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Integration of Monomials over the Unit Sphere and Unit Ball in $\mathbb{R}^n$

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Abstract

We compute the integral of monomials of the form $x^{2\beta}$ over the unit sphere and the unit ball in $\mathbb{R}^n$ where $\beta = (\beta_1, \ldots, \beta_n)$ is a multi–index with real components $\beta_k > -1/2$, $1 \le k \le n$, and discuss their asymptotic behavior as some, or all, $\beta_k \to \infty$. This allows for the evaluation of integrals involving circular and hyperbolic trigonometric functions over the unit sphere and the unit ball in $\mathbb{R}^n$. We also consider the Fourier transform of monomials x^α restricted to the unit sphere in $\mathbb{R}^n$, where the multi–indices α have integer components, and discuss their behaviour at the origin.

Keywords . Integration over the Unit Sphere in $\mathbb{R}^n$, Integration over the Unit Ball in $\mathbb{R}^n$

1. Introduction. In his most influential mathematical work, the *Arithmetica Infinitorum*, published in 1656, Wallis introduced the well-known formulas

$$\int_0^{\pi/2} \sin^{2k}(\theta)\, d\theta = \int_0^{\pi/2} \cos^{2k}(\theta)\, d\theta = \frac{\sqrt{\pi}}{2}\frac{(2k-1)!!}{(2k)!!}, \quad k = 0, 1, 2, \ldots,$$

where $k!! = k(k-2)\cdots 2$ if k is even, and $k!! = k(k-2)\cdots 1$, if k is odd, and determined upper and lower bounds for the *Wallis ratios*, w_k, defined by

$$w_k = \frac{(2k-1)!!}{(2k)!!} = \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}, \quad k = 1, 2, \ldots,$$

where $\Gamma(s)$ is the Gamma function defined by $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$.

These results continue to attract interest today. In his paper *On Wallis' formula* [1], N. D. Kazarinoff observes that "In the course of mathematical progress new truths are discovered while older ones are sometimes more precisely articulated and often generalised. Because of their elegance and simplicity, however, some classical statements have been left unchanged. As an example, I have in mind the celebrated formula of John Wallis

$$\frac{1}{\sqrt{(\pi n)}} > \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)} > \frac{1}{\sqrt{\pi(\{n+\frac{1}{2}\})}}, \; n = 1, 2, \ldots,$$

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which for more than a century has been quoted by writers of textbooks ... Unquestionably, inequalities similar to this one can be improved indefinitely but at a sacrifice of simplicity, which is why they have survived so long."

Kazarinoff then proceeds to improve Wallis estimates; his approach entails incorporating negative powers in Wallis formula [1, p.19], specifically, with $-1 < \beta < \infty$,

$$\int_0^{\pi/2} \sin^\beta(\theta)\, d\theta = \int_0^{\pi/2} \cos^\beta(\theta)\, d\theta = \frac{\sqrt{\pi}}{2}\, \frac{\Gamma\big(\frac{\beta}{2}+\frac{1}{2}\big)}{\Gamma\big(\frac{\beta}{2}+1\big)}.$$

We could find no proof of this statement in the literature. Now, resting on an idea of Herman Weyl [2], one may show that for $\beta_1, \beta_2 > -1$,

$$\int_0^{\pi/2} \cos^{\beta_1}(\theta) \sin^{\beta_2}(\theta)\, d\theta = \frac{1}{2}\, \frac{\Gamma\big(\frac{\beta_1}{2}+\frac{1}{2}\big)\,\Gamma\big(\frac{\beta_2}{2}+\frac{1}{2}\big)}{\Gamma\big(\frac{\beta_1}{2}+\frac{\beta_2}{2}+1\big)} \tag{1.1}$$

where the integral is interpreted in the sense of an improper Riemann integral.

And also that, applying Stirling's formula

$$\Gamma(x+a) \sim \sqrt{2\pi x}\, x^x x^{a-1} e^{-x}, \quad a > 0,$$

asymptotically one has

$$\int_0^{\pi/2} \cos^{\beta_1}(\theta) \sin^{\beta_2}(\theta)\, d\theta \sim \sqrt{\pi}\, \frac{\beta_1^{\beta_1/2} \beta_2^{\beta_2/2}}{\big(\beta_1+\beta_2\big)^{(\beta_1+\beta_2+1)/2}}$$

as $\beta_1, \beta_2 \to \infty$.

