Journal homepage http://revistas.unitru.edu.pe/index.php/SSMM

SELECCIONES MATEMÁTICAS
Universidad Nacional de Trujillo
ISSN: 2411-1783 (Online)
2025; Vol.12(1):44-61.

Traveling waves in a delayed reaction-diffusion SVIR epidemic model with generalized incidence function and imperfect vaccination

Rassim Darazirar

Received, Mar. 3, 2025; *Accepted, Jun. 20, 2025;* *Published, Jul. 27, 2025*

How to cite this article:
Darazirar R. *Traveling waves in a delayed reaction-diffusion SVIR epidemic model withgeneralized incidence function and imperfect vaccination* . Selecciones Matemáticas. 2025;12(1):44–61. http://dx.doi.org/10.17268/sel.mat.2025.01.04

Abstract

This paper is concerned with traveling wave solutions for a delayed reaction-diffusion SVIR epidemic model that includes both general incidence function and imperfect vaccination. In the model, the spread of infection in space is explicitly taken into account by using a heterogeneous environment; it takes into consideration the delay in immune response and inefficiency in vaccinations. The analysis carried out below shows that the basic reproduction number R_0 will be a critical value for determining the existence of traveling waves. More precisely, when $R_0 > 1$ there exists a minimal wave speed $\rho^* > 0$ such that the system admits nontrivial traveling wave solutions for $\rho \geq \rho^*$ whereas no such solutions exist for $\rho < \rho^*$. On the other hand, if $R_0 \leq 1$, there are no traveling wave solutions. The introduction of delays and imperfect vaccination adds richness and complexity to the dynamics, such as possible wave speed adjustments and pattern formations, which are hallmarks of complex systems. This work develops a theoretical framework that shall guide the understanding of how delays, spatial spread, and control measures interact in epidemic systems and offers insights applicable to real-world infectious disease dynamics. Numerical simulations for some typical nonlinear incidence functions are given in the last to illustrate the existence of traveling waves.

Keywords . Imperfect vaccination; minimal wave speed; delay; basic reproduction number.

1. Introduction. Infectious diseases often present with very complex spatial and temporal behaviors. Examples include traveling wave phenomena: the spatial spread of infection fronts through a heterogeneous environment, as a function of dynamic interactions in disease transmission, recovery, immunity, and movement [1, 2, 3, 4, 5]. Mathematical models that capture such dynamics are particularly important for understanding and prediction regarding infectious diseases spread through space and time. According to Murray (2002) [6], Shigesada & Kawasaki, 1997 [7].

The SVIR model is one of the most widespread frameworks in epidemiology [8], dividing a population into four compartments: Susceptible, Vaccinated, Infected, and Recovered. Liu et al. [9] formulated the following system of ordinary differential equations:

$$\begin{cases} \frac{dS}{dt} &= \Lambda - (\mu+\alpha)S(t) - \beta_1 S(t)I(t)), \\ \frac{dV}{dt} &= \alpha S(t) - \beta_2 V(t)I(t) - (\mu+\kappa)V(t), \\ \frac{dI}{dt} &= \beta_1 S(t)I(t) + \beta_2 V(t)I(t) - (\mu+\delta+\xi)I(t). \\ \frac{dR}{dt} &= \delta I(t) + \kappa V(t) - \mu R(t). \end{cases}$$

*Department of Mathematics, Faculty of Exact Sciences and Informatics,
Hassiba Benbouali University, Chlef 02000, Algeria. Laboratory of Mathematics and Applications, Faculty of Exact Sciences and Informatics, Hassiba Benbouali University of Chlef, Hay Essalam, Chlef 02000, Algeria. **Correspondence author** (rassimrassim269@gmail.com).

see also [10, 11, 12]. However, although vaccination remains one of the most important tools for the control of infectious diseases, the real world contains complicating factors like vaccine inefficacy, waning immunity, and delayed responses. Coupling these factors with spatial diffusion in the SVIR model allows for more realistic modeling of epidemic spread and helps in determining optimal intervention strategies. This was noted by Diekmann et al. (2013) [13] and Hethcote (2000) [14].

Traveling waves are solutions of epidemic models that describe the development of an infection front in time. Such waves typically appear for reaction-diffusion models whose reaction term describes local dynamics of a disease, such as infection, recovery, and vaccination, while the diffusion term describes spatial spread due to mobility [15, 16, 17, 18, 19, 20, 21]. In this respect, within the framework of the SVIR model, traveling wave solutions might indicate the speed of epidemic fronts, persistence of infection in spatial domains, and the role of vaccination coverage in mitigating disease spread. This was supported by such works as those of [22, 23].

Policy or other delays in vaccination, and even response times in vaccine efficacy, further complicate the dynamics of epidemic waves. Most of these delays lead to richer dynamics such as oscillations, bistability, and the possibility of backward bifurcation. This problem is further complicated by imperfect vaccinations, which can be described by partial immunity or incomplete coverage; indeed, imperfect vaccinations may allow infection to persist in vaccinated populations or result in traveling waves with slower or fragmented fronts (Wang et al., 2012 [24]; Liu & Wu, 2020 [25]).

In this paper, the authors consider traveling waves in a generalized diffusive SVIR epidemic model with delay and imperfect vaccination, which includes spatial diffusion, time delays in the vaccination effect, and the reduction of vaccine efficacy. The conditions for traveling waves to arise, the effects of delays on wave speed, and the role of vaccination in the control of disease spread over space were considered both analytically and numerically. These findings have a critical implication for epidemic management when vaccinations are poorly effective or late. In light of the aforementioned issues, we provide the following diffusive delay SVIR epidemic model with a nonlinear incidence rate.

$$\begin{cases} \frac{\partial S}{\partial t} &= d_1\Delta S + \Lambda - (\mu+\alpha)S(x,t) - F_1(S(x,t), I(x,t)), \\ \frac{\partial V}{\partial t} &= d_2\Delta V + \alpha S(x,t) - F_2(V(x,t), I(x,t)) - (\mu+\kappa)V(x,t), \\ \frac{\partial I}{\partial t} &= d_3\Delta I + F_1(S(x,t-\tau), I(x,t-\tau)) + F_2(V(x,t-\tau), I(x,t-\tau)) - (\mu+\delta+\xi)I(x,t), \\ \frac{\partial R}{\partial t} &= d_4\Delta R + \delta I(x,t) + \kappa V(x,t) - \mu R(x,t). \end{cases} \tag{1.1}$$

with $t > 0$ and $x \in \mathbb{R}$. Therefore, $S(x,t), V(x,t), I(x,t)$ are the density of the susceptible, vaccinated, infected population at time t and position x, respectively. The non-dependence of the three equations of the system (1.1) to $R(x,t)$ allows for the omission of the equation of the recovered population (the fourth equation of (1.1)), and the asymptotic behaviour of this equation can be inferred by examining the evolution of the solution of the first three equations. The constant that enters the S-class is Λ. The constant natural death rate is μ. The transmission functions for people in the S and V classes are denoted by $F_j (j = 1, 2)$, respectively. The immunisation rate is α. The approximate amount of time spent in V-class prior to gaining immunity is $\frac{1}{\kappa}$ and ξ is death rate of the disease. The average duration of the infectious period is $\frac{1}{\delta}$. It is demonstrated that the basic reproduction number (BRN) R_0 can exhibit threshold behaviour in the model (1.1). We presume

(A) d_j are positive, and $\mu, \tau, \delta, \gamma, \kappa, \xi > 0$ for $j = 1, 2, 3, 4$. Also, we suppose that $F_1, F_2 \in C^2(\mathbb{R}^2)$, and satisfies

(H) : $F_1(S,0) = F_1(0,I) = 0$, $\frac{\partial F_1(S,I)}{\partial I} > 0$, $\frac{\partial^2 F_1(S,I)}{\partial I^2} < 0$ and $\frac{\partial F_1(S,I)}{\partial S} > 0$ for all $S, I > 0$.

(D) : $F_2(V,0) = F_2(0,I) = 0$, $\frac{\partial F_2(V,I)}{\partial I} > 0$, $\frac{\partial^2 F_2(V,I)}{\partial I^2} < 0$ and $\frac{\partial F_2(V,I)}{\partial V} > 0$ for all $V, I > 0$.

The organization of this paper is as follows. In section 2, we proved the existence of traveling wave solutions of (1.1) for by applying Schauder's fixed point theorem and Lyapunov method. In section 3, we show that the existence of traveling wave solutions of (1.1) for $\rho = \rho^*$. Furthermore, we investigate the nonexistence of traveling wave solutions under some conditions in section 4. At last, there is a brief numerical simulation.

2. Existence of traveling wave solutions for $\rho > \rho^*$. The existence of traveling wave solutions of system (1.1) is examined in this section. We must examine the following subsystem of (1.1) as we have assumed that the recovered have acquired permanent immunity and that $R(x,t)$ is decoupled from other equations.

$$\begin{cases} \frac{\partial S}{\partial t} &= d_1\Delta S + \Lambda - (\mu+\alpha)S(x,t) - F_1(S(x,t),I(x,t)), \\ \frac{\partial V}{\partial t} &= d_2\Delta V + \alpha S(x,t) - F_2(V(x,t),I(x,t)) - (\mu+\kappa)V(x,t), \\ \frac{\partial I}{\partial t} &= d_3\Delta I + F_1(S(x,t-\tau),I(x,t-\tau)) + F_2(V(x,t-\tau),I(x,t-\tau)) - (\mu+\delta+\xi)I(x,t), \end{cases} \tag{2.1}$$

We must determine the constant equilibria of (2.1) in order to examine its TWS. $(S^0,V^0,0) = (\frac{\Lambda}{\mu+\alpha}, \frac{\alpha\Lambda}{(\mu+\alpha)(\mu+\kappa)}, 0)$ is the DFE of (2.1), and it is always present. To achieve the positive equilibrium, the following ODE system is comparable and should be taken into account.

$$\begin{cases} \frac{dS}{dt} &= \Lambda - (\mu+\alpha)S(t) - F_1(S(t),I(t)), \\ \frac{dV}{dt} &= \alpha S(t) - F_2(V(t),I(t)) - (\mu+\kappa)V(t), \\ \frac{dI}{dt} &= F_1(S(t-\tau),I(t-\tau)) + F_2(V(t-\tau),I(t-\tau)) - (\mu+\delta+\xi)I(t). \end{cases} \tag{2.2}$$

The corresponding (BRN) R_0 is given as follows

$$R_0 = \frac{\frac{\partial F_1(S^0,0)}{\partial I} + \frac{\partial F_2(V^0,0)}{\partial I}}{\mu+\delta+\xi}.$$

It worth noting that if $R_0 > 1$, then (2.2) admits a unique endemic equilibrium $E^* = (s^*, v^*, i^*)$. For the proof of the existence and uniqueness of E^* we refer [26, 27] .

We consider $(S^0,V^0,0)$ to be the initial DFE. This particulate $(s(\zeta),v(\zeta),i(\zeta))$, with $\zeta = x+\rho t$ fulfills

$$\begin{cases} \rho s'(\zeta) &= d_1 s''(\zeta) + \Lambda - (\mu+\alpha)s(\zeta) - F_1(s(\zeta),i(\zeta)), \\ \rho v'(\zeta) &= d_2 v''(\zeta) + \alpha s(\zeta) - F_2(v(\zeta),i(\zeta)) - (\mu+\kappa)v(\zeta), \\ \rho i'(\zeta) &= d_3 i''(\zeta) + F_1(s(\zeta-\rho\tau),i(\zeta-\rho\tau)) + F_2(v(\zeta-\rho\tau),i(\zeta-\rho\tau)) - (\mu+\delta+\xi)i(\zeta), \end{cases} \tag{2.3}$$

with the boundary conditions

$$(s,v,i)(-\infty) = (S^0,V^0,0), \quad (s,v,i)(+\infty) = (S^*,,V^*,I^*). \tag{2.4}$$

We intend to establish a positive solution of (2.3) that satisfies the boundary condition (2.4). Linearizing the second equation of the system (2.3) at E_0, to obtain

$$-\rho i'(\zeta) + d_2 i''(\zeta) + \frac{\partial F_1(S^0,0)}{\partial i} i(\zeta-\rho\tau) + \frac{\partial F_2(V^0,0)}{\partial i} i(\zeta-\rho\tau) - (\mu+\delta+\xi)i(\zeta) = 0 \tag{2.5}$$

Plugging $i(\zeta) = \exp^{\lambda\zeta}$ into (2.5) to get the following characteristic equation

$$G(\lambda,\rho) := -\rho\lambda + d_2\lambda^2 + \left[\frac{\partial F_1(S^0,0)}{\partial i} + \frac{\partial F_2(V^0,0)}{\partial i}\right] \exp^{-\rho\tau\lambda} - (\mu+\delta+\xi) = 0. \tag{2.6}$$

