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Some Variants of Wayment's Mean Value Theorem for Integrals

German Lozada-Cruz

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Abstract

This note deals with some variants of Wayment's Mean Value Theorem for integrals. Our approach is rather elementary and does not use advanced techniques from analysis. The simple auxiliary functions were used to prove the results.

Keywords . Flett's theorem, Myers' theorem, Wayment's theorem.

1. Introduction. The Mean Value Theorems are fundamental tools in mathematical analysis. The first mean value theorem we typically encounter is Lagrange's Mean Value Theorem (see e.g. [1, Theorem 2.3] or [2, Theorem 4.12]). A well-known variant of this result was provided by T.M. Flett [3], known as Flett's Mean Value Theorem:

Theorem 1.1 (Flett's Theorem). *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on the interval $[a, b]$ with $f'(a) = f'(b)$. Then, there exists a point $\eta \in (a, b)$ such that*

$$f(\eta) - f(a) = f'(\eta)(\eta - a). \tag{1.1}$$

Geometrically, Flett's Theorem asserts that if the curve $(t, f(t))$ is smooth on the interval $[a, b]$, and the tangent lines at the endpoints $(a, f(a))$ and $(b, f(b))$ are parallel, then there exists a point $\eta \in (a, b)$ such that the tangent line to the graph of f at $(\eta, f(\eta))$ also intersects the point $(a, f(a))$.

Physically, Flett's Theorem states that if a particle follows a smooth trajectory $(t, f(t))$ over the time interval $[a, b]$ and its initial and final velocities are equal, then there exists a moment $\eta \in (a, b)$ at which the particle's instantaneous velocity equals the average velocity of its motion from the initial time a up to η.

Later, R.E. Myers [4] established a slight variation of Flett's theorem:

Theorem 1.2 (Myers' Theorem). *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on $[a, b]$ with $f'(a) = f'(b)$. Then, there exists $\eta \in (a, b)$ such that*

$$f(b) - f(\eta) = f'(\eta)(b - \eta). \tag{1.2}$$

Another important result is the classical Integral Mean Value Theorem, which can be derived directly from Lagrange's theorem and the First Fundamental Theorem of Calculus. It guarantees that every continuous function defined on a closed and bounded interval attains its average value at least once within the interval.

*Departamento de Matemática, Instituto de Biociências, Letras e Ciências Exatas (IBILCE) - Universidade Estadual Paulista (UNESP), 15054-000 São José do Rio Preto, São Paulo, Brazil. **Correspondence author** (german.lozada@unesp.br).

Theorem 1.3 (Integral Mean Value Theorem). *Let $f : [a,b] \to \mathbb{R}$ be a continuous on the interval $[a,b]$. Then, there exists a point $\xi \in (a,b)$ such that*

$$(b-a)f(\xi) = \int_a^b f(x)dx. \tag{1.3}$$

The Mean Value Theorem for Integrals is essential for various applications in mathematics, physics, engineering, and other fields, where the average behavior of functions over intervals is of interest. Also, the mean value theorem for integrals is a powerful tool that enhances our understanding of functions and their integrals, making it a cornerstone of calculus and its applications in various disciplines.

The main motivation for this note comes from the paper by L. Bougoffa [5], where a generalization of Wayment's Integral Mean Value Theorem was established (Theorem 2.1).

In this note, we present some further variants of integral mean value theorems (see Theorems 2.6 and 2.7 in Section 2). To prove these results, we employ simple auxiliary functions that allow us to extend and generalize previous theorems.

2. Variants of Wayment's mean value theorem. In this section, we present and prove our main results.

In 1970, S.G. Wayment [6] established an integral version of Flett's Mean Value Theorem.

Theorem 2.1 (Wayment's Theorem). *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on the interval $[a,b]$ with $f(a) = f(b)$. Then, there exists a point $\xi \in (a,b)$ such that*

$$f(\xi)(\xi - a) = \int_a^\xi f(x)dx. \tag{2.1}$$

In 2021, Lozada-Cruz [7] proved a slight variation of Wayment's Mean Value Theorem.

Theorem 2.2 ([7, Theorem 2.4]). *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on the interval $[a,b]$ with $f(a) = f(b)$. Then, there exists a point $\eta \in (a,b)$ such that*

$$f(\eta)(b - \eta) = \int_\eta^b f(x)dx. \tag{2.2}$$

In 2024, L. Bougoffa [5] proved the following generalization of Wayment's Mean Value Theorem.

