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Surfaces with mean of the hyperbolic curvature radii of double harmonic type

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Abstract

In this paper, we define surfaces with mean of the hyperbolic curvature radii of double harmonic type (in short DHRMC-surfaces) in the hyperbolic space, these surfaces include the generalized Weingarten surfaces of the harmonic type (HGW-surfaces). We give a characterization of DHRMC-surfaces. Given a real function, we will present a family of DHRMC-surfaces that depend on two holomorphic functions. Moreover, we classify the DHRMC-surfaces of rotation.
Keywords . hyperbolic curvature radii, holomorphic functions, Weingarten surfaces.

1. Introduction. The theory of minimal surfaces has made great progress due to the connection between the study of them and complex analysis. From the Weierstrass representation it is possible to obtain examples of these surfaces, taking advantage of a pair of holomorphic functions. There are Weierstrass representations for some classes of surfaces, among them are certain classes of Weingarten surfaces, objects that we investigate in this paper.

Corro, in [1] introduce a large class of generalized Weingarten surfaces of Bryant type (in short, BGW-surfaces) in the hyperbolic 3-space, whose mean curvature H, Gaussian curvature K_I and radius function h satisfy a relation of the form

$$2ach^{2\frac{c-1}{c}}(H-1)+(a+b-ach^{2\frac{c-1}{c}})K_I=0,$$

where $a+b\neq 0$ and $c\neq 0$.

In [2], the authors study the immersions $X : M \longrightarrow \mathbb{H}^3$, where M is a Riemann surface whose mean curvature H and Gaussian curvature K_I satisfies the relation

$$2(H-1)Ce^{2\mu}+K_I(1-Ce^{2\mu})=0.$$

where μ is a harmonic function with respect to the quadratic form $\sigma=-K_I I+2(H-1)II$, $C\in\mathbb{R}$. The surfaces which satisfy the above relation are called *Generalized Weingarten surfaces of harmonic type in hyperbolic space (in short HGW-surfaces)*.

Given a hypersurface in $\mathbb{R}^{n+1}$ with principal curvatures k_i, $1\leq i\leq n$, define the curvature radii R_i and the mean of the curvature radii H_R as

$$R_i=\frac{1}{k_i}\quad\text{and}\quad H_R=\frac{1}{n}\sum_{i=1}^{n}R_i.$$

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In [4], Espinar, Gálvez and Mira extend the problem to the hyperbolic case $\mathbb{H}^{n+1}$, i.e. they define the *hyperbolic curvature radii* and the *mean of the hyperbolic curvature radii* as

$$\widetilde{R_i} = \frac{1}{1-k_i} \quad \text{and} \quad \widetilde{H_R} = \frac{1}{n}\sum_{i=1}^{n} \widetilde{R}_i,$$

and propose the Christoffel problem where it says that given a diffeomorphism of $G : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ and a differentiable function $T : \mathbb{S}^n \longrightarrow \mathbb{R}$, it is possible to find hypersurfaces with Gauss application G and T prescribed as the mean of the hyperbolic curvature radii of this hypersurface, i.e., $T = \widetilde{H}_R$.

In this paper, we define the *surfaces with mean of the hyperbolic curvature radii of double harmonic type (in short DHRMC-surfaces)*, these surfaces are such that the mean of the hyperbolic curvature radii depends on a real function C and two harmonic functions μ and ν, that is,

$$\widetilde{H}_R = \frac{1}{2}\left(1 - C(\mu)e^{2\nu}\right),$$

these surfaces include the generalized Weingarten surfaces of the harmonic type (HGW-surfaces). We give a characterization of DHRMC-surfaces. Given the real function, we will present a family of DHRMC-surfaces that depend on two holomorphic functions. Moreover, we classify the DHRMC-surfaces of rotation.