Since $R_0 > 1$,as easy calculations, we have

$$G(0,\rho) = \left[\frac{\partial F_1(S^0,0)}{\partial i} + \frac{\partial F_2(V^0,0)}{\partial i}\right] - (\mu+\xi+\delta) = (\mu+\xi+\delta)[R_0 - 1] > 0,$$

$$\left.\frac{\partial G(\lambda,\rho)}{\partial\lambda}\right|_{\lambda=0} = -\rho - \rho\tau\left[\frac{\partial F_1(S^0,0)}{\partial i} + \frac{\partial F_2(V^0,0)}{\partial i}\right] < 0,$$

means that G is convex in λ. We note that

$$\frac{\partial^2 G(\lambda,\rho)}{\partial\lambda^2} = 2 + \rho^2\tau^2\lambda^2\left[\frac{\partial F_1(S^0,0)}{\partial i} + \frac{\partial F_2(V^0,0)}{\partial i}\right] \exp^{-\rho\tau\lambda} > 0,$$

and

$$G(\lambda,0) = \lambda^2 + \left[\frac{\partial F_1(S^0,0)}{\partial i} + \frac{\partial F_2(V^0,0)}{\partial i}\right] - (\mu+\xi+\delta) = \lambda^2 + (\mu+\xi+\delta)[R_0 - 1] > 0,$$

$$\frac{\partial G(\lambda,\rho)}{\partial\rho} = -\lambda - \tau\lambda\left[\frac{\partial F_1(S^0,0)}{\partial i} + \frac{\partial F_2(V^0,0)}{\partial i}\right] \exp^{-\rho\tau\lambda} < 0, \;\; \forall\lambda > 0.$$

Given the aforementioned characteristics of the function $G(\lambda,\rho)$, we may make the following observations: $\rho > 0$.

$$\frac{\partial G(\lambda,\rho)}{\partial \lambda} = 2\lambda - \rho - \rho\tau\left[\frac{\partial F_1(S^0,0)}{\partial i} + \frac{\partial F_2(V^0,0)}{\partial i}\right]\exp^{-\rho\tau\lambda},$$

and

$$\frac{\partial^2 G(\lambda,\rho)}{\partial \lambda^2} = 2 + \rho^2\tau^2\left[\frac{\partial F_1(S^0,0)}{\partial i} + \frac{\partial F_2(V^0,0)}{\partial i}\right]\exp^{-\rho\tau\lambda} > 0,$$

with

$$\lim_{\lambda\to+\infty} G(\lambda) = +\infty \text{ and } G(0) = \left[\frac{\partial F_1(S^0,0)}{\partial i} + \frac{\partial F_2(V^0,0)}{\partial i}\right] - (\mu+\delta+\xi) = (\mu+\xi+\delta)[R_0-1] > 0.$$

where

$$\frac{\partial G(\lambda,\rho)}{\partial \lambda}|_{\lambda=\lambda^*} = 2\lambda^* - \rho - \rho\tau\left[\frac{\partial F_1(S^0,0)}{\partial i} + \frac{\partial F_2(V^0,0)}{\partial i}\right]\exp^{-\rho\tau\lambda^*} = 0,$$

hence

$$\frac{2\lambda^* - \rho}{\rho\tau} = \left[\frac{\partial F_1(S^0,0)}{\partial i} + \frac{\partial F_2(V^0,0)}{\partial i}\right]\exp^{-\rho\tau\lambda^*}$$

we remplace the resulte in $F(\lambda^*)$, we get

$$G(\lambda^*) = -\rho\lambda^* + \lambda^{*2} + \frac{2\lambda^* - \rho}{\rho\tau} - (\mu+\delta+\xi) \geq 0,$$

hence,

$$\rho^2\tau\lambda^* + \rho(\tau(\mu+\delta+\xi) - \lambda^{*2}\tau + 1) - 2\lambda^* = 0.$$

The sign rule of Descartes states that there exists $\rho^* > 0$. Consequently, the following outcomes are obtained by examining the characteristic equation (2.6).

Lemma 2.1. *Suppose* $R_0 = \dfrac{\left[\frac{\partial F_1(S^0,0)}{\partial i} + \frac{\partial F_2(V^0,0)}{\partial i}\right]}{\mu+\delta+\xi} > 1$, $\exists\, \rho^* > 0$ *and* $\lambda^* > 0$ *such that*

$$\left.\frac{\partial G(\lambda,\rho)}{\partial \lambda^2}\right|_{(\lambda^*,\rho^*)} = 0 \ \text{ and } \ G(\lambda^*,\rho^*) = 0$$

Furthermore, the following alternatives hold:

(i) If $0 < \rho < \rho^*$*, thus* $G(\lambda,\rho) > 0$ *for all* $\lambda \in (0,\lambda_\rho)$*, with* $\lambda_\rho \in [0,+\infty[$*,*

(ii) If $\rho > \rho^*$*, hence* $G(\lambda;\rho) = 0$ *has two positive distinct real roots* $\lambda_1(\rho) < \lambda_2(\rho)$ *that satisfy*

$$G(\lambda,\rho)\begin{cases} > \ 0 & \lambda \in (0,\lambda_1(\rho)) \ \cup \ (\lambda_2(\rho),\infty), \\ < \ 0 & \lambda \in (\lambda_1(\rho),\lambda_2(\rho)). \end{cases}$$

Where

$$\rho^* = \sup\{\rho > 0 | G(\lambda,\rho) > 0, \forall\lambda \in \mathbb{R}\},$$

exists and positive.

2.1. Upper-lower solution. We use an iterative process to construct a pair of super and sub solutions of (2.3) for $\rho > \rho^*$. Specifically, we construct the s, v-components of the supper solution s^+, v^+ first, and then the i-component of the supper solution i^+ using that equation. Using i^+ in turn yields the lower solution s^- for s, v-components. Finally, we use s^-, v^- to construct the i-component of the lower solution i^-. The idea underlying such a structure is

Definition 2.1. (s^+,v^+,i^+) *and* (s^-,v^-,i^-) *denote the upper and lower solutions of* (2.3)*, respectively, and fulfil*

$$-\rho(s^+)'(\zeta) + d_1(s^+)''(\zeta) + \Lambda - (\mu+\alpha)(s^+)(\zeta) - F_1((s^+)(\zeta),(v^-)(\zeta)) \leq 0, \tag{2.7}$$

$$-\rho(s^-)'(\zeta) + d_1(s^-)'' + \Lambda - (\mu+\alpha)(s^-)(\zeta) - F_1((s^-)(\zeta),(v^+)(\zeta)) \geq 0, \tag{2.8}$$

$$-\rho(v^+)'(\zeta) + d_2(v^+)''(\zeta) + \alpha(s^+)(\zeta) - F_2((v^+)(\zeta),(i^-)(\zeta)) - (\mu+\kappa)v^+(\zeta) \leq 0, \tag{2.9}$$

$$-\rho(v^-)'(\zeta) + d_2(v^-)'' + \alpha(s^-)(\zeta) - F_2((v^-)(\zeta),(i^+)(\zeta)) - (\mu+\kappa)v^-(\zeta) \geq 0 \tag{2.10}$$

$$\begin{aligned}-\rho(i^+)'(\zeta) + d_3(i^+)'' + F_1((s^+)(\zeta-\rho\tau),(i^+)(\zeta)) + F_2((v^+)(\zeta-\rho\tau),(i^+)(\zeta-\rho\tau))&\\ -(\mu+\delta+\xi)i^+(\zeta) \leq 0,&\end{aligned} \tag{2.11}$$

$$\begin{aligned}-\rho(i^-)'(\zeta) + d_3(i^-)''F_1((s^-)(\zeta-\rho\tau),(i^-)(\zeta)) + F_2((v^-)(\zeta-\rho\tau),(i^-)(\zeta-\rho\tau))&\\ -(\mu+\delta+\xi)i^-(\zeta) \geq 0,&\end{aligned} \tag{2.12}$$

except for finite points of $\zeta \in \mathbb{R}$.

Lemma 2.2.

$$\begin{aligned} &s^+(\zeta) = S^0, && i^+(\zeta) = \min\{\exp^{\lambda_1\zeta}, B_1\} \\ &v^+(\zeta) = V^0, && s^-(\zeta) = \max\left\{S^0 - M_1 \exp^{\vartheta\zeta}, 0\right\}, \\ &i^-(\zeta) = \max\{\exp^{\lambda_1\zeta}(1 - Le^{\varrho\zeta}), 0\}, && v^-(\zeta) = \max\left\{V^0 - M_2 \exp^{\vartheta\zeta}, 0\right\} \end{aligned} \tag{2.13}$$

for some positive constants ϑ, M_j, L $(j = 1, 2)$*, then* (2.7)-(2.12) *are satisfied.*

We can make sure that B *is the unique positive root of* $F_1(s, \xi) + F_2(v, \xi) = \xi \frac{\partial F_1(S^0,0)+\partial F_2(V^0,0)}{R_0}$, *where* $\partial F_1(S^0, 0) = \frac{\partial F_1(S^0,0)}{\partial i}$ *and* $\partial F_2(V^0, 0) = \frac{\partial F_2(V^0,0)}{\partial i}$. *The following proposition is thus obtained.*

Proposition 2.1. *The following algebraic equation*

$$F_1(S^0, B) + F_2(V^0, B) - (\mu + \delta + \xi)B = 0, \tag{2.14}$$

admits at least one positive solution.

Proof: The proof is obtained by the following points.

(i): Simply $s^+(\zeta) = S^0$ and $v^+(\zeta) = V^0$ satisfies

$$\begin{aligned} -\rho(s^+)'(\zeta) + d_1(s^+)'' + \Lambda - \mu s^+(\zeta) - F_1(s^+(\zeta), i^-(\zeta)) &\leq 0, \\ -\rho(v^+)'(\zeta) + d_2(v^+)''(\zeta)\alpha(s^+)(\zeta) - F_2((v^+)(\zeta), (i^-)(\zeta)) - (\mu + \kappa)v^+(\zeta) &\leq 0 \end{aligned} \tag{2.15}$$

therefore, (2.7) and (2.9) are clear to proof.

(ii) Obviously, for $\zeta > \zeta_0$, with $\zeta_0 = \frac{lnB}{\lambda_1}$, we have $i^+(\zeta) = B_1$, and then $i^+(\zeta - \rho\tau) \leq B$. So, we get

$$d_2(i^+)'' + F_1((s^+)(\zeta - \rho\tau), (i^+)(\zeta)) + F_2((v^+)(\zeta - \rho\tau), (i^+)(\zeta - \rho\tau)) - (\mu + \xi + \delta)i^+(\zeta) - \rho(i^+)'(\zeta)$$
$$\leq \left[\partial F_1(S^0, 0) + \partial F_2(V^0, 0)\right] B - (\mu + \xi + \delta)B = 0.$$

For $\zeta < \zeta_0$, we obtain $i^+(\zeta) = \exp^{\lambda_1\zeta}$, we show that $i^+(\zeta)$ fulfills (2.9). Therefore, we have

$$\begin{aligned} & d_2(i^+)''(\zeta)F_1((s^+)(\zeta - \rho\tau), (i^+)(\zeta)) + F_2((v^+)(\zeta - \rho\tau), (i^+)(\zeta - \rho\tau)) \\ & \qquad -(\mu + \xi + \delta)(i^+)(\zeta) - \rho(i^+)'(\zeta) \\ \leq \; & d_2(i^+)''(\zeta) + \left[\partial F_1(S^0, 0) + \partial F_2(V^0, 0)\right] i^+(\zeta - \rho\tau) - (\mu + \xi + \delta)(i^+)(\zeta) - \rho(i^+)'(\zeta), \\ \leq \; & -\rho(i^+)'(\zeta) + d_2(i^+)''(\zeta) + \left[\partial F_1(S^0, 0) + \partial F_2(V^0, 0)\right] i^+(\zeta - \rho\tau) - (\mu + \xi + \delta)i^+(\zeta), \\ = \; & d_2\lambda_1^2 \exp^{\lambda_1\zeta} + \left[\partial F_1(S^0, 0) + \partial F_2(V^0, 0)\right] \exp^{\lambda_1(\zeta - \rho\tau)} - (\mu + \delta + \xi) \exp^{\lambda_1\zeta} - \rho\lambda_1 \exp^{\lambda_1\zeta}, \\ = \; & \exp^{\lambda_1\zeta} G(\lambda_1, \rho), \\ = \; & 0, \end{aligned}$$

by the definition of λ_1 .

