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A note about a Morse's conjecture

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Abstract

In dynamical systems it is known that metrically transitive systems are topologically transitive. M. Morse in 1946 ([1]) conjectured that if the dynamical system has some degree of regularity, then the converse is true.

In this article, we will study the Morse conjecture for $\mathbb{R}^2$-actions on three-dimensional manifolds and prove the following two results: The conjecture is false if we do not impose restrictions on the singular set and we will prove that The Morse's Conjecture is valid for locally free actions.

Keywords . Topological Transitive, Group Action, Metrically Transitive.

1. Introduction. The study of transitive dynamical systems has a long tradition and it addresses two initial problems: First, what are the topological restrictions on the manifold for the existence of a dynamical system with a dense orbit? This problem is known as the admissibility problem. A second question is what dynamic implications does a transitive system have?

The results of classifying transitive manifolds was systematically presented by Smith and Thomas in 1988 in papers [2, 3]. Essentially, they characterize the compact, connected surfaces that admit transitive flows as being different the sphere $\mathbb{S}^2$, the projective plane $\mathbb{P}^2$, or the Klein bottle $\mathbb{B}^2$. In dimensions greater than or equal to three, the authors proved that every compact, connected, smooth manifold admits a transitive flow. The problems of characterizing which non-compact surfaces or manifolds are transitive are still open.

On the other hand, in relation to the study of the second problem, in [1] Morse defined a dynamical system *Metrically Transitive* as being a flow such that if K is a compact, invariant set then $m(K)m(N \setminus K) = 0$, where m denoted the Lebesgue measure. He conjectured that a vector field defined in a admissible manifold (manifolds that admit a flow with a dense orbit), with some regularity about the singular set, is metrically transitive.

About the Morse's Conjecture on surfaces exist many works, [4, 5, 6, 7]. In the first three articles, the authors prove, using different methods, that every real analyticv and transitive flow over an admissible surface is metrically transitive. In the Fourth article, the author proved that every C^2-transitive flow on an admissible surface with a finite number of singular orbits is metrically transitive.

When the system is defined by actions of $\mathbb{R}^k$ with $k \geq 3$ on closed manifolds n-dimensional, the author does not know results for the two problems mentioned above, except for some initial results. In [8], the author started the study of these two problems when the singular set is non-empty and the partial results on our two problems are described in [9, 10].

The aim of this article is to present some answers to Morse's conjecture for some dynamical systems defined by actions of $\mathbb{R}^2$ on 3-closed manifolds.

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To describe the results that we will present in this article, we will start with initial and known notations and definitions for group actions on manifolds. Let ϕ be a C^2 action of $\mathbb{R}^2$ on a C^∞ closed manifold N of dimension 3. For any p in N, the set $\mathcal{O}_p = \{\phi(p,r) : r \in \mathbb{R}^2\}$ is called the *orbit* by ϕ of p and $G_p = \{u \in \mathbb{R}^2; \phi(u,p) = p\}$ is called *the isotropy group of* p. Observe that groups isomorphic to $\mathbb{R}\times\mathbb{R}$, $\mathbb{R}\times\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}\times\mathbb{Z}$ and $\{0\}$, are respectively isotropy groups of orbits homeomorphic to single point, circle, line, cylinder, torus and plane. We denote by $\mathcal{F}_\phi$ the C^2 foliation on N (eventually, with singularities, see [11]) whose leaves are the orbits of ϕ. The *singular set*, denoted by $\mathrm{Sing}(\phi)$, is the set of points in N whose orbits have dimension less than 2. If $\mathrm{Sing}(\phi) = \emptyset$ we say that ϕ is *locally free*.

For each basis $\{w_1, w_2\}$ of $\mathbb{R}^2$ we have two commutative vector fields X_{w_1} and X_{w_2} whose flows associated are, respectively, $\phi_1^t = \phi(tw_1, \cdot)$ and $\phi_2^t = \phi(tw_2, \cdot)$. This vector fields are called *infinitesimal generators* of the action ϕ. If $w \in \mathbb{R}^2$ is a element of a base, X_w is called *infinitesimal generator for* $p \in N$ if exist $0 < t_0 \in \mathbb{R}$ such that $\phi^{t_0}(p) = \phi(t_0 w, p) = p$. and $\phi(p) \neq p$ for $0 < t < t_0$.