Theorem 2.3 (Bougoffa's Theorem). *Let $f, g : [a,b] \to \mathbb{R}$ be continuous functions on the interval $[a,b]$ with $g(x) \neq 0$ for all $x \in [a,b]$. If $f(a) = f(b)$, then there exists a point $\xi \in (a,b)$ such that*

$$\int_a^\xi f(x)g(x)dx = f(\xi)\int_a^\xi g(x)dx. \tag{2.3}$$

Remarks:
(i) If $g(x) = 1$ for all $x \in [a,b]$, we recover Wayment's theorem from (2.3).
(ii) Bougoffa's theorem can also be seen as a new variation of the Second Mean Value Theorem for Integrals (see [8, Theorem 3.3.16]) with Rolle's condition $f(a) = f(b)$.

Bougoffa's proof employs the following auxiliary result:

Theorem 2.4 (Wachnicki's Theorem [9]). *Let $F, G : [a,b] \to \mathbb{R}$ be a differentiable functions on the interval $[a,b]$ with $G'(x) \neq 0$ for all $x \in [a,b]$, and assume that*

$$\frac{F'(a)}{G'(a)} = \frac{F'(b)}{G'(b)}. \tag{2.4}$$

Then, there exists a point $\eta \in (a,b)$ such that

$$\frac{F'(\eta)}{G'(\eta)} = \frac{F(\eta) - F(a)}{G(\eta) - G(a)}. \tag{2.5}$$

In 2020, Lozada-Cruz [10, Theorem 2.4] presented a variant of Wachnicki's theorem.

Theorem 2.5 ([10, Theorem 2.4]). *Let $F, G : [a, b] \to \mathbb{R}$ be a differentiable functions on the interval $[a, b]$ with $G'(x) \neq 0$ for all $x \in [a, b]$, and assume that*

$$\frac{F'(a)}{G'(a)} = \frac{F'(b)}{G'(b)}. \tag{2.6}$$

Then, there exists a point $\eta \in (a, b)$ such that

$$\frac{F'(\eta)}{G'(\eta)} = \frac{F(b) - F(\eta)}{G(b) - G(\eta)}. \tag{2.7}$$

We now present our main result: a variant of Bougoffa's theorem.

Theorem 2.6. *Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions on $[a, b]$ with $g(x) \neq 0$ for all $x \in [a, b]$. If $f(a) = f(b)$, then there exists $\eta \in (a, b)$ such that*

$$\int_\eta^b f(x)g(x)dx = f(\eta)\int_\eta^b g(x)dx. \tag{2.8}$$

Remarks:
(i) For $g(x) = 1$, Theorem 2.6 recovers Theorem 2.2.
(ii) Theorem 2.6 can also be interpreted as a new variation of the Second Mean Value Theorem for Integrals (see [8, Theorem 3.3.16]) under the condition $f(a) = f(b)$.

Proof of Theorem 2.6. Define the functions $F, G : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_x^b f(t)dt \ \text{ and } \ G(x) = \int_x^b g(t)dt.$$

Since f and g are continuous functions on $[a, b]$, both F and G are differentiable on $[a, b]$ with $F'(x) = -f(x)$ and $G'(x) = -g(x)$ for all $x \in [a, b]$.

Thus,

$$F'(a) = -f(a),\ F'(b) = -f(b),\ G'(a) = g(a),\ G'(b) = -g(b).$$

Applying Theorem 2.5 to F and G, under the assumption $\frac{f(a)}{g(a)} = \frac{f(b)}{g(b)}$, then there exists at least one point $\eta \in (a, b)$ such that

$$\frac{F'(\eta)}{G'(\eta)} = \frac{F(b) - F(\eta)}{G(b) - G(\eta)} \Leftrightarrow \frac{f(\eta)}{g(\eta)} = \frac{\int_\eta^b f(x)dx}{\int_\eta^b g(x)dx}. \tag{2.9}$$

Replacing f by fg in (2.9), we obtain (2.8). $\square$

We now consider whether the endpoint derivative condition ($f'(a) = f'(b)$) in Flett's theorem can be replaced with an inequality of the form:

$$f'(a) < f(x) < f'(b) \ \text{ for all } \ x \in [a, b].$$

This possibility is suggested by simple examples, such as $f(x) = x^4$ on $[-1, 1]$, or $f(x) = e^{x^4}$ on $[0, 1]$. $\square$

This observation motivates the next result, which extends Theorem 2.6 under this inequality condition.

