametric-parametric surfaces Spacelike or Timelike. The vector tangent is

$$\alpha'(t) \quad = \quad S^A_u u'(t) + S^A_v v'(t) = S^B_p p'(t) + S^B_q q'(t) = \mathcal{R}_{type}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}). \tag{5.19}$$

The projection of the vector α'' onto $\mathbf{N}^A$ and $\mathbf{N}^B$ produces the equation.

$$\begin{gathered} \left\langle \mathbf{N}^A, \alpha'' \right\rangle_L = \left\langle \mathbf{N}^B, \alpha'' \right\rangle_L, \\ e^A(u')^2 + 2f^A u'v' + g^A(v')^2 = e^B(p')^2 + 2f^B p'q' + g^B(q')^2. \end{gathered} \tag{5.20}$$

The u', v' and p', q' values can be found in terms of the rotation angle θ.

$$\begin{aligned} u'(\theta) &= \frac{\left\langle \mathcal{R}_{type}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N}) \times_L S_v^A, \mathbf{N}\right\rangle_L}{\langle S_u^A \times_L S_v^A, \mathbf{N}\rangle_L}, \\ v'(\theta) &= \frac{\left\langle \mathcal{R}_{type}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N}) \times_L S_u^A, \mathbf{N}\right\rangle_L}{\langle S_v^A \times_L S_u^A, \mathbf{N}\rangle_L}. \end{aligned} \tag{5.21}$$

$$\begin{aligned} p'(\theta) &= \frac{\left\langle \mathcal{R}_{type}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N}) \times_L S_q^B, \mathbf{N}\right\rangle_L}{\left\langle S_p^B \times_L S_q^B, \mathbf{N}\right\rangle_L}, \\ q'(\theta) &= \frac{\left\langle \mathcal{R}_{type}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N}) \times_L S_p^B, \mathbf{N}\right\rangle_L}{\left\langle S_q^B \times_L S_p^B, \mathbf{N}\right\rangle_L}. \end{aligned} \tag{5.22}$$

Substituting these solutions $u'(t) = u'(\theta), v'(t) = v'(\theta), p'(t) = p'(\theta),\ and, q'(t) = q'(\theta)$ in (5.21) yields a trigonometric equation:

$$e^A(u'(\theta))^2 + 2f^A u'(\theta)v'(\theta) + g^A(v'(\theta))^2 = e^B(p'(\theta))^2 + 2f^B p'(\theta)q'(\theta) + g^B(q'(\theta))^2. \tag{5.23}$$

5.2.2. Parametric-parametric surfaces Lightlike. The projection of the vector α'' onto $\mathbf{N}^A$ and $\lambda\mathbf{N}^B$ produces the equation.

$$\begin{aligned} \left\langle \mathbf{N}^A, \alpha''\right\rangle_L &= \left\langle \lambda\mathbf{N}^B, \alpha''\right\rangle_L, \\ \bar{e}^A(u')^2 + 2\bar{f}^A u'v' + \bar{g}^A(v')^2 &= \lambda\left(\bar{e}^B(p')^2 + 2\bar{f}^B p'q' + \bar{g}^B(q')^2\right). \end{aligned} \tag{5.24}$$

The u', v' and p', q' values can be found by solving a linear system

$$\begin{aligned} u'(\theta) &= \frac{\left\langle \mathcal{R}_{type}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N}) \times_E S_v^A, \mathbf{N}\right\rangle_L}{\langle S_u^A \times_E S_v^A, \mathbf{N}\rangle_L}, \\ v'(\theta) &= \frac{\left\langle \mathcal{R}_{type}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N}) \times_E S_u^A, \mathbf{N}\right\rangle_L}{\langle S_v^A \times_E S_u^A, \mathbf{N}\rangle_L}. \end{aligned} \tag{5.25}$$

$$\begin{aligned} p'(\theta) &= \frac{\left\langle \mathcal{R}_{type}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N}) \times_E S_q^B, \mathbf{N}\right\rangle_L}{\left\langle S_p^B \times_E S_q^B, \mathbf{N}\right\rangle_L}, \\ q'(\theta) &= \frac{\left\langle \mathcal{R}_{type}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N}) \times_E S_p^B, \mathbf{N}\right\rangle_L}{\left\langle S_q^B \times_E S_p^B, \mathbf{N}\right\rangle_L}. \end{aligned} \tag{5.26}$$

Substituting these solutions $u'(t) = u'(\theta), v'(t) = v'(\theta), p'(t) = p'(\theta),\ and, q'(t) = q'(\theta)$ in (5.24) yields a trigonometric equation:

$$\bar{e}^A(u'(\theta))^2 + 2\bar{f}^A u'(\theta)v'(\theta) + \bar{g}^A(v'(\theta))^2 = \lambda(\bar{e}^B(p'(\theta))^2 + 2\bar{f}^B p'(\theta)q'(\theta) + \bar{g}^B(q'(\theta))^2). \tag{5.27}$$

5.2.3. Solution of the Equations. Theorem 8. *Let S^A and S^B be timelike surfaces that intersect tangentially at a point $P_0 = S^A(u_0, v_0) = S^B(p_0, q_0)$, i.e., $N^A(u_0, v_0) \parallel N^B(p_0, q_0)$ at P_0. Since the surfaces are timelike, the normal vector $\mathbf{N}$ is spacelike. Therefore, the corresponding rotation is of spacelike type: $\mathcal{R}_{sp}(\theta, \mathbf{N})$. As a result, the transformation $\mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(N)$ maps lightlike vectors to lightlike vectors.*

Proof:

See theorem 1 of the [4] article. □

Theorem 9.

Let S^A and S^B be lighlike surfaces that intersect tangentially at a point $P_0 = S^A(u_0, v_0) = S^B(p_0, q_0)$, i.e., $N^A(u_0, v_0) \parallel N^B(p_0, q_0)$ at P_0. Since the surfaces are lighlike, the normal vector $\mathbf{N}$ is lighlike. Therefore, the corresponding rotation is of lighlike type: $\mathcal{R}_{lg}(\theta, \mathbf{N})$. As a result, the transformation $\mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(N)$ maps lightlike vectors to lightlike vectors.