The action ϕ is called *topologically transitive* on N if for every open sets $U, V \subset N$ exist $v \in \mathbb{R}^2$ such that

$$\phi^v(V) \cap U \neq \emptyset.$$

The action ϕ is called *metrically transitive* if the only closed, non-empty, ϕ-invariant set and with positive Lebesgue measure is N.

In Section 2 of this paper, we will present four examples of topologically transitive $\mathbb{R}^2$-actions on closed 3-manifolds that are metrically transitive; in the first and third examples the dense orbit has the topological type of a cylinder and in the second and fourth examples the dense orbit has the topological type of a plane. In this section we will also present preliminary results so that, finally, in the third section we will present the proofs of our two main results:

Theorem A. Let $N^{n+1} = \mathbb{T}^n \times \mathbb{S}^1$, $n \geq 2$, then exist an C^r $\mathbb{R}^2$-action topological transitive but without metric transitivity on N with $r \geq 1$.

Theorem B. Let N^3 be a compact closed manifold and ϕ be a C^2 locallly free $\mathbb{R}^2$-action on N. If ϕ is topological transitive then ϕ is metric transitivity.

2. Some examples and preliminaries. Let N be a closed and connected 3-manifold. We always consider ϕ be a C^2-action on N. We shall now describe some $\mathbb{R}^2$-action on 3-manifolds. For Examples 1 and Example 2 consider $N = \mathbb{T}^3$ and a, b be real number which are algebraically independent over $\mathbb{Z}$.

Example 2.1. *considering the basis $\{w_1, w_2\}$ of $\mathbb{R}^2$ with $w_1 = (1,0,0)$ and $w_2 = (0,1,b)$ then the vector fields X_{w_1} and X_{w_2} are the infinitesimal generators of an action ϕ of $\mathbb{R}^2$ on the manifold $\mathbb{T}^3$ such that all the orbits defined by this actions are cylinders and each orbit is dense in $\mathbb{T}^3$.*

Example 2.2. *Considering the basis $\{w_1, w_2\}$ of $\mathbb{R}^2$ with $w_1 = (1,a,0)$ and $w_2 = (0,1,b)$ then the vector fields X_{w_1} and X_{w_2} are the infinitesimal generators of an action ϕ of $\mathbb{R}^2$ on the manifold $\mathbb{T}^3$ such that all the orbits defined by this actions are homeomorphic to $\mathbb{R}^2$ and each orbit is dense in $\mathbb{T}^3$.*

Note that in the previous examples it is easy to verify that the actions are metrically transitive. We should also highlight that the singular set is emptyset.

In [7], the author proved that a C^2-flow topologically transitive on a admissible surface Σ that has only finitely many equilibrium states then, the flow is metrically transitive. If X is the vector fild associated with this flow. With this notation, we have the following example.

Example 2.3. *Let $N = \Sigma \times \mathbb{S}^1$ be a 3-manifold. Consider the infinitesimal generators of a $\mathbb{R}^2$-action ϕ of the form:*

$$X_1(p,\theta) = X(p),$$

$$X_2(p,\theta) = b(\theta)\frac{\partial}{\partial\theta}.$$

Therefore ϕ is an $\mathbb{R}^2$-action on N with all its regular orbits of the cylinder type and with a finite number of singular orbits of the circle type. If $K \subset N$ is an ϕ-invariant set then $\tilde{K} = P(K) \subset \Sigma$, where $P : N \to \Sigma$ is the first projection, is a X^t-invariant set.

But, X^t is metric transitive we have $m(\tilde{K})m(\sigma \setminus \tilde{K}) = 0$, where m is the Lebesgue measure on Σ. Finally, we have $m(K)m(\sigma \setminus K) = 0$, so ϕ is metric transitive.