(iii) Taking $0 < \vartheta < \min\left\{\lambda_1, \frac{\rho}{d_1}\right\}$. Suppose that $\zeta \neq \frac{1}{\vartheta} ln \frac{1}{M_1} := \zeta^*$, and we claim that s^- satisfies

$$-\rho(s^-)'(\zeta) + d_1(s^-)''(\zeta) + \Lambda - (\mu + \alpha)(s^-)(\zeta) - F_1(s^-(\zeta), i^+(\zeta)) \geq 0.$$

To illustrate this claim, we first make the assumption that $\zeta > \zeta^*$. This implies that $s^-(\zeta) = 0$ in (ζ^*, ∞), and the inequality is thus instantly true. $s^-(\zeta) = s^0 - M_1 e^{\vartheta\zeta}$ is obtained if

$\zeta < \zeta^*$. We obtain $F_1(s(\zeta), i(\zeta)) \leq \frac{\partial F_1(\frac{\Lambda}{\mu+\alpha},0)}{\partial i} i(\zeta)$ by the concavity of $F_1(s(\zeta), i(\zeta))$. Next, we have

$$\begin{aligned} & -\rho(s^-)'(\zeta) + d_1(s^-)''(\zeta) + \Lambda - (\mu+\alpha)(s^-)(\zeta) - F_1(s(\zeta), i(\zeta)) \geq 0, \\ \geq \quad & \rho M_1 \vartheta \exp^{\vartheta\zeta} + d_1 M_1 \vartheta^2 \exp^{\vartheta\zeta} + \Lambda - (\mu+\alpha)(s^0 - M_1 \exp^{\vartheta\zeta}) - \frac{\partial F_1(S^0,0)}{\partial i} \exp^{\lambda_1\zeta}, \\ = \quad & \exp^{\vartheta\zeta} \left[\rho M_1 \gamma \exp^{\vartheta\zeta} - d_1 M_1 \vartheta^2 \exp^{\vartheta\zeta} - \frac{\partial F_1(S^0,0)}{\partial i} \left(\frac{S^0}{M_1}\right)^{\frac{\lambda_1-\vartheta}{\vartheta}} \right]. \end{aligned}$$

Here we use

$$\exp^{\vartheta\zeta} < \left(\frac{S^0}{M_1}\right)^{\frac{\lambda_1-\vartheta}{\vartheta}} \quad \text{for} \quad \zeta < \zeta^*.$$

Letting $M_1 \to \infty$ for any $M_1 > S^0$ big enough and ϑ small enough, while maintaining $\vartheta M_1 = 1$, gives us

$$-\rho(s^-)'(\zeta) + d_1(s^-)''(\zeta) + \Lambda - (\mu+\alpha)(s^-)(\zeta) - F_1(s^-(\zeta), i^+(\zeta)) \geq 0.$$

Then the equation (2.8) is hold.

(iv) Taking $0 < \vartheta < \min\left\{\lambda_1, \frac{\rho}{d_2}\right\}$. Suppose that $\zeta \neq \frac{1}{\vartheta} ln \frac{1}{M_2} := \zeta^{**}$, and we claim that s^- satisfies

$$-\rho(v^-)'(\zeta) + d_1(v^-)''(\zeta) + \alpha s^-(\zeta) - (\mu+\kappa)(v^-)(\zeta) - F_2(v^-(\zeta), i^+(\zeta)) \geq 0.$$

The proof is similar with $S^-(\zeta)$.

(v) We take $0 < \eta < \min\{\lambda_2 - \lambda_1, \lambda_1\}$, and $L_1 > 0$ sufficiently large. Therefore, we claim that $i^-(\zeta)$ satisfies

$$\begin{aligned} -\rho(i^-)'(\zeta) + d_2(i^-)''(\zeta) + F_1(s(\zeta-\rho\tau), i(\zeta-\rho\tau)) + F_2(v(\zeta-\rho\tau), i(\zeta-\rho\tau)) \\ -(\mu+\xi+\delta) i^-(\zeta) \geq 0, \end{aligned} \tag{2.16}$$

with $\zeta \neq \zeta_2 := \frac{-lnL}{\varrho}$.

For the two different cases, $\zeta < \zeta_2$ and $\zeta > \zeta_2$, respectively, we prove this assertion. $i^-(\zeta) = 0$ indicates that (2.16) is fulfilled if $\zeta > \zeta_2$. $i^-(\zeta) = e^{\lambda_1\zeta}(1 - Le^{\varrho\zeta})$ is acquired in the event that $\zeta < \zeta_2$. Here, we show that for sufficiently large L_1, which will be determined later, (2.16) holds. Note that the inequality (2.16) may be expressed as follows.

$$\begin{aligned} & \left[\partial F_1(S^0,0) + \partial F_2(V^0,0)\right] i^+(\zeta-\rho\tau) - F_1((s^+)(\zeta-\rho\tau), (i^+)(\zeta)) + \\ & \quad + F_2((v^+)(\zeta-\rho\tau), (i^+)(\zeta-\rho\tau)) \\ \leq \quad & -\rho(i^-)'(\zeta) + d_2(i^-)''(\zeta) + \left[\partial F_1(S^0,0) + \partial F_2(V^0,0)\right] i^+(\zeta-\rho\tau) \\ & \quad -(\mu+\delta+\xi) i^-(\zeta) \\ \leq \quad & -L_1 G(\lambda_1+\varrho, \rho) \exp^{(\lambda_1+\varrho)\zeta}. \end{aligned} \tag{2.17}$$

For any $\xi \in (0,1)$, $F_j(y,i)/i$ are a decreasing function on $(0,\infty)$. There exists $\delta_0 > 0$ such that $0 < i^- < \delta_0$ for every $\zeta < \zeta_2$. Since i^- is a bounded function for $\zeta < \zeta_2$, this is the result. $\xi > 0$ must exist as tiny as necessary to guarantee that the following inequality applies since $\partial F_j(y,0) > 0$ and i^-, v^- are constrained for $\zeta < \zeta_2$.For every $0 < i^- < \delta_0$,

$$F_j(y, i^-)/i^- \geq 1 - \xi > 0.$$

With the knowledge that $0 < v^- < \delta_0$, we obtain

$$i - \left[F_1((s,i) + F_2(v,i)\right] = i\left(1 - \frac{F_1(s,i)+F_2(v,i)}{i}\right) \leq \left(\frac{i+1-\frac{F_1(s,i)+F_2(v,i)}{i}}{2}\right)^2 \leq (1-\xi)^2.$$

Since any small $\xi > 0$ can satisfy the aforementioned inequality, we get

$$i - \left[F_1((s,i) + F_2(v,i)\right] \leq i^2, \quad \forall 0 < i \leq \delta_0.$$

Additionally, we suppose that L is sufficiently enough so that $0 < i^-(\zeta) < \delta_0$. By making (2.17)'s left side simpler

$$i - \Big[F_1((S^0,0) + F_2(V^0,0)\Big] \exp^{\lambda_1-\eta} + M_2 \exp^{(\vartheta-\varrho)\zeta} \varepsilon \exp^{\lambda_1-\rho} \leq -LG(\lambda_1+\eta,\rho) \exp^{(\lambda_1+\varrho)\zeta}. \tag{2.18}$$

The inequality (2.18) holds for sufficiently big L as both sides trend to 0 as $\zeta \to -\infty$ and are limited for all $\zeta < \zeta_2$. The proof is finished.

□

2.2. Truncated problem. Next, for this subsection we take $\rho > \rho^*$, we let the truncated problem

$$\begin{cases} \rho s'(\zeta) &= d_1 s''(\zeta) + \Lambda - (\mu+\alpha)s(\zeta) - F_1(s(\zeta), i(\zeta)), \quad \zeta \in I_l = (-l,l) \\ \rho v'(\zeta) &= d_2 v''(\zeta) + \alpha s(\zeta) - F_2(v(\zeta), i(\zeta)) - (\mu+\kappa)v(\zeta), \quad \zeta \in I_l = (-l,l) \\ \rho i'(\zeta) &= d_3 i''(\zeta) + F_1(s(\zeta-\rho\tau), i(\zeta-\rho\tau)) + F_2(v(\zeta-\rho\tau), i(\zeta-\rho\tau)) - (\mu+\delta+\xi)i(\zeta), \\ & \qquad \zeta \in I_l = (-l,l) \\ s(\zeta) &= s^-(\zeta), v(\zeta) = v^-(\zeta), i(\zeta) = i^-(\zeta), \quad \zeta \in \mathbb{R} \setminus I_l, \end{cases} \tag{2.19}$$

where $l > -\zeta_2$. We define the following spaces

$$\mathcal{X} = C(\mathbb{R}) \times C(\mathbb{R}) \times C(\mathbb{R}) \ \text{ and } \ \mathcal{Y} = C^1(I_l) \times C^1(I_l) \times C^1(I_l).$$

The Schauder fixed point theorem will be utilized to demonstrate the existence of a pair of functions $(s,i,v) \in \mathcal{X} \cap \mathcal{Y}$ that fulfill (2.19). Firstly, we define

$$\mathcal{E} = \Big\{(s,v,i) \in \mathcal{X} / s^- \leq s(\zeta) \leq s^+ \ , \ v^- \leq v(\zeta) \leq v^+ \ \text{and} \ i^- \leq i(\zeta) \leq i^+ \ \text{in} \ \mathbb{R}\Big\}, \tag{2.20}$$

that is a closed convex set $\mathcal{X}$ equipped with the norm $\|(f_1,f_2,f_3)\|_{\mathcal{X}} = \|f_1\|_{C(\mathbb{R})} + \|f_2\|_{C(\mathbb{R})} + \|f_3\|_{C(\mathbb{R})}$. Then, we let $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ such that for all $(s_0, v_0, i_0) \in \mathcal{E}$,

$$\mathcal{F}(s_0, v_0, i_0) = (s, v, i),$$

with $(s,v,i) \in \mathcal{X} \cap \mathcal{Y}$ that solves

$$\begin{cases} \rho s'(\zeta) &= d_1 s''(\zeta) + \Lambda - (\mu+\alpha)s(\zeta) - F_1(s_0(\zeta), i(\zeta)), \quad \zeta \in I_l = (-l,l) \\ \rho v'(\zeta) &= d_2 v''(\zeta) + \alpha s(\zeta) - F_2(v_0(\zeta), i(\zeta)) - (\mu+\kappa)v(\zeta), \quad \zeta \in I_l = (-l,l) \\ \rho i'(\zeta) &= d_3 i''(\zeta) + F_1(s_0(\zeta-\rho\tau), i_0(\zeta-\rho\tau)) + F_2(v_0(\zeta-\rho\tau), i_0(\zeta-\rho\tau)) \\ & \qquad -(\mu+\delta+\xi)i(\zeta), \quad \zeta \in I_l = (-l,l) \\ s(\zeta) &= s^-(\zeta), v(\zeta) = v^-(\zeta), i(\zeta) = i^-(\zeta), \quad \zeta \in \mathbb{R} \setminus I_l. \end{cases} \tag{2.21}$$

Any fixed point of $\mathcal{F}$ is the pair $(s,v,i) \in \mathcal{X} \cap \mathcal{Y}$ that fulfill (2.19). Here, we shall confirm that the $\mathcal{F}$ meets the Schauder fixed point theorem's conditions.

Lemma 2.3. *For any $(s_0, v_0, i_0) \in \mathcal{E}$, there is a unique solution $(s,v,i) \in \mathcal{X} \cap \mathcal{Y}$ fulfilling (2.21). Furthermore, $(s,v,i) \in \mathcal{E}$.* *Proof:* As (2.21) is a system of decoupled inhomogeneous linear equations, then the existence and uniqueness of solutions to the (2.21) can be obtained from Theorem 3.1 in Chapter 12 of [28]. Furthermore, as $-\rho s'(\zeta) + d_1 s''(\zeta) - (\mu+\alpha)s(\zeta) - F_1(s(\zeta), i(\zeta)) = -\Lambda \leq 0$ on I_l and $s(\pm l) = s^-(\pm l) \geq 0$, thus $s > 0$ on I_l (by the maximum principle). Similarly, we get $i > 0$ over I_l. Next, we prove that $s^- \leq s(\zeta) \leq s^+$ in I_l. By the first equation of (2.19) and $i_0 \leq i^+$, we arrive at

$$-\rho s'(\zeta) + d_1 s''(\zeta) + \Lambda - (\mu+\alpha)s(\zeta) - F_1(s(\zeta), i^+(\zeta)) \ \leq \ 0,$$

Together with (2.15), we notice that $w_1 = s - s^-$ verifies $d_1 w_1''(\zeta) - \rho w_1'(\zeta) - \Big[\mu + \alpha + \frac{F_1(w_1(\zeta), i^+(\zeta))}{w_1(\zeta)}\Big] w_1(\zeta) \leq 0$. In addition, from the third line of (2.21) and since $s(\zeta_1) > 0$ and $s^-(\zeta_0) = 0$, it is known that $w_1(\zeta) > 0$ and $w_1(l) = 0$. Thus, the maximum principle gives $w_1 \geq 0$

in $(-l,\zeta_0)$, that implies $s^- \leq s(\zeta)$. Together with $s^- = 0$ in $[\zeta_1, l)$, yields $s^* \leq s(\zeta)$ in I_l. Next, showing that $s \leq s^+$ in I_l. Since $i_0 \geq i^-$, then,

$$-\rho s'(\zeta) + d_1 s''(\zeta) + \Lambda - (\mu+\alpha)s(\zeta) - F_1(s(\zeta), i(\zeta)) \quad \geq \quad 0 \;\text{ in }\; I_l, \tag{2.22}$$