Proof: In fact, since the rotation $\mathcal{R}_{lg}(\theta, \mathbf{N}(P_0))\mathcal{D}_L(\mathbf{N}(P_0))$ maps lightlike vectors to lightlike vectors (7), and since T_pM is a lightlike subspace, then there is only one lightlike vectors at $p \in M$, T_pM. If the vector tangent $\alpha'(s_0)$ is lightlike and $\mathcal{D}_L(\mathbf{N}(P_0))$ is lightlike, then $\mathcal{R}_{lg}(\theta, \mathbf{N}(P_0))\mathcal{D}_L(\mathbf{N}(P_0))$ is lightlike and belongT_pM , therefore $\mathcal{R}_{lg}(\theta, \mathbf{N}(P_0))\mathcal{D}_L(\mathbf{N}(P_0)) = \alpha'(s_0)$ for all θ.

Since

$$\left\langle \mathbf{N}^i, \alpha''(t) \right\rangle_L = -\left\langle (\mathbf{N}^i)', \alpha'(t) \right\rangle_L,$$

if θ_0 is a solution of

$$\left\langle \mathbf{N}^A((u_0, v_0)), \alpha''(\theta) \right\rangle_L - \left\langle \lambda \mathbf{N}^B((p_0, q_0)), \alpha''(\theta) \right\rangle_L = 0,$$
$$\left\langle (\mathbf{N}^A((u_0, v_0)))', \alpha'(\theta) \right\rangle_L - \left\langle \lambda (\mathbf{N}^B((p_0, q_0)))', \alpha'(\theta) \right\rangle_L = 0, \ for\ all\ \theta.$$

therefore

$$\left\langle \mathbf{N}^A((u_0, v_0)), \alpha''(\theta) \right\rangle_L - \left\langle \lambda \mathbf{N}^B((p_0, q_0)), \alpha''(\theta) \right\rangle_L \equiv 0,$$

□

Remark 5.2.

To analyze the solutions of the trigonometric equations (5.23,5.27) in the variable θ, we need to separate into three cases: When $\mathbf{N}$ is timelike or is lightlike or is spacelike.

- *If $\mathbf{N}$ is **timelike**, we have the following cases depending upon the number of solutions:*
  - *(a) If equation has no solution, then P is the isolated contact point.*
  - *(b) If equation has one simple solution, then we have one intersection curve passing through P.*
  - *(c) If equation has several simple solutions, then P is a branch point, i.e. we have another branch passing through P.*
  - *(d) If equation vanishes, then surfaces have at least second order contact at P.*
- *If $\mathbf{N}$ is **spacelike** in $p \in M$, T_pM is a timelike plane, then T_pM contains **two** linearly independent lightlike vectors, timelike and spacelike vector, therefore the vector tangent can be spacelike, timelike or lightlike. Since the rotation $\mathcal{R}_{sp}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N})$ transform the timelike vectors to timelike vectors, the spacelike vectors to spacelike vectors and the lightlike vectors to lightlike vectors, we must choose four vector $\mu_{\mathbf{i}}$ $i \in \{1, 2, 3, 4\}$ for $\mathcal{D}_L(N) = \mu_{\mathbf{i}} \times_L \mathbf{N}$. We can choose the vector $\mu_{\mathbf{1}}$ such that $\mathcal{D}_L(N) = \mu_{\mathbf{1}} \times_L \mathbf{N}$ be lightlike and $\mu_{\mathbf{2}}$ such that $\mathcal{D}_L(N) = \mu_{\mathbf{2}} \times_L \mathbf{N}$ lightlike and $\mu_{\mathbf{3}}$ such that $\mathcal{D}_L(N) = \mu_{\mathbf{3}} \times_L \mathbf{N}$ spacelike and $\mu_{\mathbf{4}}$ such that $\mathcal{D}_L(N) = \mu_{\mathbf{4}} \times_L \mathbf{N}$ timelike.*

$$eq1 = \left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L, \ for\ \mu_{\mathbf{1}}; \qquad eq2 = \left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L, \ for\ \mu_{\mathbf{2}};$$

$$eq3 = \left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L, \ for\ \mu_{\mathbf{3}}; \qquad eq4 = \left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L, \ for\ \mu_{\mathbf{4}}.$$