Example 2.4. *Consider the singular flow of $\mathbb{S}^2$ whose regular orbits are the meridians and the singular ones are the poles P_1 and P_2, and form the product $\mathbb{S}^2 \times [0,1]$. Now, If $\psi : \mathbb{S}^2 \to \mathbb{S}^2$ is the* *rotation, fixing the poles, of angle α such that the numbers α and 2π are linearly independent over $\mathbb{Q}$. identify each $(x,1)$ with $(\psi(x),0)$. In this way one obtains a action ϕ of $\mathbb{R}^2$ on $N = \mathbb{S}^2 \times \mathbb{S}^1$ by dense planes such that* $\mathrm{Sing}(\phi) = (\{p_1\} \times \mathbb{S}^1) \cup (\{p_2\} \times \mathbb{S}^1)$. *Again, we get an example of a metrically transitive action.*

It is known that a locally free and topologically transitive $\mathbb{R}^2$-action on a 3-manifold cannot simultaneously have cylindrical (not dense, by Lemma 2.1) and planar orbits. However, we cannot yet state that this feature still holds for singular actions. In fact, in [8] we conjecture that this is not possible.

Definition 2.1. The *limit set* of $\mathcal{O}_p$ is a ϕ-invariant compact set given by $\lim \mathcal{O}_p = \cap_{i=1}^{\infty} \mathrm{cl}(\mathcal{O}_p \setminus K_i)$, where K_i is a compact subset of $\mathcal{O}_p$, $K_i \subset K_{i+1}$ and $\mathcal{O}_p = \cup_{i=1}^{\infty} K_i$. It is not difficult to show that $\mathrm{cl}(\mathcal{O}_p) = \mathcal{O}_p \cup \lim \mathcal{O}_p$.

The notions of minimal and exceptional minimal sets that we use here are the standard ones.

In this section we introduce some tools useful for our purposes.

Lemma 2.1. *If $\mathcal{O}_q \in \lim \mathcal{O}_p$, then $G_p \subset G_q$.*

Proof: Let $u \in G_p$, observe that $\phi(u, \phi(v,p)) = \phi(v,p)$ for all $v \in \mathbb{R}^2$; for other hand q is an acumulation point of some sequences $p_n \in \mathcal{O}_p$ and $v_n \in \mathbb{R}^2$ suct that $\phi(v_n, p) = p_n$ then by definition of actions and by continuity of ϕ we conclude that $\phi(u, \phi(v_n, p))$ converge to $\phi(u,q)$ and q, then $u \in G_q$. ☐

Remark 2.1. *Ussing the above lemma, if ϕ is an $\mathbb{R}^2$-action on N with a cylindrical dense orbit then the regular orbits are homeomorphic to torus or cylinder.*

2.1. Local structure in a neighborhood of an orbit diffeomorphic to $\mathbb{T}^2$. Let T be a compact orbit of ϕ which is diffeomorphic to $\mathbb{T}^2$ and let Γ be a cylindrical orbit whose limit set contains T, i.e. $T \subset \lim(\Gamma)$. let G_T^0 and H subgroups of G_T such that $G_\Gamma \subset G_T^0$ and $G_T = H \oplus G_T^0$. We can choose $\{w_1, w_2\}$ a basis of $\mathbb{R}^2$ such that w_1 and w_2 are generator of G_Γ and H, respectively.

Let $X_1 = X_{w_1}$ and $X_2 = X_{w_2}$ be the infinitesimal generators associated to w_1 and w_2. Clearly, by our choice, all orbits of the vector field X_1 through points in $T \cup \Gamma$ are periodics of period equal to 1.

We consider the open ring $A(\varepsilon) = \{(\theta, r) : \theta \in [0, 2\pi] \text{ and } r \in (-\varepsilon, \varepsilon)\}$ and the open interval $I(\delta) = (-\delta, \delta)$. There exist $\varepsilon > 0$ and $\delta > 0$ such that the application $h : A(\varepsilon) \times I(\delta) \to N$, defined by $h(\theta, r, t) = X_2^t(\theta, r)$, satisfies:

- h is a diffeomorphism onto its image V;
- $\gamma = h(\mathbb{S}^1 \times \{0\} \times \{0\})$ is a orbit of X_1 through a point q in T;
- $h(A(\varepsilon) \times \{0\})$ is transverse to T
- Moreover, in the coordinates (V, h^{-1}), the infinitesimal generators of ϕ are of the form:

$$\begin{aligned} X_1(\theta, r, t) &= a(\theta, r)\frac{\partial}{\partial t} + b(\theta, r)\frac{\partial}{\partial \theta} + c(\theta, r)\frac{\partial}{\partial r}, \\ X_2(\theta, r, t) &= \frac{\partial}{\partial t}. \end{aligned} \tag{2.1}$$

  The vector field $\widehat{X}_1(\theta, r) = b(\theta, r)\frac{\partial}{\partial \theta} + c(\theta, r)\frac{\partial}{\partial r}$ defines a local flow on $A(\epsilon)$ having $\mathbb{S}^1 \times \{0\}$ as a periodic orbit of period equal to 1.

Remark 2.2. *Note that all the orbits $\widehat{X}_1$ by points in $A(\varepsilon) \cap \Gamma$ are periodics.*

2.2. Some properties of actions having a dense cylindrical orbit. Let p be an element of N whose orbit Γ_p is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. Then there exist a base $\{u_1, u_2\}$ of $\mathbb{R}^2 \setminus \{0\}$ such that the isotropy subgroup of the orbit Γ_p is $\mathbb{Z}u_1$, denote by X_1 and X_2 the onfinetisimal gerators of ϕ. We denote by R and L the half-planes which are defined by the straight line $\mathbb{R}u_1$. We consider the following subsets $\Gamma_p^+ = \{\phi(v,p); v \in L\}$ e $\Gamma_p^- = \{\phi(v,p); v \in R\}$ of Γ_p, which, we call of *half-orbits* of p by the action ϕ.

Definition 2.2. We say that the action ϕ is *half topologically transitive,* if ϕ has a cylindrical orbit Γ_p for which one of his half-orbits is dense in N. If the half-orbit Γ_p^- (resp. Γ_p^+) is dense in N, we say that the action ϕ *is half topologically transitive with respect to R (resp. L).*

As in the case of dynamical systems defined by diffeomorphisms and flows, we have the following natural result:

Proposition 2.1. *Suppose that $H = R, L$. The following statements are equivalent:*

(i) ϕ is half topologically transitive with respect to H;

(ii) *every closed set E in N which is invariant by all diffeomorphism $\phi^v(\cdot) = \phi(v, \cdot)$, $v \in H$; is either N or has empty interior;*

(iii) *for every pair of non-empty open sets U and V in N, there is $v \in H$ such that*

$$\phi^{-v}(U) \cap V \neq \emptyset.$$

Proof: We will prove to case in that $H = R$, the proof in the other case is analogous. We start proving that (i) implies (ii). Suppose that ϕ is half topologically transitive with respect to R, and take E a closed set in N with interior non-empty which is ϕ^v-invariant for all $v \in L$. Then, we will show that $E = N$. In fact, there is a non-empty open set U in N such that $U \subset E$. Since ϕ is half topologically transitive with respect to R, we can assume that there is a point p of N, whose orbit Γ_p is a cylinder, such that Γ_p^+ is dense in N. Hence, there exists $v_0 \in L$ such that $q = \phi(v_0, p)$ belongs to U. Now, we consider the compact ring $C_p = \{\phi(tu + sv_0, p) : 0 \leq s, t \leq 1\}$. It is easy see that $\Gamma_p^+ = C_p \cup \Gamma_q^+$. Hence, being C_p a compact surface and Γ_p^+ dense in N, we obtains that Γ_q^+ is also dense in N. But, as E is a closed set ϕ^v-invariant for all $v \in L$, we have that $\overline{\Gamma_q^+} \subset E \subset N$. Therefore, whereas Γ_q^+ is dense in N, we conclude that $E = N$.

Now, we assume that (ii) is true. Let U, V two non-empty open sets in N. Then $\cup_{v\in L}\phi^{-v}(U)$ is a non-empty open set in N which, by definition, is ϕ^v-invariant for all $v \in L$. Consequently, the statement (ii) implies that this set is dense in N. Therefore, there is $v \in L$ such that

$$\phi^{-v}(U) \cap V \neq \emptyset.$$

This shows that (iii) follows.