Thus, we conclude that these surfaces give us some solutions to the classic Christoffel problem, where

$$T = \frac{1}{2}\left(1 - C(\mu)e^{2\nu}\right).$$

2. Preliminary. In this section, we present some concepts and definitions that we will use throughout this work. In this paper the inner product $\langle , \rangle : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ is defined by

$$\langle f, g\rangle = f_1 g_1 + f_2 g_2, \text{ where } f = f_1 + if_2,\ g = g_1 + ig_2,$$

are holomorphic functions. In the computation we use the following properties:

If $f, g, h : \mathbb{C} \to \mathbb{C},\ z = u_1 + iu_2 \in \mathbb{C}$ are holomorphic functions then

$$\begin{gathered}\langle f, g\rangle_{,u_1} = \langle f', g\rangle + \langle f, g'\rangle,\ \ \langle f, g\rangle_{,u_2} = \langle if', g\rangle + \langle f, ig'\rangle,\ \ \langle fg, h\rangle = \langle g, \overline{f}h\rangle,\\ \triangle\langle f, g\rangle = 4\langle f', g'\rangle,\ \ \langle f, g\rangle + i\langle f, ig\rangle = f\overline{g},\ \ \langle 1, f\rangle\langle 1, if\rangle = \frac{1}{2}\langle 1, if^2\rangle, \\ \langle 1, f\rangle^2 - \langle 1, if\rangle^2 = \langle 1, f^2\rangle.\end{gathered} \tag{2.1}$$

Let $M \subset \mathbb{R}^3$ a regular surface and $X : U \longrightarrow M$ a parameterization of M where U is an open set of $\mathbb{R}^2$.

Let N the unit normal vector field on M, then

$$N_{,i} = \sum_{j=1}^{2} W_{ij}(u)X_{,j}, 1 \le i \le 2, \tag{2.2}$$

where $u = (u_1, u_2) \in U \subset \mathbb{R}^2$, $X_{,j}$ denotes the partial derivative of X with respect to u_j and $W = (W_{ij})$ is called the ***Weingarten matrix***.

Consider $\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}$ e $Y : U \subset \mathbb{R}^2 \longrightarrow \Pi$ an orthogonal local parameterization of Π, that is, $g_{ij} = \langle Y_i, Y_j\rangle$, $1 \le i, j \le 2$, such that

$$g_{12} = g_{21} = 0 \quad \textit{and} \quad g_{ii} \neq 0, \quad 1 \le i, j \le 2.$$

Furthermore,

$$Y_{,ij} = \sum_{k=1}^{2} \Gamma_{ij}^{k} Y_{,k} \quad 1 \le i, j \le 2. \tag{2.3}$$

Using the fact that the metric is diagonal, we have

$$\Gamma_{ij}^{i} = \frac{g_{ii,j}}{2g_{ii}} \quad \forall\ i, j, \tag{2.4}$$

and

$$\Gamma_{ii}^{j} = -\frac{g_{ii}}{g_{jj}}\Gamma_{ij}^{i} \quad 1 \leq i, j \leq 2. \tag{2.5}$$

Definition 2.1. A *sphere congruence* in $\mathbb{R}^3$ is a two-parameter family of spheres with a differentiable *radius function* , whose centers lie on a surface.

Definition 2.2. An *envelope* of a sphere congruence is a surface M of $\mathbb{R}^3$ such that each point of M is tangent to a sphere of the sphere congruence.

Theorem 2.1. *A surface $M \subset \mathbb{R}^3$ is an envelope of congruence of spheres, in which the other envelope is contained in the plane $\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}$ if and only if there is an orthogonal local parameterization of Π, $Y : U \subset \mathbb{R}^2 \longrightarrow \Pi$ and a differentiable function $h : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$, such that $X : U \subset \mathbb{R}^2 \longrightarrow M$, given by*

$$X(u) = Y(u) - \frac{2h(u)}{S}\left[\frac{h_{,1}}{g_{11}}(u)Y_{,1}(u) + \frac{h_{,2}}{g_{22}}(u)Y_{,2}(u) - e_3\right] \tag{2.6}$$

is the parameterization of M, with $e_3 = (0, 0, 1)$, $g_{ii} = \langle Y_{,i}, Y_{,i}\rangle$, $1 \leq i \leq 2$ and

$$S = \frac{(h_{,1}(u))^2}{g_{11}} + \frac{(h_{,2}(u))^2}{g_{22}} + 1. \tag{2.7}$$