eq1	eq2	eq3	eq4	case	solution
$eq1 \neq 0$	$eq2 \neq 0$	$eq3 \neq 0$	$eq4 \neq 0$	a	If equation has no solution, then P is the isolated contact point.
$eq1 \equiv 0$	$eq2 \neq 0$	$eq3 \neq 0$	$eq4 \neq 0$	b	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_\mathbf{1} \times_L \mathbf{N}$ for any θ.
$eq1 \neq 0$	$eq2 \equiv 0$	$eq3 \neq 0$	$eq4 \neq 0$	b	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_\mathbf{2} \times_L \mathbf{N}$ for any θ.
$eq1 \neq 0$	$eq2 \neq 0$	$eq3 = 0$	$eq4 \neq 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_\mathbf{3} \times_L \mathbf{N},\ if \theta_i$ is solutions to the $eq3 = 0$.
$eq1 \neq 0$	$eq2 \neq 0$	$eq3 \neq 0$	$eq4 = 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_\mathbf{4} \times_L \mathbf{N}$, if θ_i is solutions to the $eq4 = 0$.
$eq1 \neq 0$	$eq2 \equiv 0$	$eq3 \neq 0$	$eq4 = 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_\mathbf{2} \times_L \mathbf{N}$ for any θ. $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_\mathbf{4} \times_L \mathbf{N}$, if θ_i is solutions to the $eq4 = 0$.
$eq1 \neq 0$	$eq2 \equiv 0$	$eq3 = 0$	$eq4 \neq 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_\mathbf{2} \times_L \mathbf{N}$, for any θ. $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_\mathbf{3} \times_L \mathbf{N}$, if θ_i is solutions to the $eq3 = 0$.
$eq1 \equiv 0$	$eq2 \equiv 0$	$eq3 \equiv 0$	$eq4 \neq 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_\mathbf{1} \times_L \mathbf{N};\ \alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_\mathbf{2} \times_L \mathbf{N},\ \forall\, \theta$.
$eq1 \equiv 0$	$eq2 \equiv 0$	$eq3 \neq 0$	$eq4 \equiv 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_\mathbf{1} \times_L \mathbf{N};\ \alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_\mathbf{2} \times_L \mathbf{N},\ \forall\, \theta$.
$eq1 \equiv 0$	$eq2 \equiv 0$	$eq3 \neq 0$	$eq4 \neq 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_\mathbf{1} \times_L \mathbf{N};\ \alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_\mathbf{2} \times_L \mathbf{N},\ \forall\, \theta$.
$eq1 \equiv 0$	$eq2 \neq 0$	$eq3 \neq 0$	$eq4 = 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_\mathbf{1} \times_L \mathbf{N}$,f or any θ $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_\mathbf{4} \times_L \mathbf{N}$, if θ_i is solutions to the $eq4 = 0$.
$eq1 \equiv 0$	$eq2 \neq 0$	$eq3 = 0$	$eq4 \neq 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_\mathbf{1} \times_L \mathbf{N}$ for any θ. $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_\mathbf{3} \times_L \mathbf{N}$, if θ_i is solutions to the $eq3 = 0$.
$eq1 \neq 0$	$eq2 \neq 0$	$eq3 = 0$	$eq4 = 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_\mathbf{3} \times_L \mathbf{N},\ if\ \theta_i$ is solutions to the $eq3 = 0$. $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_\mathbf{4} \times_L \mathbf{N},\ if\ \theta_i$ is solutions to the $eq4 = 0$.
$eq1 \equiv 0$	$eq2 \neq 0$	$eq3 = 0$	$eq4 = 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_\mathbf{1} \times_L \mathbf{N}$, for any θ. $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_\mathbf{3} \times_L \mathbf{N},\ if\ \theta_i$, is solutions to the $eq3 = 0$. $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_\mathbf{4} \times_L \mathbf{N},\ if\ \theta_i$ is solutions to the $eq4 = 0$.
$eq1 \neq 0$	$eq2 \equiv 0$	$eq3 = 0$	$eq4 = 0$	b,c	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta, \mathbf{N})\ \mu_\mathbf{2} \times_L \mathbf{N}$ for any θ. $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_\mathbf{3} \times_L \mathbf{N},\ if \theta_i$ is solutions to the $eq3 = 0$ $\alpha'(t_0) = \mathcal{R}_{sp}(\theta_i, \mathbf{N})\ \mu_\mathbf{4} \times_L \mathbf{N},\ if \theta_i$ is solutions to the $eq4 = 0$
$eq1 \equiv 0$	$eq2 \equiv 0$	$eq3 \equiv 0$	$eq4 \equiv 0$	d	have at least second order contact at P

Table 5.1: Solutions.

Where the cases are:

(a) If equation has no solution, then P is the isolated contact point.
(b) If equation has one simple solution, then we have one intersection curve passing through P.
(c) If equation has several simple solutions, then P is a branch point, i.e. we have another branch passing through P.
(d) If equation vanishes, then surfaces have at least second order contact at P.

- *If $\mathbf{N}$ is **lightlike** in $p \in M$, T_pM is a lightlike plane, then by proposition (2.5) the T_pM contains only one vector lightlike vector and spacelike vector, but not a timelike one, therefore vector tangent can be spacelike or lightlike.*
  Since $\mathcal{D}_L(\mathbf{N})$ can be spacelike or lightlike, the rotation $\mathcal{R}_{lg}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N})$ transform the $\mathcal{D}_L(\mathbf{N})$ spacelike vector to spacelike or lighlike vectors and $\mathcal{D}_L(\mathbf{N})$ lightlike vectors to lightlike vectors, we must choose $\mu_\mathbf{1}$ such that $\mathcal{D}_L(N) = \mu_\mathbf{1} \times_L \mathbf{N}$ be lightlike and $\mu_\mathbf{2}$ such that $\mathcal{D}_L(N) = \mu_\mathbf{2} \times_L \mathbf{N}$ spacelike. For the choice of μ, see Proposition (4.1).

$$eq1 = \left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L,\ for\ \mu_\mathbf{1};$$

$$eq2 = \left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L, \; for \; \mu_{\mathbf{2}}.$$

eq1	eq2	case	solutions
$eq1 \neq 0$	$eq2 \neq 0$	a	If equation has no solution, then P is the isolated contact point
$eq1 \equiv 0$	$eq2 \neq 0$	b	$\alpha'(t_0) = \mathcal{R}_{lg}(\theta, \mathbf{N}) \; \mu_{\mathbf{1}} \times_L \mathbf{N}, \forall \theta.$
$eq1 \equiv 0$	$eq2 = 0$	c	$\alpha'(t_0) = \mathcal{R}_{lg}(\theta, \mathbf{N}) \; \mu_{\mathbf{1}} \times_L \mathbf{N}, \forall \theta.$ $\alpha'(t_0) = \mathcal{R}_{lg}(\theta_i, \mathbf{N}) \; \mu_{\mathbf{2}} \times_L \mathbf{N}$ if θ_i is solutions of the equation $eq2 = 0$.
$eq1 \neq 0$	$eq2 = 0$	c	$\alpha'(t_0) = \mathcal{R}_{lg}(\theta_i, \mathbf{N}) \; \mu_{\mathbf{2}} \times_L \mathbf{N}$ if θ_i is solutions of the equation $eq2 = 0$.
$eq1 \equiv 0$	$eq2 \equiv 0$	d	have at least second order contact at P.

Table 5.2: Solutions

Where the cases are:
(a) If equation has no solution, then P is the isolated contact point.
(b) If equation has one simple solution, then we have one intersection curve passing through P.
(c) If equation has several simple solutions, then P is a branch point, i.e. we have another branch passing through P.
(d) If equation vanishes, then surfaces have at least second order contact at P.