The reader will have no difficulty in proving these statements, as they follow along the lines that of Proposition 2.1. In fact, one may think of these results as the 2-dimensional version of Proposition 2.1.

Kazarinoff's result attracted quite a bit of attention, and his prediction that the improvements of Wallis inequality would come at the expense of simplicity proved correct for 50 years. Then Ch.-P. Chen and F. Qi determined the best bounds in Wallis' inequality [3], and shortly after the proof was simplified to the extent that proved Kazarinoff's assessment wrong [4, 5, 6].

Now, since $\sin(\theta)$ and $\cos(\theta)$ are the coordinates of a point in the unit sphere in $\mathbb{R}^2$, Kazarinoff's formula and (1.1) may be interpreted as integrating over the unit sphere in $\mathbb{R}^2$, thus suggesting possible extensions of Wallis formulas to higher dimensions. Baker and Namazi, independently, considered the integral of monomials over the unit sphere in $\mathbb{R}^n$ and using an inductive argument combined with the divergence theorem discussed what Namazi called a generalized Wallis formula [7, 8].

Polynomial integration over the unit sphere is used in a variety of applications. In Physics, for instance, they include those situations that involve integrands containing a Green's function over the unit sphere. For example, Mura used a Fourier transform applied to an anisotropic Green's function to integrate the displacement field inside an ellipsoidal domain over the unit sphere [9]. And, Asaro and Barnett integrated the strain field inside an ellipsoidal domain subjected to stress free strain (a polynomial of degree M) over the unit sphere and observed that the formulation becomes complex for $M > 1$, [10].

Othmani discussed these and other examples, including the analytical expressions for the finite Eshelby tensors, and by means of an inductive argument combined with the divergence theorem, concluded that (in dimensions 2 and 3) the integral of monomials of order k over the unit sphere vanishes if k is odd, and satisfies an iterative relation if k is even [11, 12]. There is a caveat here. By symmetry considerations, the integral over the unit sphere of a monomial of even order that contains an odd power of one of its space variables in its expression, also vanishes.

Higher dimensional results were considered by Bochniak and Sitarz when addressing the spectral interaction between universes [13]. And Wang, when discussing the small sphere limit of quasilocal energy in higher dimensions along lightcone cuts, raised the possibility of integrating the reciprocal of monomials over the unit sphere [14].

As for our results, we begin by introducing some notations. Given a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative integers and a scalar λ, let

$$\lambda\,\alpha = (\lambda\,\alpha_1, \dots, \lambda\,\alpha_n), \quad \alpha! = \alpha_1! \cdots \alpha_n!\,, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

And, for $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$, let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Now, for a multi-index $\beta = (\beta_1, \dots, \beta_n)$ of real numbers, since for real numbers t, δ we interpret $t^{2\delta} = \Big(\big(t^2\big)^{1/2}\Big)^{2\delta} = |t|^{2\delta}$, we have $x^{2\beta} = x_1^{2\beta_1} \cdots x_n^{2\beta_n} = |x_1|^{2\beta_1} \cdots |x_n|^{2\beta_n}$.

We will assume that n is greater than or equal to 3, and denote with $d\sigma$ the element of surface area on the unit sphere $\partial B(0,1)$ in $\mathbb{R}^n$, and with $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ the surface area of the unit sphere in $\mathbb{R}^n$.

Symmetry plays a role in our discussion. Indeed, the integral over the unit sphere of any monomial x^α where α_j is odd for some j, is 0. And, since

$$\int_{\partial B(0,1)} x_j^2\, d\sigma(x) = \int_{\partial B(0,1)} x_k^2\, d\sigma(x)\,, \quad 1 \le j, k \le n,$$

it readily follows that

$$\frac{1}{\omega_n} \int_{\partial B(0,1)} x_k^2\, d\sigma(x) = \frac{1}{n}, \quad 1 \le k \le n.$$

This expresses the most elementary form of a generalized Wallis formula in higher dimensions.