Finally, we assume (iii). Let $\{U_n\}_{n=1}^{\infty}$ be a countable base of N. The statement (iii) implies that:

- there is $v_1 \in L$ such that
$$\phi^{-v_1}(U_1) \cap U_1 \neq \emptyset.$$
Take V_1 a non-empty open set in N such that $\overline{V_1}$ is a compact set contained in $\subset \phi^{-v_1}(U_1) \cap U_1$.
- there is $v_2 \in L$ such that:
$$\phi^{-v_2}(U_2) \cap V_1 \neq \emptyset.$$
Take a non-empty open set V_2 in N such that $\overline{V_2}$ is a compact set contained in $\subset \phi^{-v_2}(U_2) \cap V_1$.
- by induction, we obtain a sequence $\{V_n\}_{n=1}^{\infty}$ of non-empty open sets in N and a sequence $\{v_n\}_{n=1}^{\infty}$ of elements in R such that: $\overline{V_{n+1}}$ is a compact set contained in $V_n \cap \phi^{-v_{n+1}}(U_{n+1})$ and $V_{n+1} \subset V_n$ for all $n \geq 1$, and $\overline{V_1} \subset U_1 \cap \phi^{-v_1}(U_1)$.

Hence, the set $V = \cap_{n=1}^{\infty}\overline{V_n}$ is a non-empty compact set. Furthermore, for any $p \in V$ we have that $\phi^{v_n}(p)$ is an element of U_n for all n. This shows that Γ_p^+ is dense in N, and concludes the proof. $\square$

Obviously, if the action ϕ is half topologically transitive then it is topologically transitive. As a consequence of previous proposition, we obtain the reciprocal for actions of $\mathbb{R}^2$.

Proposition 2.2. *Assume that the action ϕ is topologically transitive and that the dense orbit is cylindrical. Then ϕ is half topologically transitive with respect to R and L.*

Proof: Let $\mathcal{O}_p$ be a dense cylindrical orbit of ϕ, and we consider U and V two non-empty open sets in N. Then, there is $v \in \mathbb{R}^2$ such that

$$\phi^v(U) \cap V \neq \emptyset.$$

We going to prove that ϕ is half topologically transitive with respect to R, the other case is analogous. In fact, we have two possibilities.

(i) *v is an element of* $\mathbb{R}u$: here, by continuity of the action ϕ, there exists $\delta > 0$ such that

$$\phi^w(U) \cap V \neq \emptyset,$$

for every w in D, the open disc in $\mathbb{R}^2$ of radius δ and center v.

Changing, if necessary, the element v for another element in $D \setminus \mathbb{R}u$ we can assume that:

(ii) v *is not in* $\mathbb{R}u$: If $v \in L$, then $w = -v$ is in R and

$$\phi^{-w}(U) \cap V \neq \emptyset,$$

Consequentely, by (iii) of Proposition 2.1, ϕ is half topologically transitive with respect to R. Hence, we can assume that $v \in R$. For $W = \phi^v(U) \cap V$, the topological transitivity implies that there is $w \in R$ such that

$$\phi^w(W) \cap W \neq \emptyset.$$

Then, there are q and q_1 two elements of W such that $q = \phi(w, q_1)$. Furthermore, by definition of W we have that $q_1 \in V$ and $q_1 = \phi(v, q_2)$ for some q_2 in U. It follows that $q_1 = \phi(w + v, q_2)$, and thence q_2 belongs to $\phi^{-v-w}(V) \cap U$. On the other hand, by (i) above, we can assume that $w \in L$. This implies that $v + w$ is also an element of H^+, consequently, by Proposition 2.1, we conclude that ϕ is half topologically transitive with respect to R. $\square$

Note that $\gamma = \phi(tu_1, p)$ is a closed orbit of the vector field X_1, defined the sets $\Gamma_t^+ = \overline{\cup_{s\geq t} X_2^s(\gamma)}$ and $\Gamma_t^- = \overline{\cup_{s<t} X_2^s(\gamma)}$ and $\omega(\Gamma) = \cap_{t\in\mathbb{R}}\Gamma_t^+$, $\alpha(\Gamma) = \cap_{t\in\mathbb{R}}\Gamma_t^-$. This last sets are called ω and α set of Γ respectively.