6. Example. In this section, we present some examples that illustrate our new methods.
Example 6.1. *Let us consider the surface S^A and S^B by the parametric equations*

$$S^A(u,v) = (cos(u)cos(v), sin(u)cos(v), sin(v)) \, . \quad S^B(p,q) = \left(\frac{1}{2}cos(p) + \frac{1}{2}, \frac{1}{2}sin(p), q\right).$$

Since the unit normal vectors of these surfaces at the intersection point $P = S^A(0,0) = S^B(0,0) = (1,0,0)$ are $N^A = N^B = N$, these surfaces intersect tangentially at P. The vectors $S_u^A(0,0) = (0,1,0)$, $S_v^A(0,0) = (0,0,1)$, $S_p^B(0,0) = (0,1/2,0)$ and $S_q^B(0,0) = (0,0,1)$, produce $\mathbf{N}^A = \frac{S_u^A \times_L S_v^A}{\|S_u^A \times_L S_v^A\|_L} = (1,0,0)$ and $\mathbf{N}^B = \frac{S_p^B \times_L S_q^B}{\left\|S_p^B \times_L S_q^B\right\|_L} = (1,0,0)$. The vector normal $\mathbf{N}^A = \mathbf{N}^B = \mathbf{N} = (1,0,0)$ are spacelikes. Let us now apply our second method to find the tangential direction.
Since $\mathbf{N} = (1,0,0)$ is spacelike, We must test the four equations:

- $eq1 = \left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L, \; for \; \mu_{\mathbf{1}}$
  Let $\mu_{\mathbf{1}} = (0,1,1)$ be lightlike, we get $\mathcal{D}_L = \mu_{\mathbf{i}} \times_L \mathbf{N} = (0,1,1)$ is lightlike. Then, from (5.19) we way write

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u' + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cosh(\theta)) & -sinh(\theta) \\ 0 & -sinh(\theta) & cosh(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} p' + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} q'.$$

i.e, we have

$$u' = -\sinh(\theta) + \cosh(\theta), \; v' = -\sinh(\theta) + \cosh(\theta),$$

$$p' = -2\sinh(\theta) + 2\cosh(\theta) \; and \; q' = -\sinh(\theta) + \cosh(\theta).$$

We have

$$e^A = -1.0, \; f^A = 0, \; g^A = -1.0, \; e^B = -0.5, \; f^B = 0 \; and \; g^B = 0.$$

If we substitute these results into (5.23), we have

$$\begin{aligned} (e^A(u'(\theta))^2 + 2f^A u'(\theta)v'(\theta) + g^A(v'(\theta))^2) - (e^B(p'(\theta))^2 + 2f^B p'(\theta)q'(\theta) + g^B(q'(\theta))^2) &= 0 \\ 0 &= 0 \end{aligned} \tag{6.1}$$

We have

$$eq1 \equiv 0.$$

- $eq2 = \left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L, \ for\ \mu_\mathbf{2}$
We need to choose $\mu_\mathbf{2}$ such that $\mathcal{D}_L(N) = \mu_\mathbf{2} \times_L \mathbf{N}$ is lightlike, but linearly independent with $\mathcal{D}_L(N) = \mu_\mathbf{1} \times_L \mathbf{N}$.
Choosing $\mu_\mathbf{2} = (0, 1, -1)$ such that $\mathcal{D}_L = \mu_\mathbf{2} \times_L \mathbf{N} = (0, -1, 1)$ is lightlike. Then, from (5.19) we way write,

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u' + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cosh(\theta)) & -sinh(\theta) \\ 0 & -sinh(\theta) & cosh(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} p' + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} q'.$$

from (5.21,5.22) we have

$$u' = -\sinh(\theta) - \cosh(\theta),\ v' = \sinh(\theta) + \cosh(\theta),$$

$$p' = -2\sinh(\theta) - 2\cosh(\theta)\ and\ q' = \sinh(\theta) + \cosh(\theta).$$

We have

$$e^A = -1.0,\ f^A = 0,\ g^A = -1.0,\ e^B = -0.5,\ f^B = 0\ and\ g^B = 0.$$

If we substitute these results into (5.23), we have

$$\begin{aligned} (e^A(u'(\theta))^2 + 2f^A u'(\theta)v'(\theta) + g^A(v'(\theta))^2) - (e^B(p'(\theta))^2 + 2f^B p'(\theta)q'(\theta) + g^B(q'(\theta))^2) &= 0 \\ 0 &= 0 \end{aligned} \tag{6.2}$$

We have

$$eq2 \equiv 0.$$

- $eq3 = \left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L, \ for\ \mu_\mathbf{3}$
Choosing $\mu_\mathbf{3} = (0, 0, 1)$ such that $\mathcal{D}_L = \mu_\mathbf{3} \times_L \mathbf{N} = (0, 1, 0)$ is spacelike. Then, from (5.19) we way write,

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u' + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cosh(\theta)) & -sinh(\theta) \\ 0 & -sinh(\theta) & cosh(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} p' + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} q'.$$

from (5.21,5.22) we have

$$u' = \cosh(\theta),\ v' = -\sinh(\theta),\ p' = 2\cosh(\theta)\ and\ q' = -\sinh(\theta).$$

We have

$$e^A = -1.0,\ f^A = 0,\ g^A = -1.0,\ e^B = -0.5,\ f^B = 0\ and\ g^B = 0.$$

If we substitute these results into (5.23), we have

$$\begin{array}{rcl} (e^A(u'(\theta))^2 + 2f^A u'(\theta)v'(\theta) + g^A(v'(\theta))^2) - (e^B(p'(\theta))^2 + 2f^B p'(\theta)q'(\theta) + g^B(q'(\theta))^2) & = & 0 \\ 1 & \neq & 0 \end{array} \quad (6.3)$$

We have

$$eq3 \neq 0.$$

- $eq4 = \langle \mathbf{N}^A, \alpha''(\theta)\rangle_L - \langle \mathbf{N}^B, \alpha''(\theta)\rangle_L$, *for* μ_4