Noting $s(\pm l) \leq s^+(\pm l)$, then, by (2.22) and the maximum principle yield $s \leq s^+$ in I_l. Next, as $-\rho v'(\zeta)+d_2 v''(\zeta)-(\mu+\kappa)v(\zeta)-F_2(v(\zeta),i(\zeta)) = -\alpha s(\zeta) \leq 0$ on I_l and $v(\pm l) = v^-(\pm l) \geq 0$, thus $v > 0$ on I_l (by the maximum principle). Similarly, we get $i > 0$ over I_l. Next, we prove that $v^- \leq v(\zeta) \leq v^+$ in I_l. By the first equation of (2.19) and $i_0 \leq i^+$, we arrive at

$$-\rho v'(\zeta) + d_2 v''(\zeta) - (\mu+\kappa)v(\zeta) - F_2(v(\zeta), i(\zeta)) + \alpha s(\zeta) \leq 0$$

Together with (2.15), we notice that $w_2 = v - v^-$ verifies $d_2 w_2''(\zeta) - \rho w_2'(\zeta) - \Big[\mu + \kappa + \frac{F_2(w_2(\zeta), i^+(\zeta))}{w_2(\zeta)}\Big] w_2(\zeta) \leq 0$. In addition, from the third line of (2.21) and since $v(\zeta_1) > 0$ and $v^-(\zeta_0) = 0$, it is known that $w_2(\zeta) > 0$ and $w_2(l) = 0$. Thus, the maximum principle gives $w_2 \geq 0$ in $(-l,\zeta_0)$, that implies $v^- \leq v(\zeta)$. Together with $v^- = 0$ in $[\zeta_1, l)$, yields $v^* \leq v(\zeta)$ in I_l. Next, showing that $v \leq v^+$ in I_l. Since $i_0 \geq i^-$, then,

$$-\rho v'(\zeta) + d_2 v''(\zeta) - (\mu+\kappa)v(\zeta) - F_2(v(\zeta), i(\zeta)) + \alpha s(\zeta) \geq 0 \;\text{ in }\; I_l, \tag{2.23}$$

Noting $v(\pm l) \leq a^+(\pm l)$, then, by (2.22) and the maximum principle yield $v \leq v^+$ in I_l. Now, claiming that $i^- \leq i \leq i^+$ in I_l. Since

$$F_1(s^-(\zeta), i^-(\zeta)) \leq F_1(s_0(\zeta), i_0(\zeta)) \leq F_1(s^+(\zeta), i^+(\zeta)),$$

and

$$F_2(v^-(\zeta), i^-(\zeta)) \leq F_2(v_0(\zeta), i_0(\zeta)) \leq F_2(v^+(\zeta), i^+(\zeta)),$$

it follows that

$$-\rho i'(\zeta) + d_3 i''(\zeta) + F_1(s(\zeta-\rho\tau), i(\zeta-\rho\tau)) + F_2(v(\zeta-\rho\tau), i(\zeta-\rho\tau)) - (\mu+\delta+\xi)i(\zeta), \leq 0$$

and

$$-\rho i'(\zeta)+d_3 i''(\zeta)+F_1(s^+(\zeta-\rho\tau), i^+(\zeta-\rho\tau))+F_2(v^+(\zeta-\rho\tau), i^+(\zeta-\rho\tau))-(\mu+\delta+\xi)i(\zeta), \geq 0, \;\; z \in I_l.$$

Let $w_3 = i - i^-$. By the second equation of (2.3) and $i(\zeta^*) > 0$, $i^-(\zeta^*) = 0$, we have $w_3(\zeta^*) > 0$, $w_3(-l) = 0$. Also, both (2.10), and

$$-\rho i'(\zeta)+d_3 i''(\zeta)++F_1(s^-(\zeta-\rho\tau), i^-(\zeta-\rho\tau))+F_2(v^-(\zeta-\rho\tau), i^-(\zeta-\rho\tau))-(\mu+\delta+\xi)i(\zeta), \leq 0$$

gives that

$$d_2 w_3''(\zeta) + \rho w_3'(\zeta) - (\mu+\delta+\xi)w_3(\zeta) \leq 0, \;\; \zeta \in (-l, \zeta_2).$$

Therefore, the maximum principle ensures that $w_2 \geq 0$ in $(-l,\zeta^*)$, which means $i^- \leq i$ in $(-l,\zeta_2)$. Together with $i^- = 0 \leq i$ in $[\zeta_2, l)$, then $i_- \leq i$ in I_l. To prove $i \leq i^+$ on I_l, we let $\bar{i}(\zeta) = \exp^{\lambda_1 \zeta}$ satisfie

$$d_3 \bar{i}''(\zeta) - \rho \bar{i}'(\zeta) + + F_1(s^+(\zeta-\rho\tau), i^+(\zeta-\rho\tau)) + F_2(v^+(\zeta-\rho\tau), i^+(\zeta-\rho\tau)) - (\mu+\delta+\xi)\bar{i}(\zeta) = 0 \;\text{ in }\; I_l.$$

Since $F_1(s_0(\zeta), i_0(\zeta)) \leq F_1(s^+(\zeta), \bar{i}(\zeta))$ and $F_2(v_0(\zeta), i_0(\zeta)) \leq F_2(v^+(\zeta), \bar{i}(\zeta))$, then

$$d_3 i''(\zeta) - \rho i'(\zeta) + F_1(s^+(\zeta-\rho\tau), \bar{i}(\zeta-\rho\tau)) + F_2(v^+(\zeta-\rho\tau), \bar{i}(\zeta-\rho\tau)) - (\mu+\delta+\xi)i(\zeta) \geq 0 \;\text{ in }\; I_l.$$

Notice that $i(\pm l) \leq \exp^{\lambda_1 \zeta}$. The maximum principle implies $i(\zeta) \leq \exp^{\lambda_1 \zeta}$ in I_l. Further, as $i^+(\zeta) = \exp^{\lambda_1 \zeta}$ in $[\zeta_0, l)$, then $i \leq i^+$ in $[\zeta_0, l)$. To show $i \leq i^+$ in $(-l, \zeta_0]$, notice that $i(-l) \leq i^+(-l)$ and $i(\zeta_0) \leq \theta_1 \exp^{\lambda_1 \zeta} = i^+(\zeta_0)$. This result, (2.1),

$$d_3 i''(\zeta) - \rho i'(\zeta) + F_1(s^+(\zeta-\rho\tau), i^+(\zeta-\rho\tau)) + F_2(v^+(\zeta-\rho\tau), i^+(\zeta-\rho\tau)) - (\mu+\delta+\xi)i(\zeta) \geq 0$$

and the maximum principle, we obtain $i \leq i^+$ in $(-l, \zeta_0]$. $\square$

Before starting on showing the existence of a fixed point, we consider an axillary result that will be helpful in the proof of the existence of the fixed point, and the TWS. Letting the following problem

$$\vartheta''(\zeta) - A\vartheta'(\zeta) + f(\zeta)\vartheta(\zeta) = h(\zeta). \tag{2.24}$$

with A is a positive constant, and $f, h \in C([a,b])$, with $[a,b]$ is an arbitrary interval of $\mathbb{R}$. The next lemma is the result of Lemma 3.1-3.3 in [29]

Lemma 2.4. *Let $\vartheta \in C([a,b]) \cap C^2((a,b))$ satisfies (2.24) in (a,b) with the boundary conditions $\vartheta(a) = \vartheta(b) = 0$. If*

$$-C_1 \leq f \leq 0 \ \text{ and } \ |h| \leq C_2 \ \text{ on} [a,b],$$

for some constants $C_1, C_2 > 0$, then there is a positive constant C_3, depending only on A, C_1, and $(b-a)$, satisfying

$$\|\vartheta\|_{C([a,b])} + \|\vartheta'\|_{C([a,b])} \leq C_3.$$

Finally, it is possible to confirm that the mapping $\mathcal{F}$ is continuous and precompact by arguing as the proofs of Lemma 4.4-4.5 in [29] and utilizing lemma 2.4. The fixed point $(s_l, i_l, v_l) \in \mathcal{X} \cap \mathcal{Y}$ for $\mathcal{F}$ is then determined by using the Schauder fixed point theorem. This pair satisfies (2.19) and $s^- \leq s_l \leq s^+$, $v^- \leq v_l \leq v^+$ and $i^- \leq i_l \leq i^+$ on $\mathbb{R}$. For the truncated problem (2.19), the existence result is as follows, based on the description above.

Lemma 2.5. *There is $(s_l, v_l, i_l) \in \mathcal{X} \cap \mathcal{Y}$ satisfying (2.19). Moreover,*

$$0 \leq s^- \leq s_l \leq s^+ = S^0, \ \ 0 \leq v^- \leq v_l \leq v^+ \leq V^0 \ \text{ and } \ 0 \leq i^- \leq i_l \leq i^+ \leq B$$

on $\mathbb{R}$.

2.3. Existence of TWS. In this subsection, we use the solution (s_l, v_l, i_l) of (2.19) to prove (s^+, v^+, i^+) (resp. (s^-, v^-, i^-)) is the upper (resp. lower) solution of (2.3), respectively. Also, we will show that $(s, v, i) \to (s^+, v^+, i^+)$ as $\zeta \to +\infty$ by constructing a Lyapunov function. At first, we show that

Lemma 2.6. *The solution (s, v, i) of (2.3) satisfies $(s, v, i) \in \mathcal{E}$ defined by (2.20). Moreover,*

$$0 < s < S^0, \ \ 0 < iv < V^0 \ \text{ and } \ 0 < i < B,$$

for all $\zeta \in \mathbb{R}$.

Proof: Let $\{l_n\}_n \in \mathbb{N}$ be an increasing sequence in (ζ_2, ∞) such that $l_1 > \max\{-\zeta_2, |\zeta_0|\}$ and $l_n \to +\infty$, and let $(s_n, v_n, i_n) \in \mathcal{X} \times \mathcal{Y}$, $n \in \mathbb{N}$, solving (2.19) with $l = l_n$ and (2.5) on $\mathbb{R}$. For any $N \in \mathbb{N}$, we have

$$\{s_n\}_{n \geq N}, \ \ \{v_n\}_{n \geq N} \ \text{ and } \ \{i_n\}_{n \geq N},$$

are uniformly bounded in $[-l_N, l_N]$, by Lemma 2.4, we ensure that

$$\{s_n'\}_{n \geq N}, \quad \{v_n'\}_{n \geq N} \ \text{ and } \ \{i_n'\}_{n \geq N}$$

are also uniformly bounded in $[-l_N, l_N]$. By (2.19), we have that s_n'', v_n'' and i_n'' can be written terms of $s_n, v_n, s_n', v_n', i_n$ and i_n'. This means that s_n'', v_n'' and i_n'' are uniformly bounded in $[-l_N, l_N]$. By a differentiation of the equations of (2.19), and utilizing the boundedness of $s_n, v_n, s_n', v_n', i_n, i_n', s_n''$, v_n'' and i_n'' , we can ensure that

$$\{s_n''\}_{n \geq N}, \{v_n''\}_{n \geq N}, \{i_n''\}_{n \geq N}, \{s_n'''\}_{n \geq N}, \{v_n'''\}_{n \geq N}, \ \text{ and } \ \{i_n'''\}_{n \geq N}$$

are uniformly bounded in $[-l_N, l_N]$. The Arzela-Ascoli theorem, and diagonal process implies that there is a subsequence $\{(s_{n_j}, v_{n_j}, i_{n_j})\}$ of $\{(s_n, i_n, v_n)\}$ satisfies

$$s_{n_j} \longrightarrow s, s_{n_j}' \longrightarrow s', s_{n_j}'' \longrightarrow s'',$$

and

$$v_{n_j} \longrightarrow v, v_{n_j}' \longrightarrow v', v_{n_j}'' \longrightarrow v'',$$

and

$$i_{n_j} \longrightarrow i, i_{n_j}' \longrightarrow i', i_{n_j}'' \longrightarrow i'',$$

uniformly in any compact interval of $\mathbb{R}$ as $n \longrightarrow \infty$, for some s, v, i in $C^2(\mathbb{R})$. the definition of $s^\pm$,$v^\pm$ and $i^\pm$ implies that $(s, v, i) \to (S^0, V^0, 0)$ as $\zeta \to -\infty$. Next, we claim that $0 < s < S^0$, $0 < v < V^0$ and $0 < i < B$ on $\mathbb{R}$. We prove this result by contradiction, we let $i(\tilde{\zeta}_2) = 0$ for some $\tilde{\zeta}_2 \in \mathbb{R}$. Thus $i'(\tilde{\zeta}_2) = 0$. Hence $i \equiv 0$ (by the uniqueness), that is a contradiction with $i \geq i^- > 0$ on $(-\infty, \zeta_2)$. To show that $s < S^0$ and $v < V^0$ on $\mathbb{R}$, assume by contradiction that $s(\tilde{\zeta}_2) = S^0$ and $v(\tilde{\zeta}_2) = V^0$ for some $\tilde{\zeta}_2 \in \mathbb{R}$. Then, $s'(\tilde{\zeta}_2) = 0$, $v'(\tilde{\zeta}_2) = 0$, $s''(\tilde{\zeta}_2) \leq 0$ and $v''(\tilde{\zeta}_2) \leq 0$. Also a contradiction with the first equation of (2.3) and $\zeta = \tilde{\zeta}_2$. The proof is achieved. □

Proposition 2.2. *Let $\rho > \rho^*$, then $-L_1 s(\zeta) < s'(\zeta) < L_2 s(\zeta)$,*

$$\begin{aligned} -L_3 v(\zeta) &< v'(\zeta) < L_3 v(\zeta) \ \ -L_5(F_1(S^0, i(\zeta)) + F_2(V^0, i(\zeta))) \\ &< i'(\zeta) < L_6(F_1(S^0, i(\zeta)) + F_2(V^0, i(\zeta))), \end{aligned}$$

for all $z \geq 0$, some constant $L_j > 0$, $\ i = 1, 2, 3, 4, 5, 6$.

