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A note on Mehler's formula

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Abstract

In this paper we propose an original proof of Mehler expansion of the Gaussian distribution in terms of probabilistic Hermite polynomials.

Keywords . Mehler's formula; Hermite polynomials

1. Introduction. The Hermite (1864) [1] polynomials are largely known because they form an orthogonal basis useful for decomposition. Their uses are broad, as in solving differential equations, in the analysis of the quantum harmonic oscillation (Andreani *et al.*, 2017) [2], in chemometrics (Shakibaei Asli and Flusser, 2018) [3], among others. In probability and statistics they play a role in the Edgeworth expansion of the Gaussian distribution (Hall, 2013) [4], hence in its discretization (Baccini *et al.*, 1994) [5].

Dealing with the multivariate Gaussian distribution, its link with the Hermite polynomials is provided by the formula by Mehler (1866) [6], given by

$$\frac{1}{\sqrt{1-\rho^2}}\,e^{\frac{2\rho xy-\rho^2\left(x^2+y^2\right)}{1-\rho^2}}=\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\widehat{H}_n\left(x\right)\widehat{H}_n\left(y\right),$$

with $|r|<1$, where $\widehat{H}_n\left(x\right)$ are the *physical* Hermite polynomials

$$\widehat{H}_n\left(x\right)=\left(-1\right)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right), n=0,1,2,...$$

The following version of Mehler formula is proposed by Watson (1933), [7]:

$$\frac{1}{\sqrt{\pi\left(1-\rho^2\right)}}e^{\frac{4xy\rho-\left(x^2+y^2\right)\left(1+\rho^2\right)}{2\left(1-\rho^2\right)}}=\sum_{n=0}^{\infty}\frac{e^{-\frac{1}{2}\left(x^2+y^2\right)}}{2^n n!\sqrt{\pi}}\rho^n\widehat{H}_n\left(x\right)\widehat{H}_n\left(y\right).$$

always involving the physical Hermite polynomials.

Kibble (1945) [8] attributes to the same Mehler (1866) [6] the following expression

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$$\frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}} = \frac{e^{-\frac{1}{2}\left(x^2+y^2\right)}}{2\pi} \sum_{n=0}^{\infty} \frac{\rho^n}{n!} H_n\left(x\right) H_n\left(y\right), \tag{1.1}$$

this time concerning the probabilistic Hermite polynomials $H_n\left(x\right)$

$$H_n\left(x\right) = \left(-1\right)^n e^{x^2/2} \frac{d^n}{dx^n}\left(e^{-x^2/2}\right), n = 0, 1, 2, ...$$

Unfortunately no proof of (1.1) is provided, therefore we consider of relevance to propose the following one.

2. Basic results. To prove Mehler's formula, the following lemma is necessary:

Lemma 2.1. *The following identity holds:*

$$e^{-\alpha^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2+2i\alpha u} du.$$

Proof: We deal with the case $\text{Im}\left(\alpha\right) = a > 0$, since the proof when $a < 0$ is similar.

The integrating function $f\left(z\right) = e^{-z^2+2i\alpha z}$ with $\alpha = a + ib \in \mathbb{C}$, is holomorphic in $\mathbb{C}$, and if γ_R is the closed path shown in the following figure

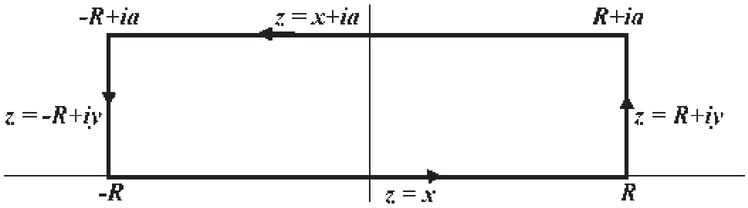

Figure 2.1: *Graph of the curve γ_R.*

we get

$$\int_{\gamma_R} f\left(z\right) dz = 0.$$

Then, we have

$$\int_{-R}^{R} e^{-x^2+2i\alpha x} dx + I_1(R) + \int_{R}^{-R} e^{-(x+b)^2-\alpha^2} dx + I_2(R) = 0, \tag{2.1}$$

where

$$I_1(R) = \int_0^a e^{-(R+iy-i\alpha)^2-\alpha^2} i dy \quad \text{and} \quad I_2(R) = \int_a^0 e^{-(-R+iy-i\alpha)^2-\alpha^2} i dy.$$

The following holds

$$|I_1(R)| \leq \int_0^a \left| e^{-(R+iy-i\alpha)^2-\alpha^2} i \right| dy \leq a e^{-R^2-2bR-a^2},$$

as well as

$$|I_2(R)| \leq \int_0^a \left| e^{-(-R+iy-i\alpha)^2-\alpha^2} i \right| dy \leq a e^{-R^2+2bR-a^2},$$

whence it follows

$$\lim_{R\to\infty} I_1\left(R\right) = \lim_{R\to\infty} I_2\left(R\right) = 0.$$