Choosing $\mu_4 = (0,1,0)$ *such that* $\mathcal{D}_L = \mu_4 \times_L \mathbf{N} = (0,0,1)$ *is timelike. Then, from (5.19) we way write*

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u' + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cosh(\theta)) & -sinh(\theta) \\ 0 & -sinh(\theta) & cosh(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} p' + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} q'.$$

i.e, we have

$$u' = -\sinh(\theta),\ v' = \cosh(\theta),\ p' = -2\sinh(\theta)\ and\ q' = \cosh(\theta).$$

We have

$$e^A = -1.0,\ f^A = 0,\ g^A = -1.0,\ e^B = -0.5,\ f^B = 0\ and\ g^B = 0.$$

If we substitute these results into (5.23), we have

$$\begin{array}{rcl} (e^A(u'(\theta))^2 + 2f^A u'(\theta)v'(\theta) + g^A(v'(\theta))^2) - (e^B(p'(\theta))^2 + 2f^B p'(\theta)q'(\theta) + g^B(q'(\theta))^2) & = & 0 \\ -1 & \neq & 0 \end{array}. \quad (6.4)$$

We have

$$eq4 \neq 0.$$

As $\langle \mathbf{N}^A, \alpha''(\theta)\rangle_L - \langle \mathbf{N}^B, \alpha''(\theta)\rangle_L \equiv 0$ *for* μ_i, $i \in \{1,2\}$ *and we have* $\langle \mathbf{N}^A, \alpha''(\theta)\rangle_L - \langle \mathbf{N}^B, \alpha''(\theta)\rangle_L \neq 0$ *for* μ_i, $i \in \{3,4\}$, *then P is a branch point. Since we can choose any value for theta, so let's choose* $\theta = 0$, *then we have:* $\alpha'(t_0) = \mathcal{D}_L(\mathbf{N}) = \mu_1 \times_L \mathbf{N} = (0,1,1)$. *and* $\alpha'(t_0) = \mathcal{D}_L(\mathbf{N}) = \mu_2 \times_L \mathbf{N} = (0,-1,1)$.

eq1	eq2	eq3	eq4	solution
$eq1 \equiv 0$	$eq2 \equiv 0$	$eq3 \neq 0$	$eq4 \neq 0$	$\alpha'(t_0) = \mathcal{R}_{sp}(\theta,\mathbf{N})\ \mu_1 \times_L \mathbf{N};\ \alpha'(t_0) = \mathcal{R}_{sp}(\theta,\mathbf{N})\ \mu_2 \times_L \mathbf{N}, \forall\theta.$

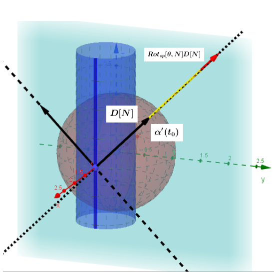


Figure 6.1: $S^A \cap S^B$.

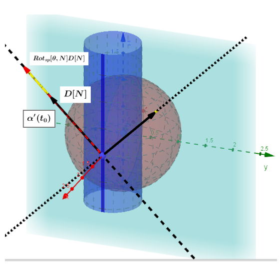


Figure 6.2: $S^A \cap S^B$.

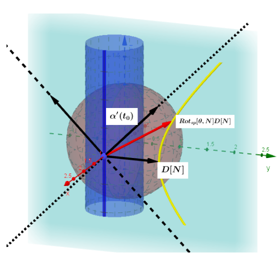


Figure 6.3: $S^A \cap S^B$.

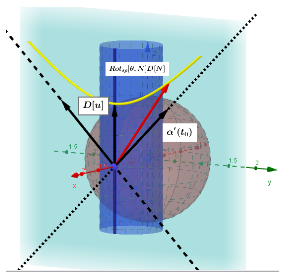


Figure 6.4: $S^A \cap S^B$.

Example 6.2.

Let us consider the surface S^A *and* S^B *by the parametric equations*

$$S^A(u,v) = (u, v^4, v)\,.\quad S^B(p,q) = (p,0,q)\,.$$

Since the unit normal vectors of these surfaces at the intersection point $P = S^A(1,0) = S^B(1,0) = (1,0,0)$ *are* $N^A = N^B = N$*, these surfaces intersect tangentially at* P*. The vectors* $S_u^A(1,0) = (1,0,0)$, $S_v^A(1,0) = (0,0,1)$, $S_p^B(1,0) = (1,0,0)$ *and* $S_q^B(1,0) = (0,0,1)$*, produce* $\mathbf{N}^A = \frac{S_u^A \times_L S_v^A}{\|S_u^A \times_L S_v^A\|_L} = (0,-1,0)$ *and* $\mathbf{N}^B = \frac{S_p^B \times_L S_q^B}{\|S_p^B \times_L S_q^B\|_L} = (0,-1,0)$*. The vector normal* $\mathbf{N}^A = \mathbf{N}^B = \mathbf{N} = (0,-1,0)$ *are spacelikes. Let us now apply our second method to find the tangential direction.*
Since $\mathbf{N} = (0,-1,0)$ *is spacelike, We must test the four equations:*

- $eq1 = \langle \mathbf{N}^A, \alpha''(\theta) \rangle_L - \langle \mathbf{N}^B, \alpha''(\theta) \rangle_L,\ for\ \mu_\mathbf{1}$
Let $\mu_\mathbf{1} = (1,0,1)$ *be lightlike, we get* $\mathcal{D}_L = \mu_\mathbf{i} \times_L \mathbf{N} = (1,0,1)$ *is lightlike. Then, from (5.19) we way write*

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u' + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v' = \begin{bmatrix} cosh(\theta) & 0 & -sinh(\theta) \\ 0 & 1 & 0 \\ -sinh(\theta) & 0 & cosh(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} p' + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} q'.$$

i.e, we have

$$u' = \cosh(\theta) - \sinh(\theta),\ v' = -\sinh(\theta) + \cosh(\theta),$$

$$p' = \cosh(\theta) - \sinh(\theta)\ and\ q' = -\sinh(\theta) + \cosh(\theta).$$