Proof:

(i) We claim that $-L_1 s(\zeta) < s'(\zeta)$, $\zeta \geq 0$, with $L_1 > 0$ sufficiently large to satisfy $L_1 > \max\left\{-\frac{s'(0)}{s(0)}, \frac{P(s^0,B_2)}{\rho\rho_1}\right\}$.We let

$$\phi_1(\zeta) = s'(\zeta) + L_1 s(\zeta).$$

We need to prove that $\phi_1(\zeta) > 0$, $\zeta \geq 0$. Clearly, $\phi_1(0) > 0$. We show this claim by contradiction. Assuming that there is $\hat{\zeta}_1 > 0$ satisfying $\phi_1(\hat{\zeta}_1) = 0$, $\phi_1'(\hat{\zeta}_1) \leq 0$. Thus, we have either

$$\phi_1(\zeta) \leq 0, \forall \zeta \geq \hat{\zeta}_1 \tag{2.25}$$

or

$$\phi_1(\hat{\zeta}_2) = 0 \text{ and } \phi_1'(\hat{\zeta}_2) \geq 0, \text{ for some } \hat{\zeta}_2 \geq \hat{\zeta}_1. \tag{2.26}$$

For (2.25), and using $L_1 \geq \frac{P(s^0,B_2)}{\rho\rho_1}$, we get

$$\rho s'(\zeta) \leq F_1(S^0, B)s(\zeta), \forall \zeta \geq \hat{\zeta}_1.$$

By $0 \leq i \leq B$ and $s < S^0$ then we have $\Lambda > (\mu+\alpha)s(\zeta)$, the first equation of (2.3) implies

$$\begin{aligned}
d_1 s''(\zeta) &= \rho s'(\zeta) - \Lambda + (\mu+\alpha)s(\zeta) + F_1(s(\zeta), i(\zeta)),\\
&\leq \rho s'(\zeta) - \Lambda + (\mu+\alpha)s(\zeta) + F_1(s(\zeta), B),\\
&\leq \rho s'(\zeta) + F_1(s(\zeta), B),\\
&\leq -L_1\rho s(\zeta) + F_1(s(\zeta), B)\\
&\leq s(\zeta)(-\rho L_1 + \tfrac{F_1(S^0,B)}{s(\zeta)})\\
&\leq s(\zeta)(-\rho L_1 + \tfrac{F_1(S^0,B)}{\rho_1}) < 0.
\end{aligned}$$

where $L_1 \geq \frac{F_1(S^0,B_2)}{\rho\rho_1}$ and for all $\zeta \geq \hat{\zeta}_1$, then s' is decreasing in $[\hat{\zeta}_1, \infty)$. Therefore $s'(\zeta) \leq s'(\hat{\zeta}_1) \leq -L_1 s(\hat{\zeta}_1) < 0$, $\zeta \geq \hat{\zeta}_1$, that is a contradiction. To see the contradiction, we know that $s'(\zeta) \leq -k_1$, with $k_1 = L_1 s(\hat{\zeta}_1)$, therefore, by integrating the previous inequality on $[\hat{\zeta}_1, \zeta]$, and letting $\zeta \to +\infty$, then we get $s(\zeta) \longrightarrow -\infty$, which contradicts the boundedness and the positivity of the solution (Lemma 2.6). for $\zeta \longrightarrow +\infty$ we get $s(\zeta) \longrightarrow -\infty$. Hence which contradicts the boundedness and the positivity of $s(\zeta)$. For the second case, (2.26) yields that

$$s'(\hat{\zeta}_2) = -L_1 s(\hat{\zeta}_2) < 0, \quad s''(\hat{\zeta}_2) = -L_1 s'(\hat{\zeta}_2) > 0. \tag{2.27}$$

By the first equation of (2.3), we deduce that

$$\begin{aligned}
0 &= d_1 s''(\hat{\zeta}_2) - \rho s'(\hat{\zeta}_2) + \Lambda - (\mu+\alpha)s(\hat{\zeta}_2) - F_1(s(\hat{\zeta}_2), i(\hat{\zeta}_2)),\\
&\geq -d_1 L_1 s'(\hat{\zeta}_2) + \rho L_1 s(\hat{\zeta}_2) - F_1(s(\hat{\zeta}_2), B),\\
&\geq \rho L_1 s(\hat{\zeta}_2) - F_1(s(\hat{\zeta}_2), B),\\
&\quad \text{by (2.27) and the fact of } s < S^0 \text{ and } i \leq B.\\
&\geq s(\hat{\zeta}_2)[\rho L_1 - \tfrac{F_1(S^0,B)}{\rho_1}],\\
&> 0, \text{ where } L_1 \geq \tfrac{F_1(S^0,B)}{\rho\rho_1}.
\end{aligned}$$

a contradiction again.

(ii) Proving that $s'(\zeta) < L_2 s(\zeta)$, $\zeta \geq 0$. Choosing $L_2 > 0$ sufficiently large to satisfy both $s'(0) < L_2 s(0)$ and $d_1 L_2^2 - \rho L_2 - (\mu+\alpha) - \frac{F_1(S^0,B)}{\rho_1} > 0$. Let

$$\phi_2 = s'(\zeta) - L_2 s(\zeta)$$

We need to prove that $\phi_2(\zeta) < 0$, $\zeta \geq 0$. Notice that $\phi_2(0) < 0$. Arguing by contradiction. Assuming that there is $\hat{\zeta}_3 \geq 0$ verifying $\phi_2(\hat{\zeta}_3) = 0$ and $\phi_2'(\hat{\zeta}_3) \geq 0$. Then

$$s'(\hat{\zeta}_3) = L_2 s(\hat{\zeta}_3), \quad s''(\hat{\zeta}_3) \geq L_2 s'(\hat{\zeta}_3) = L_2^2 s(\hat{\zeta}_3).$$

By $i \leq B$ and $s \leq S^0$, the first equation of (2.3) implies

$$\begin{aligned}
0 &= d_1 s''(\hat{\zeta}_3) - \rho s'(\hat{\zeta}_3) + \Lambda - (\mu+\alpha)s(\hat{\zeta}_3) - F_1(s(\hat{\zeta}_3), i(\hat{\zeta}_3)),\\
&\geq s(\hat{\zeta}_3)\left[d_1 L_2^2 - \rho L_2 - (\mu+\alpha) - \tfrac{F_1(S^0,B)}{\rho_1}\right] > 0,
\end{aligned}$$

that is a contradiction. By the same way we proof $-L_3 v(\zeta) < v'(\zeta) < L_3 v(\zeta)$.

(iii) Now, we show $-L_5(F_1(S^0,i(\zeta))+F_2(V^0,i(\zeta)))<i'(\zeta)$, $\zeta\geq 0$, where $L_5>0$ sufficiently large to verify $-L_5(F_1(S^0,i(\zeta))+F_2(V^0,i(\zeta)))<i'(0)$ and $L_5\geq \frac{\mu+\delta+\xi}{\rho\left(\partial F_1(S^0,0)+\partial F_2(V^0,0)\right)}$.

Let
$$\phi_3(\zeta)=i'(\zeta)+L_5(F_1(S^0,i(\zeta))+F_2(V^0,i(\zeta))).$$
We now prove $\phi_3(\zeta)>0$, $\forall\zeta\geq 0$, clearly $\phi_3(0)>0$. Again, we argue by contradiction, we suppose that there is $\hat{\zeta}_4>0$ that satisfies $\phi_3(\hat{\zeta}_4)=0$, $\phi_3'(\hat{\zeta}_4)\leq 0$. Thus, either we have
$$\begin{aligned}&\phi_3(\zeta)\leq 0,\ \text{for all } \zeta\geq\hat{\zeta}_4,\\ &\text{or } \phi_3(\hat{\zeta}_5)=0\ \text{ and }\ \phi_3'(\hat{\zeta}_5)\geq 0,\ \text{for some } \hat{\zeta}_5\geq\hat{\zeta}_4.\end{aligned}\tag{2.28}$$
For the first, (2.28) gives
$$i'(\zeta)\leq -L_5((F_1(S^0,i(\zeta))+F_2(V^0,i(\zeta)))),\forall z\geq\hat{z}_4.$$
By the third equation of (2.3) and
$$F_1(s(\zeta),i(\zeta))\leq\partial F_1(S^0,0)i(\zeta),\tag{2.29}$$
and
$$F_2(v(\zeta),i(\zeta))\leq\partial F_2(V^0,0)i(\zeta),\tag{2.30}$$
we deduce that
$$\begin{aligned}d_3i''(\zeta)&=\rho i'(\zeta)-F_1(s(\zeta-\rho\tau),i(\zeta-\rho\tau))-F_2(v(\zeta-\rho\tau),i(\zeta-\rho\tau))\\&\qquad\qquad+(\mu+\delta+\xi)i(\zeta),\\&\leq -\rho L_5(F_1(S^0,i(\zeta))+F_2(V^0,i(\zeta)))-F_1(s(\zeta-\rho\tau),i(\zeta-\rho\tau))\\&\quad -F_2(v(\zeta-\rho\tau),I_2(\zeta-\rho\tau))+(\mu+\delta+\xi)i(\zeta),\\&\leq 0,\end{aligned}$$
hence $i'(\zeta)$ decreases in $[\hat{\zeta}_4,\infty)$. Therefore, $i'(\zeta)\leq i'(\hat{\zeta}_4)=-L_5(F_1(S^0,i(\hat{\zeta}_4))+F_2(V^0,i(\hat{\zeta}_4)))<0$, $\forall\zeta\geq\hat{\zeta}_4$, that also a contradiction. For (2.28) and $\partial F_1(V^0,\xi_1)>0$ and $\partial F_2(V^0,\xi_1)>0$ for $\xi_1>0$ gives
$$i'(\hat{\zeta}_5)=-L_5(F_1(S^0,i(\hat{\zeta}_5))-F_2(V^0,i(\hat{\zeta}_5)))<0.\tag{2.31}$$
Thus, by the third equation of (2.3), (2.29) and (2.30) we get
$$\begin{aligned}0&=-d_3i''(\hat{\zeta}_5)+\rho i'(\hat{\zeta}_5)-F_1(s(\hat{\zeta}_5-\rho\tau),i(\hat{\zeta}_5-\rho\tau))-F_2(v(\hat{\zeta}_5-\rho\tau),i(\hat{\zeta}_5-\rho\tau))\\&\quad+(\mu+\delta+\xi)i(\hat{\zeta}_5),\\&\leq\rho i'(\hat{\zeta}_5)+(\mu+\delta+\xi)i(\hat{\zeta}_5),\\&\leq -\rho L_5(F_1(S^0,i(\hat{\zeta}_5))+F_2(V^0,i(\hat{\zeta}_5)))+(\mu+\delta+\gamma)i(\hat{\zeta}_5).\end{aligned}$$
Then, we get
$$\begin{aligned}&-\rho L_5(F_1(S^0,i(\hat{\zeta}_5))+F_2(V^0,i(\hat{\zeta}_5)))+(\mu+\delta+\xi)i(\hat{\zeta}_5)\\&\leq(\mu+\delta+\xi)i(\hat{\zeta}_5)\left[-\frac{\rho L_5(F_1(S^0,i(\hat{\zeta}_5))+F_2(V^0,i(\hat{\zeta}_5)))}{(\mu+\delta+\xi)i(\hat{\zeta}_5)}+1\right],\\&\leq(\mu+\delta+\xi)i(\hat{\zeta}_5)\left[-\frac{\rho L_5(F_1(S^0,B)+F_2(V^0,B))}{(\mu+\delta+\xi)B}+1\right]<0,\end{aligned}$$
where $L_5\geq\frac{(\mu+\delta+\xi)B}{\rho(F_1(S^0,B)+F_2(V^0,B))}$, a contradiction.