Then from (2.1) we obtain,

$$\int_{-\infty}^{\infty} e^{-x^2+2i\alpha x} dx = e^{-\alpha^2} \int_{-\infty}^{\infty} e^{-(x+b)^2} dx,$$

and from the well-known result $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$, the thesis follows

$$e^{-\alpha^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2+2i\alpha u} du.$$

3. Mehler's formula. We now give the proof of the main result.

Theorem 3.1. *The standardized bivariate distribution with correlation coefficient* $\rho \in \langle -1, 1 \rangle$ *defined by*

$$F(x,y,\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{x^2+y^2-2xy\rho}{2(1-\rho^2)}},$$

can be expanded as the sum

$$\frac{e^{-\frac{1}{2}\left(x^2+y^2\right)}}{2\pi} \sum_{n=0}^{\infty} \frac{\rho^n}{n!} H_n(x)\, H_n(y),$$

where $H_n(t) = (-1)^n e^{t^2/2} \frac{d^n}{dt^n}\left(e^{-t^2/2}\right)$ *are the probabilistic Hermite polynomials.*

Proof: Applying Lemma 2.1 to the exponential $e^{-x^2/2}$ in $H_n(x)$ and interchanging differentiation and integration, we get

$$e^{-x^2/2} H_n(x) = \frac{(-1)^n}{\sqrt{\pi}} \frac{d^n}{dx^n}\left(\int_{-\infty}^{\infty} e^{-u^2+\sqrt{2}ixu} du\right) = \frac{\left(-\sqrt{2}i\right)^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} u^n e^{-u^2+\sqrt{2}ixu} du$$

$$e^{-y^2/2} H_n(y) = \frac{(-1)^n}{\sqrt{\pi}} \frac{d^n}{dy^n}\left(\int_{-\infty}^{\infty} e^{-v^2+\sqrt{2}iyv} dv\right) = \frac{\left(-\sqrt{2}i\right)^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} v^n e^{-v^2+\sqrt{2}iyv} dv$$

and it results

$$\frac{1}{2\pi} \sum_{n=0}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} \frac{\rho^n}{n!} H_n(x)\, H_n(y) = \frac{1}{2\pi^2} \sum_{n=0}^{\infty} \iint_{\Re^2} \frac{(-2\,\rho\, u\, v)^n}{n!} e^{-u^2-v^2+\sqrt{2}i(xu+yv)} dudv. \tag{3.1}$$

For every $t \in \Re$, let

$$I(t) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2+2tuv-v^2+2au+2bv} dvdu,$$

making the substitution $u = \frac{1}{\sqrt{2}}(w-z)$ and $v = \frac{1}{\sqrt{2}}(w+z)$, we have

$$I(t) = \int_{-\infty}^{\infty} e^{-(1-t)w^2+\sqrt{2}(a+b)w} dw \int_{-\infty}^{\infty} e^{-(1+t)z^2+\sqrt{2}(-a+b)z} dz,$$

which is convergent when $|t| < 1$. This implies that for $\rho < 1$ in (3.1), the sum and the integrals can be interchanged without variation in value. Therefore, we have

$$\frac{1}{2\pi} \sum_{n=0}^{\infty} e^{-\frac{x^2+y^2}{2}} \frac{\rho^n}{n!} H_n(x)\, H_n(y) = \frac{1}{2\pi^2} \iint_{\Re^2} e^{-2\,\rho\, u\, v} e^{-u^2-v^2+\sqrt{2}i(xu+yv)} dudv.$$

Now, according to Fubini's theorem, the second member becomes

$$\frac{1}{2\pi}\left\{\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2+\sqrt{2}ixu} \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2+2i\left(\frac{y}{\sqrt{2}}+i\rho u\right)v} dv\right) du\right\}.$$

Again, using lemma (2.1), we obtain

$$\begin{aligned}
\frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{\rho^n}{n!} H_n(x)\, H_n(y) &= \frac{\sqrt{\pi}}{2\pi^2} \int_{-\infty}^{\infty} e^{-u^2+\sqrt{2}ixu} \left(e^{-\left(\frac{y}{\sqrt{2}}+i\rho u\right)^2}\right) du \\
&= \frac{e^{-\frac{1}{2}y^2}}{2\pi\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\left(1-\rho^2\right)u^2+i\sqrt{2}(x-\rho y)u} du.
\end{aligned}$$

By changing $w = \sqrt{1-\rho^2}\,u$ the last integral becomes

$$\frac{\sqrt{\pi}}{2\pi^2} e^{-\frac{1}{2}y^2} \frac{1}{\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-w^2+2i\frac{1}{\sqrt{2}}(x-\rho y)\frac{w}{\sqrt{1-\rho^2}}} dw,$$

and applying the lemma (2.1) once more, we get eventually

$$\begin{aligned}
\frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{\rho^n}{n!} H_n(x)\, H_n(y) &= \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2}y^2-\left(\frac{(x-\rho y)}{\sqrt{2}\sqrt{1-\rho^2}}\right)^2} \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}} = F(x,y,\rho).
\end{aligned}$$

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