Remark 2.3. *For a cylindrical orbit* Γ, *the* ω *and* α *set of* Γ *are* ϕ*-invariant, closed and non-empty. Also* $\lim \Gamma = \omega(\Gamma) \cup \alpha(\Gamma)$.

The following result is a fundamental tool to prove the Theorems B.

Proposition 2.3. *Let* ϕ *a action of* $\mathbb{R}^2$ *on a compact, connected 3-manifold* N. *Suppose* ϕ *is topologically transitive with dense orbit cylinder. Then there are no compact orbits.*

Proof: As ϕ is topologically transitive, we can assume, by Proposition 2.2, that there is a orbit Γ topologically transitive on one side. Let T, γ, $A(\varepsilon)$, X_1, X_2 and $\widehat{X}_1$ as above, then:

Given $x \in T$, as $T \subset \lim(\Gamma)$, exist $q_n = (\theta_n, r_n, t_n)$ on V, $n \geq 1$ such that $q_n \to x$. Take $\widetilde{q}_n = (\theta_n, r_n)$ on $A(\varepsilon)$, we have that the orbit C_n of $\widetilde{q}_n$, by vector field $\widehat{X}_2$ is periodic for all $n \geq 1$.

Consider the subsets A_n de $A(\varepsilon)$ homeomorphic to the open annulus $\mathbb{S}^1 \times (-1, 1)$ such that $\partial A_n = C_n \cup \gamma$. Observe that as Y is transverse to $A(\varepsilon)$, for any pair of points p_1 and p_2 on A_n, $Y(p_1)$ and $Y(p_2)$ are parallel and have the same direction.

Changing the direction of the field X_2, if necessary, we have A_n is contained in one connected component of $A(\varepsilon) \setminus \gamma$, para todo $n \geq 1$, e $A_1 \supset A_2 \supset ... \supset A_i \supset A_{i+1} \supset .....$

Fixed $i_0 \in \mathbb{N}$, consider the topological torus T_{i_0} which is the union of the following sets: the cylinder contained in Γ between C_{i_0} and C_{i_0+1} and the annulus $A_{i_0} \setminus A_{i_0+1}$. The torus T and T_{i_0} are the bord of an open submanifold N' ϕ^v-invariant for all $v \in L$. This contradiction with the item (iii) of the Proposition 2.1 concludes the demonstration. $\square$

Corollary 2.1. *If* $\phi \in \mathcal{A}$ *then* $\mathcal{O}_i$ *are the only minimal set of* ϕ, *where* $\mathcal{O}_i \in Sing(\phi)$.

Proof: Let μ a minimal set sucht that $\mu \neq \mathcal{O}_i$ for $\mathcal{O}_i \in Sing(\phi)$. Consider the locally free action $\phi' : \mathbb{R}^2 \times N' \to N'$ where $N' = N \setminus Sing(\phi)$ is given by $\phi' = \phi|_{\mathbb{R}^2\times N'}$. Observe that ϕ' has no exceptional minimal set (see [?, Theorem 8]), then $\mu = \mathcal{O}_p$ or $\mu = N'$. If $\mu = \mathcal{O}_p$, then $\mathcal{O}_p$ is a compact orbit of ϕ, contradicting the above proposition. If $\mu = N'$, then the clausure of μ in N, contain some $\mathcal{O}_i \in Sing(\phi)$; this contradiz the fact that μ is a minimal set of ϕ. $\square$

3. Proof of Theorem A and Theorem B.

In this section we will present the proofs of our main results, initially we will prove that a necessary condition for the validity of Morse's conjecture is the existence of some type of control over the singular set of the action.

Theorem 3.1 (A). *Let* $N^{n+1} = \mathbb{T}^n \times \mathbb{S}^1$, $n \geq 2$, *then exist an* C^r $\mathbb{R}^2$*-action topological transitive but without metric transitivity on* N *with* $r \geq 1$.