Also, the normal vector is given by

$$N(u) = e_3 + \frac{2}{S}\left[\frac{h_{,1}}{g_{11}}(u)Y_{,1}(u) + \frac{h_{,2}}{g_{22}}(u)Y_{,2}(u) - e_3\right] \tag{2.8}$$

and the Weingarten matrix is

$$W = 2V\,(SI - 2hV)^{-1}, \tag{2.9}$$

where the matrix $V = (V_{ij})$ is given by

$$V_{ij} = \frac{1}{g_{jj}}\left(h_{,ij} - \sum_{l=1}^{2}\Gamma_{ij}^{l}h_{,l}\right), \quad 1 \leq i, j \leq 2, \tag{2.10}$$

end

$$P = S^2 - 2hStr(V) + 4h^2\det(V) \neq 0. \tag{2.11}$$

In [5], Machado and Riveros, generalize this result to hypersurfaces. The parameterization is analogous to (2.6) for the case $M \subset \mathbb{R}^{n+1}$.

From this, we have the following result.

Corollary 2.1. *Let $X : U \subset \mathbb{R}^n \longrightarrow M^n \subset \mathbb{R}^{n+1}$ be a parameterization of a hypersurface M^n. Then the following conditions are equivalent:*

i) X is parameterized by lines of curvature,

ii) $V_{ij} = 0$ for $i \neq j$,

iii) $N_{,i} = -k_iX_{,i}$,

where $k_i = \dfrac{2V_{ii}}{2hV_{ii} - S}$, $1 \leq i \leq n$, are the principal curvatures of X.

In what follows, we consider $\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ as the upper half-space model with the metric $ds^2 = \dfrac{1}{x_3^2}(dx_1^2 + dx_2^2 + dx_3^2)$ and ideal boundary $C_\infty = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\} \cup \{\infty\}$.

The following Lemma was obtained in [3].

Lemma 2.1. *Let $M \subset \mathbb{R}^3$ be a regular surface. Consider the immersion $X : M \longrightarrow \mathbb{H}^3$ given by $X(u) = \sum_{j=1}^{3} x_j(u)e_j$, where $u = (u_1, u_2) \in U$ and $N(u) = \sum_{j=1}^{3} N_j(u)e_j$ the normal vector of*

M. Using the connection induced by the hyperbolic space metric the Weingarten matrix of X in $\mathbb{H}^3$ is given by

$$\widetilde{W} = x_3 W - N_3 I, \tag{2.12}$$

where W is the Weingarten matrix on $\mathbb{R}^3$ and I is the identity matrix.

The following Lemma obtained in [2] characterize locally the surfaces M in $\mathbb{H}^3$.

Lemma 2.2. *Let M be a connected Riemann surface whose hyperbolic Gauss map is an immersion. Then $X : M \longrightarrow \mathbb{H}^3$ is a $\mathcal{C}^2$ immersion if, and only if, there exist functions $g : M \longrightarrow \mathbb{C}_\infty$ holomorphic and $h : M \longrightarrow \mathbb{R}^+_\infty$, such that $X(M)$ is locally parameterized by*

$$X(z) = (g, 0) - \frac{2h}{S}\left[\frac{g'}{\|g'\|^2}\left(h_{,1} + ih_{,2}\right), -1\right], \tag{2.13}$$

where $z = u_1 + iu_2 \in \mathbb{C}$, $\mathbb{R}^+_\infty = \mathbb{R}^+ \cup \{+\infty\}$ and $g'(z) \neq 0$ for all $z \in \mathbb{C}$.

$$S = \frac{(h_{,1})^2 + (h_{,2})^2}{\|g'\|^2} + 1. \tag{2.14}$$

The hyperbolic Gauss map is given by

$$\eta(z) = \frac{2h}{S}\left\{(0,0,1) + \frac{2}{S}\left[\frac{g'}{\|g'\|^2}(h_{,1} + ih_{,2}), -1\right]\right\}. \tag{2.15}$$