In 1939 Hermann Weyl proved what he called a well–known formula for calculating the mean value of the monomial $x^{2\alpha}$ over the unit sphere in $\mathbb{R}^n$, [2, (12), p. 465], to wit,

$$\frac{1}{\omega_n} \int_{\partial B(0,1)} x^{2\alpha}\, d\sigma(x) = \frac{\prod_{\alpha_k \ne 0, k=1}^{n} (2\alpha_k - 1)!!}{(n + 2|\alpha| - 2) \cdots (n+2)n}. \tag{1.2}$$

We refer to (1.2) as *Wallis n–dimensional formula*.

And, a similar result holds for the unit ball in $\mathbb{R}^n$. Indeed, passing to polar coordinates, since $v_n = \omega_n/n$, from (1.2) it follows that

$$\begin{aligned} \frac{1}{v_n} \int_{B(0,1)} x^{2\alpha}\, dx &= n \int_0^1 r^{n+2|\alpha|-1} dr\, \frac{1}{\omega_n} \int_{\partial B(0,1)} x^{2\alpha}\, d\sigma(x) \\ &= \frac{\prod_{\alpha_k \ne 0, k=1}^{n} (2\alpha_k - 1)!!}{(n + 2|\alpha|)(n+2|\alpha| - 2) \cdots (n+2)}. \end{aligned} \tag{1.3}$$

Weyl's result, and approach, was revisited in [15], where it is remarked that Wallis formula remains valid for multi–indices $\beta = (\beta_1, \dots, \beta_n)$ of non-negative real numbers β_k, $1 \le k \le n$.

Wallis n–dimensional formula is the underlying principle of this note, which is organized as follows. In Section 2 we show that Weyl's approach extends to monomials $x^{2\beta}$ where the multi–indices β have real components $\beta_k > -1/2$, $1 \le k \le n$, and discuss the asymptotic behaviour as some, or all, $\beta_k \to \infty$. And, similar results hold for the unit ball in $\mathbb{R}^n$. In Section 3 we develop the tools to address the Fourier transform of monomials x^α restricted to the unit sphere in $\mathbb{R}^n$ when the multi–indices α have integer components, which is then carried out in Section 4; these transforms, when evaluated at the origin, yield Wallis n–dimensional formula. And, in Section 5 we apply Wallis n–dimensional formula to evaluate integrals involving circular and hyperbolic trigonometric functions over the unit sphere and the unit ball in $\mathbb{R}^n$. The Bernoulli and Euler numbers appear naturally in this context.

2. Generalized Wallis Formula in $\mathbb{R}^n$. *Plane wave functions*, that is, functions on $\mathbb{R}^n$ that are constant along the hyperplanes perpendicular to a fixed direction in $\mathbb{R}^n$, are an important tool when integrating over the unit sphere. Fritz John considers in particular plane wave functions $g(y \cdot x)$ of $x \in \mathbb{R}^n$, where $g(s)$ is a continuous function of the scalar variable s and y a fixed vector in $\mathbb{R}^n$, which are constant along the hyperplanes perpendicular to the direction of y, and obtains the fundamental identity

$$\int_{\partial B(0,1)} g\big(y\cdot x\,\big)\,d\sigma(x)=\omega_{n-1}\int_{-1}^{1}\big(1-t^2\big)^{(n-3)/2}g(|y|\,t)\,dt$$

for the integral of $g(x\cdot y)$ over the unit sphere [16, (1.2), p. 8].

In the particular case that $g(t)=|t|^k$, where k is an even integer, Fritz John observes that

$$\int_{\partial B(0,1)}\big|\,y\cdot x\,\big|^k\,d\sigma(x)=c_{k,n}\,|y|^k,$$

and (1.2) may be derived from this setting $k=2\,|\alpha|$.