(iv) We prove that $i'(\zeta)<L_6(F_1(S^0,i(\zeta))+F_2(V^0,i(\zeta)))$, $\forall\zeta\geq 0$. Selecting $L_6>0$ large enough to satisfy the inequalities $i'(0)<L_6(F_1(S^0,i(\zeta))+F_2(V^0,i(\zeta)))$ and $d_2L_6^2(\partial F_1(S^0,i(\hat{\zeta}_6))+\partial F_2(V^0,i(\hat{\zeta}_6)))-\rho L_6-(\mu+\delta+\xi)\frac{1}{\partial F_1(S^0,0)+\partial F_2(V^0,0)}>0$. Let
$$\phi_4(\zeta)=i'(\zeta)-L_6(F_1(S^0,i(\zeta))+F_2(V^0,i(\zeta))).$$

It remain to prove that $\phi_4(\zeta) < 0, \forall \zeta \geq 0$. As $\phi_4(0) < 0$, and for obtaining a contradiction, we suppose that there is $\hat{\zeta}_6 > 0$ verifies $\phi_4(\hat{\zeta}_6) = 0$ and $\phi_4'(\hat{\zeta}_6) \geq 0$. Then,

$$i'(\hat{\zeta}_6) = L_6(F_1(s(\hat{\zeta}_6), i(\hat{\zeta}_6)) + F_2(v(\hat{\zeta}_6), i(\hat{\zeta}_6))),$$

and

$$\begin{array}{rcl} d_3 i''(\hat{\zeta}_6) & \geq & L_6(\partial F_1(s(\hat{\zeta}_6), i(\hat{\zeta}_6)) + \partial F_2(v(\hat{\zeta}_6), i(\hat{\zeta}_6)))i'(\hat{\zeta}_6), \\ & = & L_6^2(\partial F_1(S^0, i(\hat{\zeta}_6)) + \partial F_2(V^0, i(\hat{\zeta}_6)))i'(\hat{\zeta}_6)(F_1(S^0, i(\hat{\zeta}_6)) + F_2(V^0, i(\hat{\zeta}_6)))i'(\hat{\zeta}_6). \end{array}$$

We deduce from the third and forth equations of (2.19) that

$$\begin{array}{rcl} 0 & = & d_3 i''(\hat{\zeta}_6) - \rho i'(\hat{\zeta}_6) + F_1(s(\zeta - \rho\tau), i(\hat{\zeta}_6 - \rho\tau)) + F_2(v(\zeta - \rho\tau), i(\hat{\zeta}_6 - \rho\tau)) \\ & & -(\mu + \delta + \xi)i(\hat{\zeta}_6), \\ & \geq & d_3 i''(\hat{\zeta}_6) - \rho i'(\hat{\zeta}_6) - (\mu + \delta + \xi)i(\zeta), \\ & \geq & d_3 L_6^2(\partial F_1(S^0, i(\hat{\zeta}_6)) + \partial F_2(V^0, i(\hat{\zeta}_6)))i'(\hat{\zeta}_6)(F_1(S^0, i(\hat{\zeta}_6)) + F_2(V^0, i(\hat{\zeta}_6)))i'(\hat{\zeta}_6) \\ & & -\rho L_6(F_1(S^0, i(\hat{\zeta}_6)) + F_2(V^0, i(\hat{\zeta}_6)))i'(\hat{\zeta}_6) - (\mu + \delta + \xi)\dfrac{(F_1(S^0, i(\hat{\zeta}_6)) + F_2(V^0, i(\hat{\zeta}_6)))i'(\hat{\zeta}_6)}{\partial F_1(S^0, 0) + \partial F_2(V^0, 0)}, \\ & \geq & (F_1(S^0, i(\hat{\zeta}_6)) + F_2(V^0, i(\hat{\zeta}_6)))i'(\hat{\zeta}_6)\Big(d_2 L_6^2(\partial F_1(S^0, i(\hat{\zeta}_6)) + \partial F_2(V^0, i(\hat{\zeta}_6)))i'(\hat{\zeta}_6) \\ & & -\rho L_6 - (\mu + \delta + \xi)\dfrac{1}{\partial F_1(S^0, 0) + \partial F_2(V^0, 0)}\Big) > 0, \end{array}$$

a contradiction. □

To prove the existence of non-critical traveling wave, we need to prove that $(s, i, v) \to (s^*, i^*, v^*)$ as $\zeta \to \infty$ by applying the Lyapunov-LaSalle Theorem. The obtained results are highlighted as follows

Lemma 2.7. $(s, i, v) \to (s^*, i^*, v^*)$ *uniformly as* $\zeta \to +\infty$

Proof: We construct the Lyapunov functional.

$$V(\zeta) = V_1(\zeta) + V_2(\zeta),$$

where

$$\begin{array}{rcl} V_1(\zeta) & = & \rho\left[s - s^* - \displaystyle\int_{s^*}^{s} \frac{F_1(s^*, i^*)}{F_1(\varepsilon, i^*)}d\varepsilon + v - v^* - \int_{v^*}^{v} \frac{F_2(v^*, i^*)}{F_2(\varepsilon, i^*)}d\varepsilon + i^* h\left(\frac{i}{i^*}\right)\right] \\ & & -d_1 s'(\zeta)\left(1 - \frac{s^*}{s(\zeta)}\right) - d_2 v'(\zeta)\left(1 - \frac{v^*}{v(\zeta)}\right) - d_3 i'(\zeta)\left(1 - \frac{i^*}{i(\zeta)}\right), \end{array}$$

$$\begin{array}{rcl} V_2(\zeta) & = & F_1(s^*, i^*)\displaystyle\int_0^\tau h\left(\frac{F_1(s(\zeta - \rho\kappa), i(\zeta - \rho\kappa))}{F_1(s^*, i^*)}\right)d\kappa \\ & & +F_2(v^*, i^*)\displaystyle\int_0^\tau h\left(\frac{F_2(v(\zeta - \rho\kappa), i(\zeta - \rho\kappa))}{F_2(v^*, i^*)}\right)d\kappa, \end{array}$$

It is evident that $h(x) > 0, \forall x > 0$, with $h(x) = x - 1 - ln(x)$; $x \in \mathbb{R}^+$. Next, we have

$$\begin{array}{rcl} \frac{dV_1(\zeta)}{d\zeta} & = & \left(1 - \frac{F_1(s^*, i^*)}{F_1(s(\zeta), i^*)}\right)\left(\rho s'(\zeta) - d_1 s''(\zeta)\right) + \left(1 - \frac{F_2(v^*, i^*)}{F_2(v(\zeta), i^*)}\right)\left(\rho v'(\zeta) - d_2 v''(\zeta)\right) \\ & & +\left(1 - \frac{i^*}{i(\zeta)}\right)\left(\rho i'(\zeta) - d_3 i''(\zeta)\right) - d_1\frac{(s'(\zeta))^2 \frac{\partial F_1(s(\zeta), i^*)}{\partial s}}{F_1(s^*, i^*)}\left(\frac{F_1(s^*, i^*)}{F_1(s(\zeta), i^*)}\right)^2 \\ & & -d_2\frac{(v'(\zeta))^2 \frac{\partial F_2(v(\zeta), i^*)}{\partial v}}{F_2(v^*, i^*)}\left(\frac{F_2(v^*, i^*)}{F_2(v(\zeta), i^*)}\right)^2 - d_3\frac{(i'(\zeta))^2}{i^*}\left(\frac{i^*}{i(\zeta)}\right)^2. \end{array}$$

Note that (s^*, v^*, i^*) satisfies

$$\begin{cases} \Lambda = (\mu + \alpha)s^* + F_1(s^*, i^*), \\ \alpha s^* - F_2(v^*, i^*) = (\mu + \kappa)v^* \\ (\mu + \delta + \xi)i^* = F_1(s^*, i^*) + F_2(v^*, i^*). \end{cases}$$

Then, we obtain

$$\begin{aligned}\frac{dV_1(\zeta)}{d\zeta} = &\left(1-\frac{F_1(s^*,i^*)}{F_1(s(\zeta),i^*)}\right)\Big((\mu+\alpha)s^*+F_1(s^*,i^*)-(\mu+\alpha)s(\zeta)-F_1(s(\zeta),i(\zeta))\Big)\\ &+\left(1-\frac{F_2(v^*,i^*)}{F_2(v(\zeta),i^*)}\right)\Big(-(\alpha s^*-F_2(v^*,i^*))\frac{v(\zeta)}{v^*}+\alpha s(\zeta)-F_2(v(\zeta),i(\zeta))\Big)\\ &+\left(1-\frac{i^*}{i(\zeta)}\right)\Big(F_1(s(\zeta-\rho\tau),i(\zeta-\rho\tau))-F_1(s^*,i^*)\frac{i(\zeta)}{i^*}+F_1(s(\zeta),i(\zeta))\\ &-F_1(s(\zeta),i(\zeta))+F_2(v(\zeta-\rho\tau),i(\zeta-\rho\tau))-F_2(v^*,i^*)\frac{i(\zeta)}{i^*}+F_2(v(\zeta),i(\zeta))\\ &-F_2(v(\zeta),i(\zeta))\Big)-d_1\frac{(s'(\zeta))^2\frac{\partial F_1(s(\zeta),i^*)}{\partial s}}{F_1(s^*,i^*)}\left(\frac{F_1(s^*,i^*)}{F_1(s(\zeta),i^*)}\right)^2-d_2\frac{(v'(\zeta))^2\frac{\partial F_2(v(\zeta),i^*)}{\partial v}}{F_2(v^*,i^*)}\left(\frac{F_2(v^*,i^*)}{F_2(v(\zeta),i^*)}\right)^2\\ &-d_3\frac{(i'(\zeta))^2}{i^*}\left(\frac{i^*}{i(\zeta)}\right)^2.\end{aligned}$$

Now, we compute $\frac{dV_2(z)}{dz}$

$$\begin{aligned}\frac{dV_2(\zeta)}{d\zeta} = &-F_1(s^*,i^*)\Bigg[\left(\frac{F_1(s(\zeta-\rho\tau),i(\zeta-\rho\tau))}{F_1(s^*,i^*)}\right)-1-ln\left(\frac{F_1(s(\zeta-\rho\tau),i(\zeta-\rho\tau))}{F_1(s^*,i^*)}\right)-\left(\frac{F_1(s(\zeta),i(\zeta))}{F_1(s^*,i^*)}\right)\\ &+1+ln\left(\frac{F_1(s(\zeta),i(\zeta))}{F_1(s^*,i^*)}\right)\Bigg]-F_2(v^*,i^*)\Bigg[\left(\frac{F_2(v(\zeta-\rho\tau),i(\zeta-\rho\tau))}{F_2(v^*,i^*)}\right)-1-\\ &-ln\left(\frac{F_2(v(\zeta-\rho\tau),i(\zeta-\rho\tau))}{F_2(v^*,i^*)}\right)-\left(\frac{F_2(v(\zeta),i(\zeta))}{F_2(v^*,i^*)}\right)+1+ln\left(\frac{F_2(v(\zeta),i(\zeta))}{F_2(v^*,i^*)}\right)\Bigg]\end{aligned}$$

By the same way with [8, 15], then we sum $\frac{dV_1(\zeta)}{d\zeta}$ and $\frac{dV_2(\zeta)}{d\zeta}$ to obtain the result

$$\frac{dV(\zeta)}{d\zeta} \leq 0.$$

Therefore, we deduce that $(s,v,i)(\infty)=(s^*,v^*,i^*)$, We let the set D corresponding to (2.3) as follows:

$$\begin{aligned}D = &\Big\{(s,v,i)|0<s<S^0,\ \ 0<v<V^0\ \ 0<i<i^+,\ \ -L_1s<s'<L_2s,\quad -L_3v<v'<L_4v\\ &-L_5(F_1(s,i)+F_2(v,i))<i'<L_6(F_1(s,i)+F_2(v,i))\Big\}.\end{aligned}$$

Then, for every $\zeta\geq 0$, proposition 2.2 suggests that D is positively invariant for (2.2). Keep in mind that the orbital derivative of L along $\Psi(\zeta)$ is non-positive. Moreover, it is evident that V is continuous and confined below on D. $\Psi(\zeta)\to(s^*,v^*,i^*)$ as $z\to\infty$, according to this and the Lyapunov-LaSalle Theorem. Consequently, $(s,v,i)\to(s^*,v^*,i^*)$ as $\zeta\to+\infty$. This brings the proof to an end. □

Note that Lemma 2.6 suggests that the solution of (2.2) satisfies $(s,v,i)\to(S^0,V^0,0)$ as $\zeta\to-\infty$, as well as $s^-\leq s\leq s^+$, $v^-\leq v\leq v^+$, and $i^-\leq i\leq i^+$. $(s,v,i)\to(s^*,v^-,i^*)$ is $\zeta\to+\infty$ according to Lemma 2.7. As a result, we conclude that the travelling wave solution of the system (2.2) is the only positive solution that admits the system and fulfils the boundary conditions (2.4).

3. Existence of a critical TWS . In this section, we want to show that (2.3) permits a TWS for $R_0>1$ and $\rho=\rho^*$.

Lemma 3.1.

$$\begin{aligned}&s^+(\zeta)=S^0, && i^+(\zeta)=\min\{\exp^{\lambda^*\zeta},B_1\}\\ &v^+(\zeta)=V^0, && s^-(\zeta)=\max\Big\{S^0-M_1\exp^{\vartheta\zeta},0\Big\},\\ &i^-(\zeta)=\max\{\exp^{\lambda^*\zeta}(1-Le^{\varrho\zeta}),0\}, && v^-(\zeta)=\max\Big\{V^0-M_2\exp^{\vartheta\zeta},0\Big\}\end{aligned}\tag{3.1}$$

for some positive constants ϑ, M_j, L $(j=1,2)$*, then* (2.7)-(2.12) *are satisfied.*

Since the proof may be accomplished similarly to the proof of Lemma 2.2, we do not provide it here. By replacing ρ by ρ^* and λ_1 by λ^*, the same process as in sections 2 is used to deduce the presence of a TWS for $\rho=\rho^*$.