Theorem 2.7. *Let $f : [a, b] \to \mathbb{R}$ be differentiable function and let $g : [a, b] \to \mathbb{R}$ be a continuous function with $g(x) > 0$ for all $x \in [a, b]$. If*

$$f'(a) < f(x) < f'(b) \ \textit{for all } x \in [a, b], \tag{2.10}$$

Then, there exists a point $\eta \in (a, b)$ such that

$$\int_\eta^b f(x)g(x)dx = f'(\eta)\int_\eta^b g(x)dx \tag{2.11}$$

Proof. From the inequality, (2.10), we have

$$f'(a)\int_x^b g(t)dt < \int_x^b f(t)g(t)dt < f'(b)\int_x^b g(t)dt, \quad a \leqslant x \leqslant b.$$

Thus,

$$f'(a) < \frac{\int_x^b f(t)g(t)dt}{\int_x^b g(t)dt} < f'(b), \quad \text{for } x \in [a,b).$$

Define the function $B : [a,b] \to \mathbb{R}$ as

$$B(x) = \begin{cases} \dfrac{\int_x^b f(t)g(t)dt}{\int_x^b g(t)dt}, & a \leqslant x < b \\ f(b), & x = b. \end{cases} \tag{2.12}$$

Now define the function $L : [a,b] \to \mathbb{R}$ by $L(x) = B(x) - f'(x)$.

Since B is continuous on $[a,b]$ follows that L is continuous on $[a,b]$, and

$$L(a) = B(a) - f'(a) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} - f'(a) > 0 \quad \text{and}$$
$$L(b) = B(b) - f'(b) = f(b) - f'(b) < 0.$$

By the Intermediate Value Theorem, there exists $\eta \in (a,b)$ such that $L(\eta) = 0$, i.e.,

$$\frac{\int_\eta^b f(x)g(x)dx}{\int_\eta^b g(x)dx} - f'(\eta) = 0 \Leftrightarrow \int_\eta^b f(x)g(x)dx = f'(\eta)\int_\eta^b g(x)dx.$$

□

Corollary 2.1. *If $g(x) = 1$ in Theorem 2.7, then for any differentiable function f on $[a,b]$ satisfying*

$$f'(a) < f(x) < f'(b) \textit{ for all } x \in [a,b],$$

there exists $\eta \in (a,b)$ such that

$$f'(\eta)(b-\eta) = \int_\eta^b f(x)dx. \tag{2.13}$$

The following example illustrates how Theorem 2.7, as presented in this note, with its dual inequalities in the hypothesis, can be employed to confirm the theorem's conclusion and obtain particular estimations.

Example 2.1. *Let $h(x) = e^{-x^2}$ on $[0,1]$. Neither Flett's nor Wayment's conditions hold for h, since $h(0) \neq h(1)$ and $h'(0) \neq h'(1)$.*

Consider $f(x) = e^{x^2}$ and $g(x) = e^{-x^2}$ on $[0,1]$, where

$$f'(0) = 0 < e^{x^2} < f'(1) = 2e \textit{ for } x \in [0,1]. \tag{2.14}$$

Applying Theorem 2.7 gives:

$$\frac{\int_\eta^1 e^{-x^2}dx}{1-\eta} = \frac{e^{-\eta^2}}{2\eta}, \textit{ for some } \eta \in (0,1). \tag{2.15}$$

A numerical approximation yields $\eta \approx 0.59$.

3. Conclusion. In this note, we have presented some variants of Wayment's Integral Mean Value Theorem. Theorem 2.6 extends Bougoffa's result. Theorem 2.7 introduces an inequality-based condition, providing an alternative to the classical endpoint derivative hypothesis. These results, while elementary, enrich the theory of mean value theorems by offering additional formulations and insights. They may be of interest both for pedagogical purposes and for further theoretical exploration in real analysis.

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Conflicts of interest. The author declare no conflict of interest.

ORCID and License

German Lozada-Cruz https://orcid.org/0000-0003-2860-954X

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