We have

$$e^A = 0.0,\ f^A = 0,\ g^A = 0.0,\ e^B = 0.0,\ f^B = 0\ and\ g^B = 0.$$

If we substitute these results into (5.23), we have

$$\begin{aligned} (e^A(u'(\theta))^2 + 2f^A u'(\theta)v'(\theta) + g^A(v'(\theta))^2) - (e^B(p'(\theta))^2 + 2f^B p'(\theta)q'(\theta) + g^B(q'(\theta))^2) &= 0 \\ 0 &= 0 \end{aligned} \tag{6.5}$$

We have

$$eq1 \equiv 0.$$

- $eq2 = \langle \mathbf{N}^A, \alpha''(\theta) \rangle_L - \langle \mathbf{N}^B, \alpha''(\theta) \rangle_L,\ for\ \mu_\mathbf{2}$
We need to choose $\mu_\mathbf{2}$ *such that* $\mathcal{D}_L(N) = \mu_\mathbf{2} \times_L \mathbf{N}$ *is lightlike, but linearly independent with* $\mathcal{D}_L(N) = \mu_\mathbf{1} \times_L \mathbf{N}$.
Choosing $\mu_\mathbf{2} = (-1,0,1)$ *such that* $\mathcal{D}_L = \mu_\mathbf{2} \times_L \mathbf{N} = (1,0,-1)$ *is lightlike. Then, from (5.19) we way write,*

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u' + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v' = \begin{bmatrix} cosh(\theta) & 0 & -sinh(\theta) \\ 0 & 1 & 0 \\ -sinh(\theta) & 0 & cosh(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} p' + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} q'.$$

from (5.21,5.22) we have

$$u' = \cosh(\theta) + \sinh(\theta),\ v' = -\sinh(\theta) - \cosh(\theta),$$

$$p' = \cosh(\theta) + \sinh(\theta)\ and\ q' = -\sinh(\theta) - \cosh(\theta).$$

We have

$$e^A = 0,\ f^A = 0,\ g^A = 0.0,\ e^B = 0,\ f^B = 0\ and\ g^B = 0.$$

If we substitute these results into (5.23), we have

$$\begin{array}{rcl} (e^A(u'(\theta))^2+2f^Au'(\theta)v'(\theta)+g^A(v'(\theta))^2)-(e^B(p'(\theta))^2+2f^Bp'(\theta)q'(\theta)+g^B(q'(\theta))^2) & = & 0, \\ 0 & = & 0. \end{array} \tag{6.6}$$

We have

$$eq2 \equiv 0.$$

- $eq3 = \left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L,\ for\ \mu_\mathbf{3}$
Choosing $\mu_\mathbf{3} = (0,0,1)$ *such that* $\mathcal{D}_L = \mu_\mathbf{3} \times_L \mathbf{N} = (1,0,0)$ *is spacelike. Then, from (5.19) we way write,*

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u' + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v' = \begin{bmatrix} cosh(\theta) & 0 & -sinh(\theta) \\ 0 & 1 & 0 \\ -sinh(\theta) & 0 & cosh(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} p' + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} q'.$$

from (5.21,5.22) we have

$$u' = \cosh(\theta),\ v' = -\sinh(\theta),\ p' = \cosh(\theta)\ and\ q' = -\sinh(\theta).$$

We have

$$e^A = 0,\ f^A = 0,\ g^A = 0,\ e^B = 0,\ f^B = 0\ and\ g^B = 0.$$

If we substitute these results into (5.23), we have

$$\begin{array}{rcl} (e^A(u'(\theta))^2+2f^Au'(\theta)v'(\theta)+g^A(v'(\theta))^2)-(e^B(p'(\theta))^2+2f^Bp'(\theta)q'(\theta)+g^B(q'(\theta))^2) & = & 0 \\ 0 & = & 0. \end{array} \tag{6.7}$$

We have

$$eq3 \equiv 0.$$

- $eq4 = \left\langle \mathbf{N}^A, \alpha''(\theta) \right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta) \right\rangle_L,\ for\ \mu_\mathbf{4}.$

Choosing $\mu_\mathbf{4} = (1,0,0)$ *such that* $\mathcal{D}_L = \mu_\mathbf{4} \times_L \mathbf{N} = (0,0,1)$ *is timelike. Then, from (5.19) we way write*

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u' + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v' = \begin{bmatrix} cosh(\theta) & 0 & -sinh(\theta) \\ 0 & 1 & 0 \\ -sinh(\theta) & 0 & cosh(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} p' + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} q'.$$

i.e, we have

$$u' = -\sinh(\theta),\ v' = \cosh(\theta),\ p' = -\sinh(\theta)\ and\ q' = \cosh(\theta).$$

We have

$$e^A = 0,\ f^A = 0,\ g^A = 0,\ e^B = 0,\ f^B = 0\ and\ g^B = 0.$$

If we substitute these results into (5.23), we have

$$\begin{array}{rcl} (e^A(u'(\theta))^2+2f^Au'(\theta)v'(\theta)+g^A(v'(\theta))^2)-(e^B(p'(\theta))^2+2f^Bp'(\theta)q'(\theta)+g^B(q'(\theta))^2) & = & 0 \\ 0 & = & 0 \end{array}. \tag{6.8}$$

We have

$$eq4 \equiv 0.$$

As $\left\langle \mathbf{N}^A, \alpha''(\theta)\right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta)\right\rangle_L \equiv 0$ *for* $\mu_{\mathbf{i}},\ i \in \{1,2,3,4\}$, *then surfaces have at least second order contact at P.*

eq1	eq2	eq3	eq4	solution
$eq1 \equiv 0$	$eq2 \equiv 0$	$eq3 \equiv 0$	$eq4 \equiv 0$	have at least second order contact at P.