4. Non existence of a TWS. The next theorem shows the case when the system (2.2) do not admits a traveling waves solutions.

Theorem 4.1. *Assume that* $R_0 > 1$ *and* $0 < \rho < \rho^*$. *Then,* (2.3) *has no traveling wave solution of the form* $(s(x+\rho t), v(x+\rho t), i(x+\rho t))$*, and satisfying the boundary conditions* (2.4)*.*

Proof: We demonstrate the theorem using a contradiction. For the system (2.3), assume that the traveling wave solution $(s(x+\rho t),\, v(x+\rho t),\, i(x+\rho t))$ satisfies the boundary conditions. Consequently, we have $\frac{F_1(s(\zeta-\rho\tau),i(\zeta-\rho\tau))}{i(\zeta-\rho\tau)} \longrightarrow \frac{\partial F_1(S^0,0)}{\partial i}$ and $\frac{F_2(v(\zeta-\rho\tau),i(\zeta-\rho\tau))}{i(\zeta-\rho\tau)} \longrightarrow \frac{\partial F_2(V^0,0)}{\partial i}$ as $\zeta \longrightarrow -\infty$, then, there is $\zeta_1 < 0$ satisfying $\frac{F_1(s(\zeta-\rho\tau),i(\zeta-\rho\tau))+F_2(v(\zeta-\rho\tau),i(\zeta-\rho\tau))}{i(\zeta-\rho\tau)} > \frac{\partial F_1(S^0,0)}{\partial i} + \frac{\partial F_2(V^0,0)}{\partial i} + \frac{(\mu+\xi+\delta)-\frac{\partial F_1((S^0,0))}{\partial i}-\frac{\partial F_2((V^0,0))}{\partial i}}{2} = \frac{\frac{\partial F_1((S^0,0))}{\partial i}+\frac{\partial F_2((V^0,0))}{\partial i}+(\mu+\xi+\delta)}{2}$ for all $\zeta < \zeta_1$. Then

$$
\begin{aligned}
\rho i'(\zeta) &= d_2 i''(\zeta) + F_1(s(\zeta-\rho\tau), i(\zeta-\rho\tau)) + F_2(v(\zeta-\rho\tau), i(\zeta-\rho\tau)) - (\mu+\xi+\delta) i(\zeta),\\
&\geq d_2 i''(\zeta) + \delta_1[i(\zeta-\rho\tau) - i(\zeta)] + \delta_2 i(\zeta), \quad \zeta \in \mathbb{R}
\end{aligned} \tag{4.1}
$$

where

$$
\delta_1 = \frac{\frac{\partial F_1((S^0,0))}{\partial i} + \frac{\partial F_2((V^0,0))}{\partial i} + (\mu+\xi+\delta)}{2},
$$

and

$$
\delta_2 = \frac{\frac{\partial F_1(S^0,0)}{\partial i} + \frac{\partial F_2((V^0,0))}{\partial i} - (\mu+\xi+\delta)}{2}
$$

Let $K(\zeta) = \displaystyle\int_{-\infty}^{\zeta} i(s)ds$, $\zeta \in \mathbb{R}$. Notice that $K(\zeta)$ is increasing in $\zeta \in \mathbb{R}$, by the integration of the two sides of (4.1) on $(-\infty, \zeta)$ where $\zeta < \zeta_1$, we get (we keep in mind $i(-\infty) = 0$, $i'(-\infty) = 0$)

$$
\rho i(\zeta) + \delta_1 \int_{\zeta-\rho\tau}^{\zeta} i(s)ds \;\geq\; d_2 i'(\zeta) + \delta_2 K(\zeta), \quad \forall \zeta < \zeta_1 \tag{4.2}
$$

Integrating the two sides of (4.2) form $-\infty$ to ζ gives

$$
\rho K(\zeta) + \delta_1 \int_{-\infty}^{\zeta} \left(\int_{\eta-\rho\tau}^{\eta} i(s)ds \right) d\eta \;\geq\; d_2 i(\zeta) + \delta_2 \int_{-\infty}^{\zeta} K(s)ds, \quad \forall \zeta < \zeta_1 \tag{4.3}
$$

Note that

$$
\begin{aligned}
\int_{-\infty}^{\zeta} \left(\int_{\eta-\rho\tau}^{\eta} i(s)ds \right) d\eta &= \rho\tau \lim\nolimits_{b \longrightarrow -\infty} \int_{b}^{\zeta} \left(\int_{0}^{1} i(\eta - \rho\tau + \rho\tau\theta) d\theta \right) d\eta\\
&= \rho\tau \lim\nolimits_{b \longrightarrow -\infty} \int_{0}^{1} \left(\int_{b}^{\zeta} i(\eta - \rho\tau + \rho\tau\theta) d\eta \right) d\theta\\
&\leq \rho\tau \int_{0}^{1} K(\zeta) d\theta = \rho\tau K(\zeta).
\end{aligned} \tag{4.4}
$$

Combining (4.3) and (4.4), we get

$$
d_2 i(\zeta) + \delta_2 \int_{-\infty}^{\zeta} K(s)ds \leq \rho K(\zeta) + \rho\tau\delta_1 K(\zeta) = \rho K(\zeta)(1+\tau\delta_1), \quad \forall \zeta < \zeta_1. \tag{4.5}
$$

Given that $K(t)$ is rising in t, which indicates that $K(t-l) < K(t)$ for any $l > 0$, we may observe that we have

$$
lK(\zeta - l) < \int_{\zeta-l}^{\zeta} K(s)ds < \int_{-\infty}^{\zeta} K(s)ds. \tag{4.6}
$$

Using (4.6) in (4.5), we take for any $l > 0$

$$
lK(\zeta - l) \leq \rho K(\zeta)(1+\tau\delta_1).
$$

Then, for $l > 0$ sufficiently large, we get

$$
K(\zeta - l) < \tfrac{1}{2} K(\zeta), \quad \forall \zeta < \zeta_1.
$$

Put $u_0 = \frac{ln2}{l}$, and let $p(\zeta) = K(\zeta)\exp^{-u_0\zeta}$. Therefore

$$p(\zeta - l) = K(\zeta - l)\exp^{-u_0(\zeta-l)} < \tfrac{1}{2}K(\zeta)\exp^{-u_0(\zeta-l)} \leq K(\zeta)\exp^{-u_0\zeta} = p(\zeta), \quad \forall z < \zeta_1.$$

hence $p(\zeta)$ is bounded as $\zeta \longrightarrow -\infty$, i.e., there is $p_0 > 0$ satisfies $p(\zeta) \leq p_0$ for all $\zeta < \zeta_1$, consequently,

$$K(\zeta) \leq p_0 \exp^{u_0\zeta}, \quad \forall \zeta < \zeta_1. \tag{4.7}$$

It follows from (4.5) that

$$i(\zeta) \leq \rho K(\zeta)(1 + \tau\delta_1), \quad \forall \zeta < \zeta_1. \tag{4.8}$$

then, from (4.7), and (4.8), we deduce $i(\zeta)\exp^{-u_0\zeta}$ is bounded for all $\zeta < \zeta_1$. By (4.2) and (4.8), we get

$$i'(\zeta) \leq \rho i(\zeta) + \delta_1[K(\zeta) - K(\zeta - \rho\tau)] \leq \rho i(\zeta) + 2\delta_1 p_0 \exp^{u_0\zeta}, \quad \forall \zeta < \zeta_1. \tag{4.9}$$

Thus,$\mid i'(\zeta) \mid \exp^{-u_0\zeta}$ is bounded for all $\zeta < \zeta_1$. Further, by (4.1), we know $\mid i''(\zeta) \mid \exp^{-u_0\zeta}$ is bounded for all $\zeta < \zeta_1$. Then,

$$i(-\infty) = 0, \quad i'(-\infty) = 0, \quad i''(-\infty) = 0, \tag{4.10}$$

implies $i(\zeta)\exp^{-u_0\zeta}$, $\mid i'(\zeta) \mid \exp^{-u_0\zeta}$ and $\mid i''(\zeta) \mid \exp^{-u_0\zeta}$ are all bounded on $\mathbb{R}$, i.e.,

$$\sup\left\{ \sup_{\zeta\in\mathbb{R}}\{i(\zeta)\exp^{-u_0\zeta}\}, \sup_{\zeta\in\mathbb{R}}\{\mid i'(\zeta) \mid \exp^{-u_0\zeta}\}, \sup_{\zeta\in\mathbb{R}} \mid\mid i''(\zeta) \mid \exp^{-u_0\zeta} \mid \right\} < \infty. \tag{4.11}$$

Besides, by the second equation of (2.3), we have for all $\zeta \in \mathbb{R}$,

$$d_2 i''(\zeta) - \rho i'(\zeta) + \left[\frac{\partial F_1((S^0,0))}{\partial i} + \frac{\partial F_2((V^0,0))}{\partial i}\right] i(\zeta - \rho\tau) - (\mu + \xi + \delta)i(\zeta) =$$
$$\left(\frac{\partial F_1((S^0,0))}{\partial i} + \frac{\partial F_2((V^0,0))}{\partial i} - \frac{F_1(s(\zeta-\rho\tau), i(\zeta-\rho\tau)) + F_2(v(\zeta-\rho\tau), i(\zeta-\rho\tau))}{i(\zeta-\rho\tau)}\right) i(\zeta - \rho\tau).$$

Then, (4.12) and (4.11) gives

$$\sup_{\zeta\in\mathbb{R}}\left\{\exp^{-u_0\zeta}\left(\tfrac{\partial F_1((S^0,0))}{\partial i} + \tfrac{\partial F_2((V^0,0))}{\partial i} - \tfrac{F_1(s(\zeta-\rho\tau),i(\zeta-\rho\tau))+F_2(v(\zeta-\rho\tau),i(\zeta-\rho\tau))}{i(\zeta-\rho\tau)}\right) i(\zeta-\rho\tau)\right\}$$

$$\begin{aligned} = \sup_{\zeta\in\mathbb{R}} \Bigg\{ & \exp^{-u_0\zeta} i''(\zeta) - \rho\exp^{-u_0\zeta} i'(\zeta) + \frac{\partial F_1((S^0,0))}{\partial i} \cdot \\ & + \frac{\partial F_2((V^0,0))}{\partial i}\exp^{-u_0\zeta} i(\zeta-\rho\tau) - (\mu+\xi+\delta)\exp^{-u_0\zeta} i(\zeta)\Bigg\} < \infty. \end{aligned} \tag{4.12}$$

For any $\lambda \in \mathbb{C}$ with $0 < Re\lambda < u_0$, we define a two-side Laplace transform of $i(t)$ by

$$L(\lambda) = \int_{-\infty}^{+\infty} \exp^{-\lambda\zeta} i(\zeta)d\zeta.$$

The resultant equation is integrated on $\mathbb{R}$ and simplified using integration by parts (while still utilising the boundary conditions (4.10)) after the two sides of (4.12) have been multiplied by $e^{-\lambda t}$.

$$\begin{aligned} G(\lambda,\rho)L(\lambda) = & \int_{-\infty}^{+\infty} \exp^{-\lambda\zeta}\left(\left(\frac{\partial F_1((S^0,0))}{\partial i} + \frac{\partial F_2((V^0,0))}{\partial i}\right.\right. \\ & \left.\left. - \frac{F_1(s(\zeta-\rho\tau), i(\zeta-\rho\tau)) + F_2(v(\zeta-\rho\tau), i(\zeta-\rho\tau))}{i(\zeta-\rho\tau)}\right) i(\zeta-\rho\tau)\right) d\zeta, \end{aligned} \tag{4.13}$$

$0 < Re\{\lambda\} < u_0$, and (2.6) defines $G(\lambda,\rho)$ for $\lambda \in \mathbb{C}$. Keep in mind that, according to lemma 2.1, $G(\lambda,\rho)$ ¿ 0 for every $\rho \in (0,\rho^*)$. For $\rho \in (0,\rho^*)$, we assert that there is no singularity in $L(\lambda)$ defined by (4.13). We provide contradictions to support this assertion. We assume that $L(\lambda)$ has no singularity for $\lambda < \lambda_0$ and a singularity at $\lambda = \lambda_0$, where $\lambda_0 > 0$. Keep in mind that