The Weingarten matrix is

$$\widetilde{W} = \frac{2h}{P}\left[2hV + (S - 2htr(V))\, I\right] - I, \tag{2.16}$$

where

$$P = -S^2 + 2S\,(1-\gamma) + 4h^2 \det(V), \tag{2.17}$$

and

$$\gamma = \frac{h\Delta h}{\|g'\|^2} - S + 1. \tag{2.18}$$

The matrix $V = (V_{ij})$ is defined by

$$V_{ij} = \frac{1}{\|g'\|^2}\left(h_{,ij} - \sum_{l=1}^{2}\Gamma^l_{ij} h_{,l}\right), \quad 1 \leq i, j \leq 2, \tag{2.19}$$

with

$$\Gamma^i_{ii} = \frac{|g'|_{,i}}{|g'|} \quad \textit{and} \quad \Gamma^i_{ij} = \frac{|g'|_{,j}}{|g'|} = -\Gamma^j_{ii}, \quad 1 \leq i \neq j \leq 2. \tag{2.20}$$

Definition 2.3. Let M be a hypersurface in the hyperbolic space $\mathbb{H}^n$. The *hyperbolic curvature radii* $\widetilde{R}_i$ and *the mean of the hyperbolic curvature radii* $\widetilde{H}_R$ of M are given respectively by

$$\widetilde{R}_i = \frac{1}{1 - k_i} \quad \textit{and} \quad \widetilde{H}_R = \frac{1}{n}\sum_{i=1}^{n}\widetilde{R}_i. \tag{2.21}$$

3. Main results. In this section we define and characterize the DHRMC-surfaces.

Definition 3.1. A hypersurface $M \subset \mathbb{H}^{n+1}$, $n \geq 2$, is said to be a *hypersurface with mean of the hyperbolic curvature radii of double harmonic type (in short DHRMC-surfaces)* if given a real function $C : \mathbb{R} \to \mathbb{R}$, the mean of the hyperbolic curvature radii is given by

$$\widetilde{H}_R = \frac{1}{2}\left(1 - C(\mu)e^{2\nu}\right), \tag{3.1}$$

where μ and ν are harmonic functions.

Remark 3.1. We observe that (3.1) is equivalent to

$$2(H-1)C(\mu)e^{2\nu} + K_I(1 - C(\mu)e^{2\nu}) = 0,$$

where μ and ν are harmonic functions with respect to the quadratic form $\sigma = -K_I I + 2(H-1)II$. Thus, when $n = 2$ and the real function C is constant we obtain HGW-surfaces, i.e. surfaces that satisfy the equation

$$2(H-1)Ce^{2\nu} + K_I(1 - Ce^{2\nu}) = 0.$$

Theorem 3.1. *Let $M \subset \mathbb{H}^3$ an envelope of a sphere congruence, that satisfies the conditions of the Lemma 2.2. Given a real function $C : \mathbb{R} \to \mathbb{R}$, μ and ν harmonic functions, then M is a DHRMC-surface if and only if*

$$h\Delta h - \|\nabla h\|^2 = C(\mu)e^{2\left(\nu + \ln\|g'\|\right)}. \tag{3.2}$$

Proof: Let $X : U \subset \mathbb{R}^2 \to M$, by the corollary 2.1 we have that the principal curvatures t_i of M are given by

$$t_i = \frac{2\sigma_i}{2h\sigma_i - S} \tag{3.3}$$

where σ_i are the eigenvalues of the matrix V_{ij}.

Now consider the immersion $X : M \to \mathbb{H}^3$, by (2.12) we conclude that

$$k_i = N_3 - X_3 t_i \tag{3.4}$$

using the equations (2.6), (2.8) and substituting the equation (3.3), we have

$$k_i \;=\; 1 - \frac{2}{S} + \frac{2h}{S}\left(\frac{2\sigma_i}{2h\sigma_i - S}\right) = 1 + \frac{2}{2h\sigma_i - S}.$$

Consider the parameterization $Y = (g, 0)$, where g is a holomorphic function. Thus, from the equation above and (2.21), we have

$$\widetilde{R}_i \;=\; \frac{1}{1-k_i} = \frac{1}{1 - 1 - \dfrac{2}{2h\sigma_i - S}} = \frac{S}{2} - h\sigma_i. \tag{3.5}$$