The formulation of Weyl's result anticipated in the Introduction is the following:

Proposition 2.1. *Let $\beta=(\beta_1,\dots,\beta_n)$ be a multi-index of real numbers with $\beta_k>-1/2$, $1\le k\le n$. Then, the integral over the unit sphere in $\mathbb{R}^n$ of the monomial $x^{2\beta}$ can be computed as*

$$\int_{\partial B(0,1)} x^{2\beta}\,d\sigma(x)=2\,\frac{1}{\Gamma(|\beta|+n/2)}\prod_{k=1}^{n}\Gamma(\beta_k+1/2).\tag{2.1}$$

In particular, when $\beta=\alpha$ is a multi–index of nonnegative integers, (1.2) holds. Furthermore, asymptotically,

$$\int_{\partial B(0,1)} x^{2\beta}\,d\sigma(x)$$
$$\sim 2(2\pi)^{(m-1)/2}e^{(\beta_{m+1}+\cdots+\beta_n)}\prod_{k=m+1}^{n}\Gamma(\beta_k+1/2)\frac{\prod_{k=1}^{m}\beta_k^{\beta_k}}{|\beta|^{|\beta|+(n-1)/2}},\tag{2.2}$$

as $\beta_1,\dots,\beta_m\to\infty$, for $1\le m\le n$.

And, if all the $\beta_j\to\infty$, we have

$$\int_{\partial B(0,1)} x^{2\beta}\,d\sigma(x)\sim 2(2\pi)^{(n-1)/2}\frac{\prod_{k=1}^{n}\beta_k^{\beta_k}}{|\beta|^{|\beta|+(n-1)/2}}.$$

Proof: Fix $\varepsilon>0$, let $B(0,\varepsilon)$ denote the ball of radius ε centered at the origin, and set

$$f(x,\varepsilon)=x^{2\beta}e^{-|x|^2}\big(1-\chi_{B(0,\varepsilon)}(x)\big),\quad x\in\mathbb{R}^n.$$

Then, passing to polar coordinates, it readily follows that

$$\int_{\mathbb{R}^n} f(x,\varepsilon)\,dx=\int_{\varepsilon}^{\infty}e^{-r^2}r^{2\,|\beta|+n-1}\,dr\int_{\partial B(0,1)}x^{2\beta}\,d\sigma(x),$$

where

$$\int_{\varepsilon}^{\infty}e^{-r^2}r^{2\,|\beta|+n-1}\,dr=\frac{1}{2}\int_{\varepsilon^2}^{\infty}e^{-r}r^{|\beta|+n/2-1/2}r^{-1/2}\,dr.$$

Hence, since by assumption $|\beta|=\beta_1+\cdots+\beta_n>-n/2$, we have

$$\lim_{\varepsilon\to0}\int_{\mathbb{R}^n} f(x,\varepsilon)\,dx=\frac{1}{2}\Gamma(|\beta|+n/2)\int_{\partial B(0,1)}x^{2\beta}\,d\sigma(x).\tag{2.3}$$

Let now $\chi_Q(0,2\varepsilon)$ denote the characteristic function of the cube of sidelength 2ε centered at the origin, and observe that

$$\int_{\mathbb{R}^n} f(x,\varepsilon)\,dx=\int_{\mathbb{R}^n}x^{2\beta}e^{-|x|^2}\big(1-\chi_{Q(0,2\varepsilon)}(x)\big)\,dx$$
$$+\int_{\mathbb{R}^n}x^{2\beta}e^{-|x|^2}\big(\chi_{Q(0,2\varepsilon)}(x)-\chi_{B(0,\varepsilon)}(x)\big)\,dx.$$

Then, with $\alpha = 1/\sqrt{2n}$ it readily follows that

$$\Big|\chi_{Q(0,2\varepsilon)}(x) - \chi_{B(0,\varepsilon)}(x)\Big| \lesssim \Big(\chi_{Q(0,2\varepsilon)}(x) - \chi_{Q(0,2\alpha\varepsilon)}(x)\Big)$$

and, consequently, that

$$\begin{aligned}\Big| \int_{\mathbb{R}^n} x^{2\beta} e^{-|x|^2} \Big(\chi_{Q(0,2\varepsilon)}(x) - \chi_{Q(0,2\alpha\varepsilon)}(x)\Big)\, dx\Big|& \\ \lesssim \sum_{k=1}^{n} \int_{2\alpha\varepsilon}^{\varepsilon} |x_k|^{2\beta_k}\, dx_k \lesssim \sum_{k=1}^{n} \varepsilon^{1+2\beta_k},&\end{aligned} \tag{2.4}$$

which, since $1 + 2\beta_k > 0$ for all k, tends to 0 as $\varepsilon \to 0$.