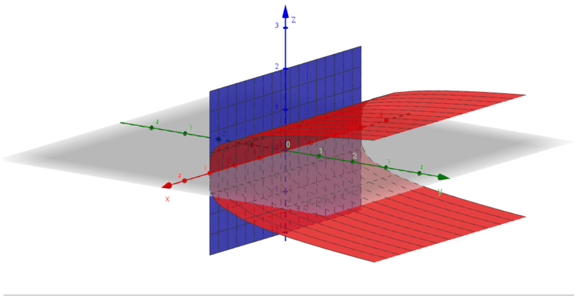


Figure 6.5: Seconde Order Contact.

Example 6.3.
Let us consider the surface S^A *and* S^B *by the parametric equations*

$$S^A(u,v) = \left(u, v, \sqrt{u^2+v^2}\right),$$

$$S^B(p,q) = (-p-q, p, q).$$

Since the unit normal vectors of these surfaces at the intersection point $P = S^A(0,1) = S^B(1,-1) = (0,1,1)$ *are* $N^A = N^B = N$, *these surfaces intersect tangentially at* $P = (0,1,1)$. *The vectors* $S^A_u(0,1) = (1,0,0)$, $S^A_v(0,1) = (0,1,1)$, $S^B_p(1,-1) = (-1,1,1)$, $S^B_q(1,-1) = (-1,0,0)$, *produce* $\mathbf{N}^A = S^A_u \times_L S^A_v = (0,-1,-1)$ *and* $\mathbf{N}^B = S^B_p \times_L S^B_q = (0,-1,-1)$. *The vector normal* $\mathbf{N}^A = \lambda \mathbf{N}^B = \mathbf{N} = (0,-1,-1)$ *are lightlikes and* $\lambda = 1$. *Let us now apply our second method to find the tangential direction. Since* $\mathbf{N}(P) = (0,-1,-1)$ *is lightlike, We must test the two equations:*

- $eq1 = \left\langle \mathbf{N}^A, \alpha''(\theta)\right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta)\right\rangle_L$, *for* $\mu_{\mathbf{1}}$
  Let $\mu_{\mathbf{1}} = (1,0,0)$ *be lightlike, we get* $\mathcal{D}_L = \mu_{\mathbf{1}} \times_L \mathbf{N} = (0,1,1)$ *is lightlike. Then, from (5.19) we way write*

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u' + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} v' = \begin{bmatrix} 1 & \theta & -\theta \\ -\theta & 1-0.5\theta^2 & 0.5\theta^2 \\ -\theta & -0.5\theta^2 & 0.5\theta^2+1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} p' + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} q',$$

or

$$\begin{aligned} u'(\theta) &= \frac{\left\langle \mathcal{R}_{lg}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N}) \times_E S^A_v, \mathbf{N}\right\rangle_L}{\langle S^A_u \times_E S^A_v, \mathbf{N}\rangle_L} = 0, \\ v'(\theta) &= \frac{\left\langle \mathcal{R}_{lg}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N}) \times_E S^A_u, \mathbf{N}\right\rangle_L}{\langle S^A_v \times_E S^A_u, \mathbf{N}\rangle_L} = 1. \end{aligned} \tag{6.9}$$

$$\begin{aligned} p'(\theta) &= \frac{\left\langle \mathcal{R}_{type}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N}) \times_E S^B_q, \mathbf{N}\right\rangle_L}{\left\langle S^B_p \times_E S^B_q, \mathbf{N}\right\rangle_L} = 1, \\ q'(\theta) &= \frac{\left\langle \mathcal{R}_{type}(\theta,\mathbf{N})\mathcal{D}_L(\mathbf{N}) \times_E S^B_p, \mathbf{N}\right\rangle_L}{\left\langle S^B_q \times_E S^B_p, \mathbf{N}\right\rangle_L} = -1. \end{aligned} \tag{6.10}$$

If we substitute these results into (5.23), we have

$$\bar{e}^A(u'(\theta))^2 + 2\bar{f}^A u'(\theta)v'(\theta) + \bar{g}^A(v'(\theta))^2 - \bar{e}^B(p'(\theta))^2 - 2\bar{f}^B p'(\theta)q'(\theta) - \bar{g}^B(q'(\theta))^2 = 0. \tag{6.11}$$

$\lambda = 1$ *and we have* $\bar{e}^A = 1,\ \bar{f}^A = 0,\ \bar{g}^A = 0, \bar{e}^B = 0,\ \bar{f}^B = 0,\ \bar{g}^B = 0.$

$$\begin{aligned} 1.0^2 + 2.(0).0.1 + 0.1^2 - 0.1^2 + 2.(0).1.(-1) - 0.(-1)^2 + &= 0 \\ 0 &= 0 \end{aligned} \tag{6.12}$$

We have

$$eq1 \equiv 0.$$

- $eq2 = \left\langle \mathbf{N}^A, \alpha''(\theta)\right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta)\right\rangle_L,\ for\ \mu_{\mathbf{2}}.$
  We need to choose $\mu_{\mathbf{2}}$ *such that* $\mathcal{D}_L(N) = \mu_{\mathbf{2}} \times_L \mathbf{N}$ *is spacelike*
  Choosing $\mu_{\mathbf{2}} = (0, 0, 1)$ *such that* $\mathcal{D}_L = \mu_{\mathbf{2}} \times_L \mathbf{N} = (1, 0, 0)$ *is spacelike. Then, from (5.19) we way write,*