From (2.21) and (3.5) we have

$$\begin{aligned} \widetilde{H}_R &= \frac{1}{2}\left(\left(\frac{S}{2} - h\sigma_1\right) + \left(\frac{S}{2} - h\sigma_2\right)\right) = \frac{1}{2}\left(S - h(\sigma_1 + \sigma_2)\right) \\ &= \frac{1}{2}\left(S - htr(V)\right). \end{aligned} \tag{3.6}$$

By (2.7) S is given by

$$S \;=\; \frac{\|\nabla h\|^2}{\|g'\|^2} + 1 \tag{3.7}$$

and from (2.19) and (2.20) we have

$$\begin{aligned} tr(V) &= \frac{1}{\|g'\|^2}\left(h_{,11} - \left(\varGamma_{11}^1 h_{,1} + \varGamma_{11}^2 h_{,2}\right) + h_{,22} - \left(\varGamma_{22}^1 h_{,1} + \varGamma_{22}^2 h_{,2}\right)\right) \\ &= \frac{\Delta h}{\|g'\|^2}. \end{aligned} \tag{3.8}$$

From (3.6), (3.7) and (3.8)

$$\widetilde{H}_R = \frac{1}{2}\left(1 + \frac{\|\nabla h\|^2 - h\Delta h}{\|g'\|^2}\right). \tag{3.9}$$

Using the definition 3.1,

$$\frac{\|\nabla h\|^2 - h\Delta h}{\|g'\|^2} = -C(\mu)e^{2\nu}. \tag{3.10}$$

Hence, we obtain that

$$\|\nabla h\|^2 - h\Delta h = -C(\mu)e^{2\nu}\|g'\|^2 = -C(\mu)e^{2\left(\nu + \ln\|g'\|\right)} \tag{3.11}$$

where the result follows. $\square$

Theorem 3.2.
Let M be an envelope of a sphere congruence, that satisfies the conditions of the Theorem 3.1. Consider l a real function, $C = ll'' - l'^2$, $\mu = \langle 1, f\rangle$ and $\nu = \ln\left\|\frac{f'}{g'}\right\|$ where f and g are holomorphic functions, then M is a DHRMC-surface.
Furthermore, the local parameterization of M is given by

$$X(z) = (g,0) - \frac{2l}{S}\left(\frac{g'l'\overline{f'}}{\|g'\|^2}, -1\right), \tag{3.12}$$

where $z = u_1 + iu_2 \in \mathbb{C}$, $g'(z) \neq 0$ for all $z \in \mathbb{C}$ and

$$S = \frac{(l')^2\|f'\|^2}{\|g'\|^2} + 1. \tag{3.13}$$

The Gauss map is given by

$$\eta(z) = \frac{2}{S^2}\left(\frac{2g'll'\overline{f'}}{\|g'\|^2}, l(S-2)\right). \tag{3.14}$$

The Weingarten matrix is

$$\widetilde{W} = \frac{2l}{P}\left[2lV + (S - 2ltr(V))I\right] - I \tag{3.15}$$

where

$$P = -S^2 + 2S(1 - C(\mu)e^{2\nu}) + 4l^2 det(V). \tag{3.16}$$

The matrix $V = (V_{ij})$ is given by

$$V_{ij} = \frac{1}{g_{jj}}\left(h_{,ij} - \sum_{l=1}^{2}\Gamma_{ij}^{l}h_{,l}\right), \quad 1 \leq i,j \leq 2 \tag{3.17}$$

with

$$\Gamma_{ii}^{i} = \frac{\|g'\|_{,i}}{\|g'\|} \quad and \quad \Gamma_{ij}^{i} = \frac{\|g'\|_{,j}}{\|g'\|} = -\Gamma_{ii}^{j} \quad 1 \leq i \neq j \leq 2. \tag{3.18}$$

Proof:
Consider $h(u) = l(\mu(u))$ such that $\Delta\mu = 0$.
Thus, differentiating h we have

$$\begin{aligned} h_{,i} &= l'(\mu)(\mu_{,i}), \\ h_{,ii} &= l''(\mu)(\mu_{,i})^2 + l'(\mu)\mu_{,ii}. \end{aligned} \tag{3.19}$$