Now, the integral

$$\int_{\mathbb{R}^n} x^{2\beta} e^{-|x|^2} \big(1 - \chi_{Q(0,2\varepsilon)}(x)\big)\, dx$$

can be computed as

$$\begin{aligned}\prod_{k=1}^{n} \int_{\mathbb{R}} e^{-x_k^2}\, x_k^{2\beta_k} \big(1 - \chi_{Q(0,2\varepsilon)}(x_k)\big) dx_k &= 2^n \prod_{k=1}^{n} \int_{\varepsilon}^{\infty} e^{-x_k^2}\, x_k^{2\beta_k} dx_k \\ &= \prod_{k=1}^{n} \int_{\varepsilon^2}^{\infty} e^{-x_k}\, x_k^{\beta_k - 1/2} dx_k\end{aligned}$$

and, therefore, we get

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} x^{2\beta} e^{-|x|^2} \big(1 - \chi_{Q(0,2\varepsilon)}(x)\big)\, dx = \prod_{k=1}^{n} \Gamma(\beta_k + 1/2) \tag{2.5}$$

Hence, combining (2.4) and (2.5) we conclude that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} f(x,\varepsilon)\, dx = \prod_{k=1}^{n} \Gamma(\beta_k + 1/2),$$

which together with (2.3) gives

$$\frac{1}{2}\Gamma(|\beta| + n/2) \int_{\partial B(0,1)} x^{2\beta}\, d\sigma(x) = \prod_{k=1}^{n} \Gamma(\beta_k + 1/2),$$

and (2.1) holds.

Now, unraveling the expression on the right–hand side of (2.1) when $\beta = \alpha$ is an integer multi–index it follows that in this case also (1.2) holds.

As for the asymptotic behavior of (2.1), suppose that $\beta_1, \dots, \beta_m \to \infty$, $1 \le m \le n$. Then by Stirling's formula we have

$$\Gamma(\beta_k + 1/2) \sim \sqrt{2\pi}\, \beta_k^{\beta_k} e^{-\beta_k}, \quad 1 \le k \le m,$$

and

$$\Gamma(|\beta| + n/2) \sim \sqrt{2\pi}\, |\beta|^{|\beta| + (n-1)/2} e^{-|\beta|},$$

and, consequently,

$$\begin{aligned}\frac{1}{\Gamma(|\beta| + n/2)} \prod_{k=1}^{m} \Gamma(\beta_k + 1/2) &\sim \frac{1}{\sqrt{2\pi}\, |\beta|^{|\beta|+(n-1)/2} e^{-|\beta|}} \prod_{k=1}^{m} \sqrt{2\pi}\, \beta_k^{\beta_k} e^{-\beta_k} \\ &\sim (2\pi)^{(m-1)/2}\, e^{(\beta_{m+1}+\cdots+\beta_n)}\, \frac{\prod_{k=1}^{m} \beta_k^{\beta_k}}{|\beta|^{|\beta|+(n-1)/2}}.\end{aligned} \tag{2.6}$$

(2.2) follows at once from (2.1) and (2.6). □

As for the integral over the unit ball in $\mathbb{R}^n$ we have:

Proposition 2.2. *Let $\beta = (\beta_1, \dots, \beta_n)$ be a multi-index of real numbers with $\beta_k > -1/2$, $1 \le k \le n$. Then, the integral of the monomial $x^{2\beta}$ over the unit ball in $\mathbb{R}^n$ can be computed as*

$$\int_{B(0,1)} x^{2\beta}\,dx = \frac{1}{\Gamma((|\beta|+1)+n/2)} \prod_{k=1}^{n} \Gamma(\beta_k + 1/2). \tag{2.7}$$