$0 < Re\{\lambda\} < u_0 + \lambda_0$ defines the right-hand side of (4.13) if $\lambda \in \mathbb{C}$. Given that (4.13) may be written like this,

$$\int_{-\infty}^{+\infty} \exp^{-\lambda\zeta}\left(G(\lambda,\rho)i(\zeta) - \left(\frac{\partial F_1((S^0,0))}{\partial i} + \frac{\partial F_2((V^0,0))}{\partial i}\right.\right.$$
$$\left.\left. - \frac{F_1(s(\zeta-\rho\tau),i(\zeta-\rho\tau)) + F_2(v(\zeta-\rho\tau),i(\zeta-\rho\tau))}{i(\zeta-\rho\tau)}\right) i(\zeta-\rho\tau)\right) d\zeta = 0, \tag{4.14}$$

and $\left(\frac{\partial F_1((S^0,0))}{\partial i} + \frac{\partial F_2((V^0,0))}{\partial i} - \frac{F_1(s(\zeta-\rho\tau),i(\zeta-\rho\tau))+F_2(v(\zeta-\rho\tau),i(\zeta-\rho\tau))}{i(\zeta-\rho\tau)}\right)i(\zeta-\rho\tau)$, $i(t)$ are all fixed for $\lambda > 0$, and $\lim_{\lambda\to+\infty} G(\lambda,\rho) = +\infty$ for any $\rho \in (0,\rho^*)$, a contradiction with the equality (4.14) is obtained. Consequently, the boundary requirement cannot be satisfied by any travelling wave solution. The evidence is now complete. □

5. Numerical Simulation. In order to confirm the TWS of (2.1) that link the system's two equilibria, we first provide a number of numerical simulations in this section. The nonlinear incidence function of Holling type II for the S compartment, $F_1(S,I) = \frac{\beta_1 SI}{1+aI}$, and the Beddington-DeAngelis incidence function for the V compartment, $F_2(V,I) = \frac{\beta_2 VI}{1+bI+cV}$, are also taken into consideration. Both of these functions satisfy the assumptions (**H**) and (**D**). The geographical heterogeneity of the environment, where two towns' modes of transmission are thought to be distinct from one another, may be better modelled using this kind of incidence function combination.to carry out this action. The following basic conditions are also taken into consideration.

$$S_0(x) = \begin{cases} 0.7 & \text{if } x \in [50,100], \\ 0.25 & \text{if } x \in [0,50], \end{cases}$$
$$V_0(x) = \begin{cases} 0.3 & \text{if } x \in [50,100], \\ 0.12 & \text{if } x \in [0,50], \end{cases}$$
$$I_0(x) = \begin{cases} 0 & \text{if } x \in [50,100], \\ 0.06 & \text{if } x \in [0,50], \end{cases}$$

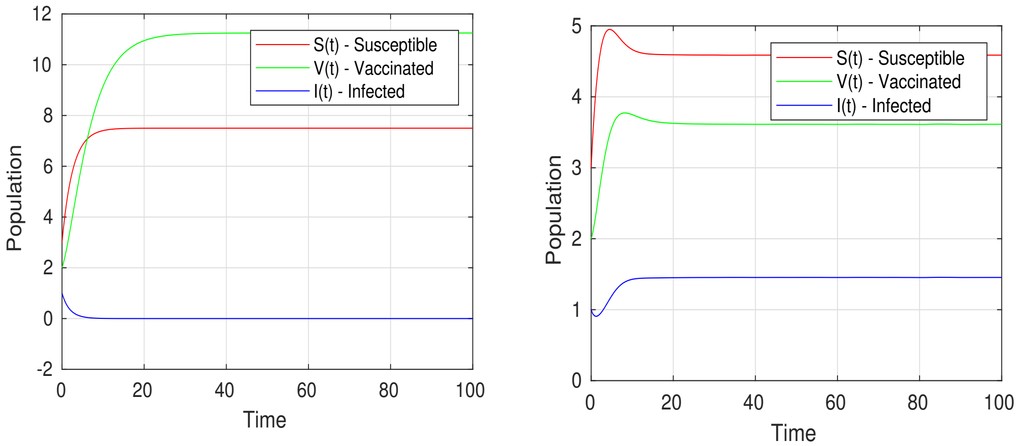

Figure 5.1: For the system (2.2), the global stability findings show that $R_0 < 1$ when we take into account $\beta_1 = 0.02$ and $\beta_2 = 0.01$ for the left-hand figure. The DFE is globally asymptotically stable in this instance. But if we take into account $\beta_1 = 0.2$ and $\beta_2 = 0.1$ for the right-hand figure, we obtain $R_0 > 1$. In this instance, the EE is globally asymptotically stable but the DFE is unstable.

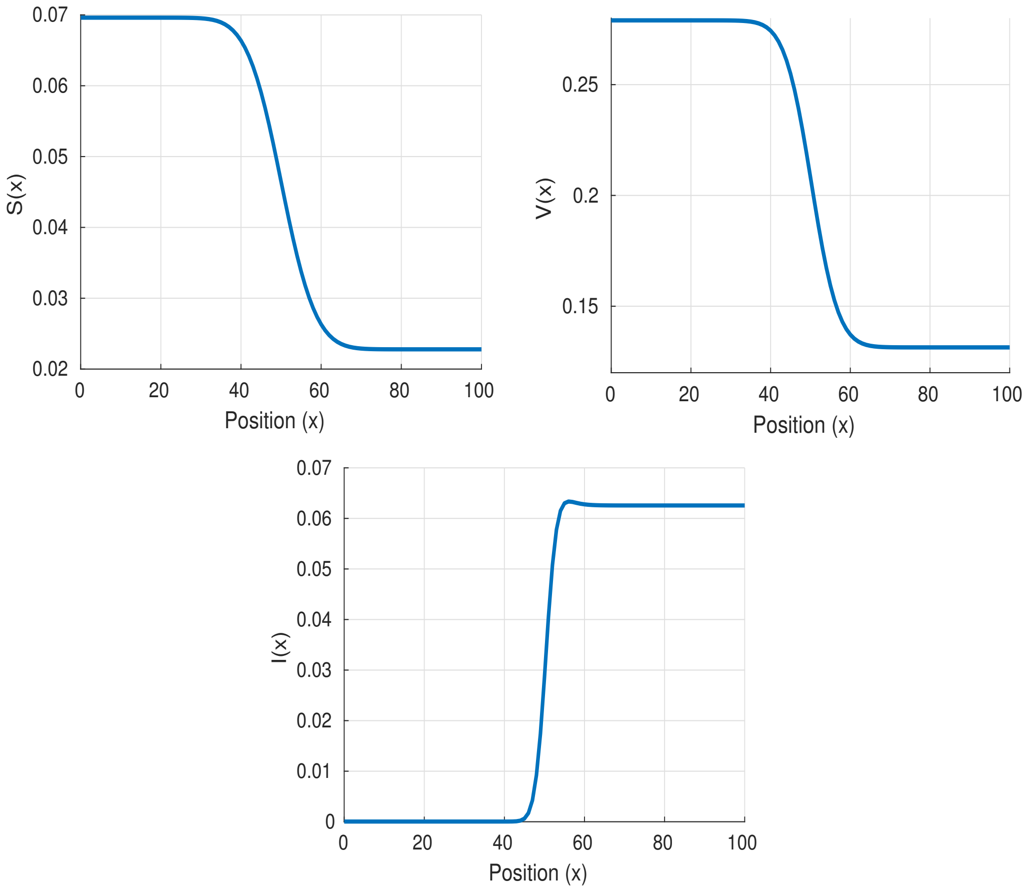

Figure 5.2: Cross section curves of the model (2.1) solutions, which guarantee the existence of a TWS.

We offer Fig. 5.1, which illustrates the global stability of equilibria in the instance of the DDE issue (2.2), in order to emphasise the global stability of (2.2). Here, however, we offer Fig. 5.2, which emphasises the presence of a TWS of (2.1), which is the positive solution of (2.1) that fulfils the boundary condition (2.4).

Conflicts of interest. The authors declare no conflict of interest.

Funding. No funding is received for this study.

ORCID and License

Rassim Darazirar https://orcid.org/0009-0009-8698-2116

References

[1] Dong F, Li W, Wang J. Propagation dynamics in a three-species competition model with nonlocal anisotropic dispersal. Nonlinear Analysis: Real World Applications. 2019;48:232-66.

[2] Tomás C, Fatini M, Pettersson M. A stochastic SIRI epidemic model with relapse and media coverage. Discrete and Continuous Dynamical Systems. 2018;23(8):3483-501.

[3] Cai Y, Wang W. Fish-hook bifurcation branch in a spatial heterogeneous epidemic model with cross-diffusion. Nonlinear Analysis: Real World Applications. 2016;30:99-125.

[4] Thieme H, Zhao X. Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction-diffusion models. Journal of Differential Equations. 2003;195(2):430-70.

[5] Crooks E, Dancer E, Hilhorst D, Mimura M, Ninomiya H. Spatial segregation limit of a competition-diffusion system with Dirichlet boundary conditions. Nonlinear Analysis: Real World Applications. 2004;5(4):645-65.

[6] Murray JD. Mathematical Biology II: Spatial Models and Biomedical Applications. Springer; 2002.

[7] Shigesada N, Kawasaki K. Biological Invasions: Theory and Practice. Oxford University Press; 1997.

[8] Djilali S, Bentout S. Global dynamics of SVIR epidemic model with distributed delay and imperfect vaccine. Results in Physics. 2021. Available from: https://doi.org/10.1016/j.rinp.2021.104569.

[9] Liu X, Takeuchi Y, Iwami S. SVIR epidemic models with vaccination strategies. Journal of Theoretical Biology. 2008;253:1-11.

[10] Xu J, Zhou Y. Global stability of a multi-group model with vaccination age, distributed delay and random perturbation. Mathematical Biosciences and Engineering. 2015;12:1083-106.

[11] Duan X, Yuan S, Li X. Global stability of an SVIR model with age of vaccination. Applied Mathematics and Computation. 2014;226:528-40.

[12] Wang J, Zhang R, Kuniya T. The dynamics of an SVIR epidemiological model with infection age. IMA Journal of Applied Mathematics. 2016;81:321-43.

[13] Diekmann O, Heesterbeek JAP, Britton T. Mathematical Tools for Understanding Infectious Disease Dynamics. Princeton University Press; 2013.

[14] Hethcote HW. The mathematics of infectious diseases. SIAM Review. 2000;42:599-653. Available from: https://doi.org/10.1137/S0036144500371907.

[15] Guenad B, Darazirar R, Djilali S, Alraddadi I. Traveling waves in a delayed reaction–diffusion SIR epidemic model with a generalized incidence function. Nonlinear Dynamics. 2024.

[16] Ran Z, Liu S. Traveling waves for SVIR epidemic model with nonlocal dispersal. Mathematical Biosciences and Engineering. 2019;16(3):1654-82.

[17] Peng R, Liu S. Global stability of the steady states of an SIS epidemic reaction-diffusion model. Nonlinear Analysis: Theory, Methods and Applications. 2009;71:239-47.

[18] Lou Y, Zhao XQ. A reaction-diffusion malaria model with incubation period in the vector population. Journal of Mathematical Biology. 2011;62:543-68.

[19] Ducrot A, Magal P. Travelling wave solutions for an infection-age structured epidemic model with external supplies. Nonlinearity. 2011;24:2891-911.

[20] Djilali S, Darazirar R, Alraddadi I. Traveling wave solution for a delayed reaction-diffusion two-group SIR epidemic model with a generalized nonlinear incidence function. Journal of Applied Mathematics and Computing. 2025. Available from: https://doi.org/10.1007/s12190-025-02474-4.

[21] Darazirar R, Yaseen RM, Mohsen AA, et al. Minimal wave speed and traveling wave in nonlocal dispersion SIS epidemic model with delay. Boundary Value Problems. 2025. Available from: https://doi.org/10.1186/s13661-025-02055-1.

[22] Weinberger HF. Long-time behavior of a class of biological models. SIAM Journal on Mathematical Analysis. 1982;13:353-96.

[23] Hosono Y, Ilyas B. Traveling waves for a simple diffusive epidemic model. Mathematical Models and Methods in Applied Sciences. 1995;5:935-66. Available from: https://doi.org/10.1142/S0218202595000431.

[24] Wang J, Modnak C, Eisenberg JN. Backward bifurcations in disease models: Applications to waterborne infections. Mathematical Biosciences. 2012;240:227-34. Available from: https://doi.org/10.1016/j.mbs.2012.07.004.

[25] Liu Y, Wu J. Traveling waves and spreading speeds for an epidemic model with spatial diffusion and delay. Journal of Mathematical Biology. 2020;80:235-59. Available from: https://doi.org/10.1007/s00285-019-01385-5.

[26] Huang G, Takeuchi Y, Ma W, Wei D. Global stability for delay SIR and SEIR epidemic models with nonlinear incidence rate. Bulletin of Mathematical Biology. 2010;72:1192-207. Available from: https://doi.org/10.1007/s11538-010-9489-6.

[27] Korobeinikov A, Maini PK. Non-linear incidence and stability of infectious disease models. Mathematical Medicine and Biology: A Journal of the IMA. 2005;22(2):113-28. Available from: https://doi.org/10.1093/imammb/dqi005.

[28] Hartman P. Ordinary Differential Equations. Society for Industrial and Applied Mathematics; 2002.

[29] Fu SC. Traveling waves for a diffusive SIR model with delay. Journal of Mathematical Analysis and Applications. 2016;435:20-37. Available from: https://doi.org/10.1016/j.jmaa.2015.10.018.