Now from equations (3.19) we concluded that

$$\|\nabla h\|^2 = (l')^2\|\nabla\mu\|^2 \tag{3.20}$$

and

$$h\Delta h = ll'' \left\| \bigtriangledown \mu \right\|^2. \tag{3.21}$$

Then, for (3.2), (3.20) and (3.21)

$$\left(ll'' - (l')^2 \right) \left\| \bigtriangledown \mu \right\|^2 = C(\mu) e^{2\left(\nu + \ln \left\| g' \right\| \right)}. \tag{3.22}$$

Using the fact that, $\nu = \ln \left\| \frac{f'}{g'} \right\|$, $\mu = \langle 1, f \rangle$, where f is a holomorphic function, it follows that M is a DHRMC-surface.
Now, differentiating $h = l(\mu) = l(\langle 1, f \rangle)$ using (2.1), we have

$$\begin{aligned} h_{,1} &= l'(\mu) \langle 1, f' \rangle, \\ h_{,2} &= l'(\mu) \langle 1, if' \rangle. \end{aligned} \tag{3.23}$$

Thus, from (3.23) we obtain

$$h_{,1} + i h_{,2} = l' \overline{f'}. \tag{3.24}$$

Substituting (3.23) and (3.24) in (2.13), (2.14), (2.15) we have (3.12), (3.13) and (3.14), respectively. Since $h = l(\mu)$ and $\gamma = C(\mu) e^{2\nu}$, from (2.16)-(2.20) we obtain the expressions (3.15)-(3.18). Thus , the proof is complete. □

4. Examples of DHRMC-Surfaces. Using the representation formula (3.12), we give some graphs of DHRMC-surfaces.

Example 4.1. Considering $f(z) = z$, $g(z) = \cos z$, $l(t) = t^2$, $C = -2t^2$ in Theorem 3.2, we obtain a DHRMC-surface

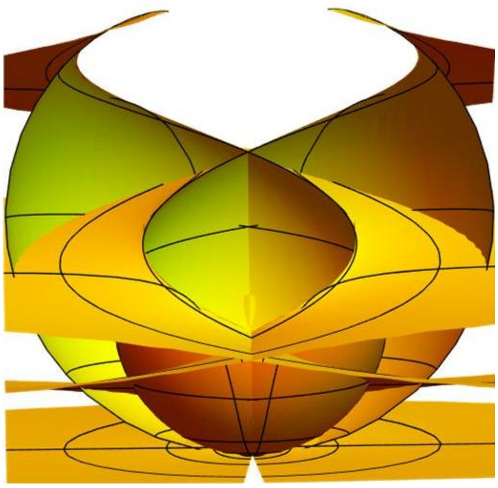

Figure 4.1: DHRMC-surface.

Example 4.2. Considering $f(z) = e^z$, $g(z) = z$, $l(t) = \ln t$, $C = -\frac{\ln t + 1}{t^2}$ in Theorem 3.2, we obtain a DHRMC-surface .

Example 4.3. Considering $f(z) = z^4$, $g(z) = z^5$, $l(t) = \sin t$, $C = -1$ in Theorem 3.2, we obtain a DHRMC-surface.

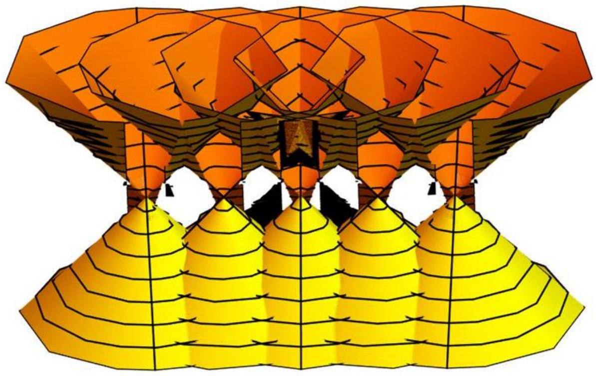

Figure 4.2: DHRMC-surface.