Proof: On Torus $\mathbb{T}^n$, cosider the constant vector field defined by:

$$X(x) = (1, \pi, \pi^2, \cdots, \pi^{n-1}) \tag{3.1}$$

It is known that this field is minimal, that is, all its orbits are dense in $\mathbb{T}^n$. Let Γ_0 be the solution of Equation 3.1 passing through the point $(0, 0, \cdots, 0) \in \mathbb{T}^n$, then $\overline{\Gamma_0} = \mathbb{T}^n$. Let us denote by φ^t the flow associated with the vector field X.

Consider the sets $S_1 = [\{(0, t_2, \cdots, t_n); t_i \in \mathbb{R}\}]$ and $S_2 = [\{(\frac{1}{2}, t_2, \cdots, t_n); t_i \in \mathbb{R}\}]$, where $[A]$ denotes the image by applying the universal covering $p : \mathbb{R}^n \to \mathbb{T}^n$ of the subset A of $\mathbb{R}^n$. Note that $\varphi^{\frac{1}{2}}(S_1) = S_2$. Let $\{p_k\}_{k\in\mathbb{Z}}$ be a sequence of points of $\Gamma_0 \cap S_1$ and let c be a positive real number such that $\displaystyle\sum_{k\geq 0} \frac{2c}{1+k^2} < \frac{1}{2}$.

Let us denote by m the Lebesgue measure on $\mathbb{T}^n$, by m^* the Lebesgue measure on S_1 and let B_k be open balls in S_1 centered at p_k such that $m^*(B_k) = \frac{c}{1+k^2}$ therefore $m^*(B) < \frac{1}{2}$ where $B = \bigcup_{k \in \mathbb{Z}} B_k$.

Considering the sets $F_1 = S_1 \setminus B$, $F_1 = \{\varphi^{\frac{1}{2}}(p); p \in F_1\}$ and $E = \{\varphi^t(p); p \in F_1, 0 \leq t \leq \frac{1}{2}\}$ then $\frac{1}{4} < m(E) < 1$.

Taking the compact $F = F_1 \cup F_2$, then there exists $\delta : \mathbb{T}^n \to [0, \infty)$ of class C^∞ such that $F = \delta^{-1}(0)$ e $\delta(x) > 0$ if $x \notin F$. Considering the new vector field

$$X_1(x) = \delta(x)X(x)$$

We observe that $Sing(X_1) = F$, Γ_0 is still a dense orbit in $\mathbb{T}^n$ of the vector field X_1 and E is a closed and invariant subset such that $\frac{1}{4} < m(E) < 1$.

Thus, the flow associated with the field X_1 is topologically transitive but not metrically transitive.

Finally, on the manifold $N = \mathbb{T}^n \times \mathbb{S}^1$, consider the coordinates (x, θ) such that and the C^∞-action ϕ on N with set of generators $\{X_1, \frac{\partial}{\partial \theta}\}$. Thus, there is $K = E \times \mathbb{S}^1$ compact and ϕ-invariante subset of N such that $0 < m(K) < 1$, where m is the Lebesgue measure on N. This proves the Theorem. ☐

Remark 3.1. *The previous result proves that for Morse's conjecture to be true for $\mathbb{R}^2$-action, we must have some control over the singular set of the action. A first way to have this control is to impose that the singular set is empty. Under these conditions we have that Morse's conjecture is true as we will prove below.*

Theorem 3.2 (B). *Let N^3 be a compact closed manifold and ϕ be a C^2 locallly free $\mathbb{R}^2$-action on N. If ϕ is topological transitive then ϕ is metric transitivity.*

Proof: Consider the regular foliation $\mathcal{F}_\phi$ defined on N induced by ϕ, then this foliations is by planes, cylinders and torus.

If there are not torus then by [12, Theorem 9] and [13] all the leaves are dense and have the same topological type, in particular this action is metrically transitive.

In exist a compact orbit and how the action is topologically transitive, by [13] exist an unique orbit torus T. By Proposition 2.3, $N_1 = N \setminus T$ is a open manifold with a C^2 foliation by planes such that $\phi_1(N_1) = \mathbb{Z}^2$ (this by [13, Item 3]). Finally, by [14, Theorem 2] the foliation in N_1 is minimal; this implies that the actions ϕ is metrically transitive. ☐

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