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u' + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} v' = \begin{bmatrix} 1 & \theta & -\theta \\ -\theta & 1-0.5\theta^2 & 0.5\theta^2 \\ -\theta & -0.5\theta^2 & 0.5\theta^2+1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} p' + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} q',$$

or

$$\begin{aligned} u'(\theta) &= \frac{\left\langle \mathcal{R}_{lg}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}) \times_E S_v^A, \mathbf{N}\right\rangle_L}{\langle S_u^A \times_E S_v^A, \mathbf{N}\rangle_L} = 1, \\ v'(\theta) &= \frac{\left\langle \mathcal{R}_{lg}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}) \times_E S_u^A, \mathbf{N}\right\rangle_L}{\langle S_v^A \times_E S_u^A, \mathbf{N}\rangle_L} = -\theta. \end{aligned} \tag{6.13}$$

$$\begin{aligned} p'(\theta) &= \frac{\left\langle \mathcal{R}_{type}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}) \times_E S_q^B, \mathbf{N}\right\rangle_L}{\left\langle S_p^B \times_E S_q^B, \mathbf{N}\right\rangle_L} = -\theta, \\ q'(\theta) &= \frac{\left\langle \mathcal{R}_{type}(\theta, \mathbf{N})\mathcal{D}_L(\mathbf{N}) \times_E S_p^B, \mathbf{N}\right\rangle_L}{\left\langle S_q^B \times_E S_p^B, \mathbf{N}\right\rangle_L} = \theta - 1. \end{aligned} \tag{6.14}$$

If we substitute these results into (5.23), we have

$$\bar{e}^A(u'(\theta))^2 + 2\bar{f}^A u'(\theta)v'(\theta) + \bar{g}^A(v'(\theta))^2 - \bar{e}^B(p'(\theta))^2 - 2\bar{f}^B p'(\theta)q'(\theta) - \bar{g}^B(q'(\theta))^2 = 0\ . \tag{6.15}$$

$\lambda = 1$ *and we have* $\bar{e}^A = 1,\ \bar{f}^A = 0,\ \bar{g}^A = 0, \bar{e}^B = 0,\ \bar{f}^B = 0,\ \bar{g}^B = 0.$

$$\begin{aligned} 1.1^2 + 2.(0).1.(-\theta) + 0.(-\theta)^2 - 0.(-\theta)^2 + 2.(-\theta).(\theta-1) - 0.(\theta-1)^2 + &= 0 \\ 1 &= 0 \end{aligned}\ . \tag{6.16}$$

We have

$$eq2 = 1 \neq 0.$$

As $\left\langle \mathbf{N}^A, \alpha''(\theta)\right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta)\right\rangle_L \equiv 0$ *for* $\mu_{\mathbf{1}}$ *and* $\left\langle \mathbf{N}^A, \alpha''(\theta)\right\rangle_L - \left\langle \mathbf{N}^B, \alpha''(\theta)\right\rangle_L \neq 0$ *for* $\mu_{\mathbf{2}}$.
The vector tangent is

$$\alpha'(t_0) = \mathcal{R}_{lg}(\theta, \mathbf{N})\mu_{\mathbf{1}} \times_L \mathbf{N}(\mathbf{N}) = \begin{bmatrix} 1 & \theta & -\theta \\ -\theta & 1-0.5\theta^2 & 0.5\theta^2 \\ -\theta & -0.5\theta^2 & 0.5\theta^2+1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = (0, 1, 1).$$

eq1	eq2	solutions
$eq1 \equiv 0$	$eq2 \neq 0$	$\alpha'(t_0) = \mathcal{R}_{lg}(\theta, \mathbf{N})\ \mu_{\mathbf{1}} \times_L \mathbf{N}$ for any θ.

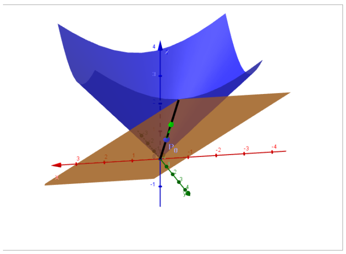


Figure 6.6: $S^A \cap S^B$.

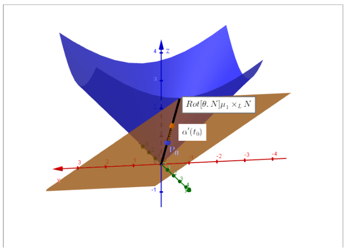


Figure 6.7: $S^A \cap S^B$.

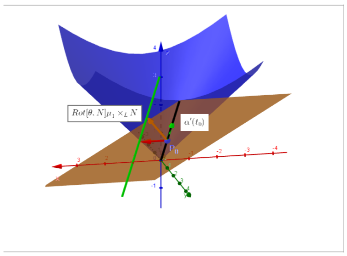


Figure 6.8: $S^A \cap S^B$.

7. Conclusion. Differently from the works in [47, 48, 49, 52], this study investigates *tangential intersection curves* instead of transversal intersections. We compute the tangent vector of tangential intersection curves formed by the intersection of two *spacelike*, *timelike*, or *lightlike* surfaces, where the surface pairs may be parametric–parametric in the three-dimensional Lorentz-Minkowski space E_1^3.

Our novel approach is based on a newly defined operator and the *Euler-Rodrigues rotation formula* in Minkowski 3-space. This methodology is applicable to both tangential and transversal intersection id the surfaces parametrics. In this work, however, we focus only on tangential intersections.

The application of the Euler-Rodrigues rotation formula in Minkowski 3-space is more intricate than the classical Rodrigues rotation formula in Euclidean 3-space. In the case of tangential intersection between two *timelike surfaces*, the tangent vector is computed by applying the rotation to all three types of vectors: spacelike, timelike, and lightlike. For tangential intersections of two *lightlike surfaces*, the rotation involves spacelike and lightlike vectors. For two *spacelike surfaces*, the computation of the tangent vector is similar to that using Rodrigues' rotation formula in Euclidean 3-space.

As future work, we intend to extend the method to transversal intersection curves in Lorentz-Minkowski space, possibly employing quaternions to simplify the computation of the tangent vector by rotating a single vector. The generalization of our method to broader settings in Lorentz-Minkowski spaces E_1^3 and E_1^4 remains an open direction for further research.

Contribuciones de los autores. Aportaciones de los autores. Todos los autores de esta publicación contribuyeron de manera equitativa en la conceptualización, investigación, análisis formal, metodología, validación, revisión y redacción del trabajo.

Agradecimiento. Agradecemos a la Universidad Federal del Triángulo Minero por el espacio y el tiempo concedidos para la investigación.

Conflicto de intereses. Los autores declaran no tener conflicto de intereses.

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