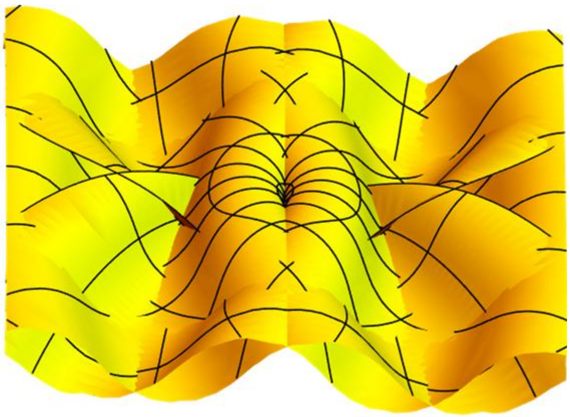

Figure 4.3: DHRMC-surface.

5. DHRMC-surfaces of rotation. The following result characterize the DHRMC-surfaces of rotation.

Theorem 5.1.

Let M be a surface of the congruence of spheres, that satisfies the conditions of the Theorem 3.1. M is a DHRMC-surface of rotation, if and only if $g = e^z$, $\mu = a_1u_1 + b_1$, $\nu = \ln|a_1| - u_1$ and l satisfy

$$ll'' - l'^2 = C(a_1u_1 + b_1). \tag{5.1}$$

Moreover, the parameterization of M is given by

$$X(z) \quad = \quad (M(u_1)\cos u_2, M(u_1)\sin u_2, N(u_1)) \,. \tag{5.2}$$

where

$$M(u_1) = e^{u_1}\left[1 - \frac{2a_1ll'}{a_1^2l'^2 + e^{2u_1}}\right] \quad \textit{and} \quad N(u_1) = \frac{2le^{2u_1}}{a_1^2l'^2 + e^{2u_1}}. \tag{5.3}$$

Proof: Let M be a DHRMC-surface of rotation and consider $h = l(\mu) = l(a_1u_1 + b_1)$. Thus, differentiating h, we have

$$h_{,1} \quad = \quad l'(\mu)a_1, \tag{5.4}$$
$$h_{,11} \quad = \quad l''(\mu)a_1^2. \tag{5.5}$$

Since $h_{,2} = 0$, from (3.2), (5.4) and (5.5) we obtain

$$h\Delta h - \|\nabla h\|^2 = (ll'' - l'^2)a_1^2. \tag{5.6}$$

On the other hand,

$$\begin{aligned} C(\mu)e^{2(\nu+\ln\|g'\|)} &= C(a_1u_1+b_1)(e^{2(\ln|a_1|-u_1+\ln e^{u_1})}), \\ &= C(a_1u_1+b_1)a_1^2. \end{aligned} \tag{5.7}$$

From (5.6) and (5.7), we obtain (5.1). Finally, the expression (5.3) it follows from (3.12), this complete the proof. □

6. Examples of DHRMC-Surfaces of rotation. Using the representation formula (5.2), we give some graphs of DHRMC-surfaces of rotation.

Example 6.1. Considering $a_1 = 1$, $b_1 = 2$, $l(t) = t+1$, $C = -1$ in Theorem 5.1, we obtain a DHRMC-surface of rotation.

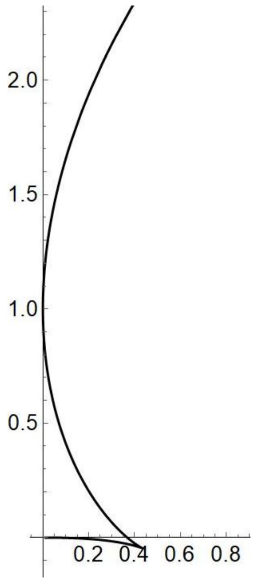

Figure 6.1: Profile curve.

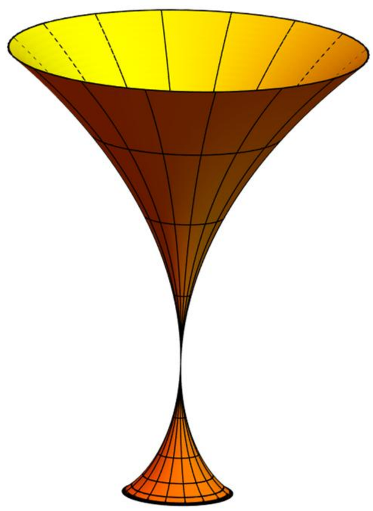

Figure 6.2: DHRMC-surface of rotation.

Example 6.2. Considering $a_1 = 1$, $b_1 = 1$, $l(t) = t^2$, $C = -2t^2$ in Theorem 5.1, we obtain a DHRMC-surface of rotation.

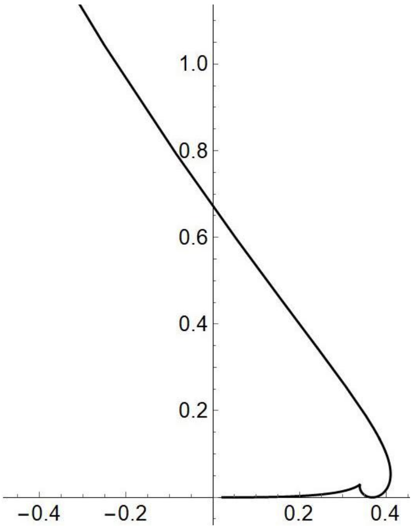

Figure 6.3: Profile curve.

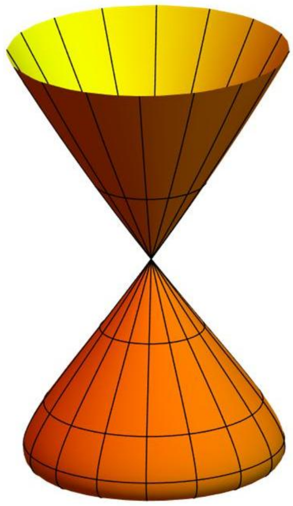

Figure 6.4: DHRMC-surface of rotation.

Example 6.3. Considering $a_1 = 1$, $b_1 = 1$, $l(t) = \sqrt[3]{t}$, $C = -\dfrac{1}{3t\sqrt[3]{t}}$ in Theorem 5.1, we obtain a DHRMC-surface of rotation.

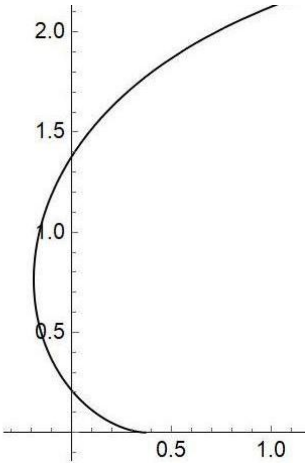

Figure 6.5: Profile curve.

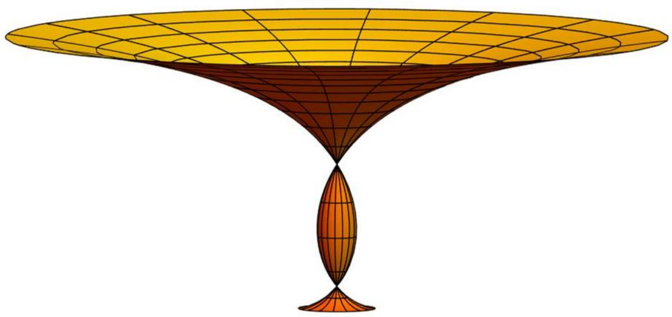

Figure 6.6: DHRMC-surface of rotation.

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References

[1] Corro AV. Generalized Weingarten Surfaces of Bryant type in hyperbolic 3-space. XIV School on Differential Geometry (Portuguese). Mat. Contemp. 2006; 30:71–89.

[2] Corro AMV, Fernandes KV, Riveros CMC. Generalized Weingarten Surfaces of harmonic type in hiperbolic 3-Space. Differential Geometry and its Applications. 2018; (58):202–226.

[3] Corro AV, Pina R, Souza M. Surfaces of Rotation with Constant Extrinsic Curvature in a Conformally Flat 3-Space. Results. Math. 2011; 60:225–234.

[4] Espinar JM, Gálvez JA, Mira P. Hypersurfaces in $\mathbb{H}^{n+1}$ and conformally invariant equations: the generalized Christoffel and Nirenberg problems. J. Eur. Math. Soc. 2009; 11(4):903–939.

[5] Machado CDF, Riveros CMC. Weingarten hypersurfaces of the spherical type in Euclidean spaces. Commentationes Mathematicae Universitatis Carolinae, vol. 2020; 61(2):213–236.