In particular, when $\beta = \alpha$ is a multi–index of nonnegative integers, (1.3) holds.
Furthermore, asymptotically,

$$\int_{B(0,1)} x^{2\beta}\,dx \sim (2\pi)^{(m-1)/2} e^{(\beta_{m+1}+\cdots+\beta_n)} \prod_{k=m+1}^{n} \Gamma(\beta_k + 1/2) \frac{\prod_{k=1}^{m} \beta_k^{\beta_k}}{|\beta|^{|\beta|+(n+1)/2}} \tag{2.8}$$

as $\beta_1, \dots, \beta_m \to \infty$, for $1 \le m \le n$.
And, if all the $\beta_j \to \infty$, we have

$$\int_{B(0,1)} x^{2\beta}\,dx \sim (2\pi)^{(n-1)/2} \frac{\prod_{k=1}^{n} \beta_k^{\beta_k}}{|\beta|^{|\beta|+(n+1)/2}}.$$

Proof: For $\varepsilon > 0$, consider

$$g(x,\varepsilon) = x^{2\beta} \chi_{B(0,1)\setminus B(0,\varepsilon)(x)}.$$

Then, passing to polar coordinates it readily follows that

$$\int_{\mathbb{R}^n} g(x,\varepsilon)\,dx = \int_{\varepsilon}^{1} r^{n+2|\beta|-1} dr \int_{\partial B(0,1)} x^{2\beta}\,d\sigma(x),$$

where, by assumption $n + 2|\beta| > 0$, and, consequently,

$$\lim_{\varepsilon\to 0} \int_{\varepsilon}^{1} r^{n+2|\beta|-1} dr = \frac{1}{(2|\beta|+n)}.$$

Thus, by (2.1) we have

$$\begin{aligned}
\int_{B(0,1)} x^{2\beta}\,dx &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} g(x,\varepsilon)\,dx \\
&= \frac{1}{(2|\beta|+n)} \int_{\partial B(0,1)} x^{2\beta}\,d\sigma(x) \\
&= \frac{1}{\Gamma((|\beta|+1)+n/2)} \prod_{k=1}^{n} \Gamma(\beta_k + 1/2),
\end{aligned}$$

which gives (2.7).

Moreover, since

$$\frac{1}{2|\beta|+n} \sim \frac{1}{2|\beta|} \quad \text{as } |\beta| \to \infty,$$

the asymptotic values (2.8) follow at once from (2.2), and we have finished. □

3. Bessel Functions. In this section we cover the preliminary material to discuss the Fourier transform of mononials restricted to the unit sphere in $\mathbb{R}^n$.

$J_\nu(x)$, the Bessel function of order ν, is defined as the solution of the second order linear equation

$$t^2\,\frac{d^2y}{dt^2}+t\frac{dy}{dt}+(t^2-\nu^2)\,y=0.$$

Basic properties of the Bessel functions follow readily from their power series expansion [17],

$$J_v(t)=\Big(\frac{t}{2}\Big)^{\nu}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\Big(\frac{t}{2}\Big)^{2t}.$$

They include:

$$\lim_{t\to0^+}\frac{J_\nu(t)}{t^\nu}=2^{-\nu}\frac{1}{\Gamma(\nu+1)},$$

and the recurrence formula

$$\frac{d}{dt}\Big(\frac{J_\nu(t)}{t^\nu}\Big)=-t^{-\nu}\,J_{\nu+1}(t).$$

In this note we will work with the function $\Psi_\nu(t)$ defined by

$$\Psi_\nu(t)=\frac{J_\nu(t)}{t^\nu}=2^{-\nu}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\Big(\frac{t}{2}\Big)^{2k}. \tag{3.1}$$

Then the above relations become, respectively,

$$\lim_{t\to0^+}\Psi_\nu(t)=2^{-\nu}\frac{1}{\Gamma(\nu+1)}\,, \tag{3.2}$$

and

$$\Psi_\nu'(t)=-t\,\Psi_{\nu+1}(t). \tag{3.3}$$

Now, repeated applications of (3.3) yield

$$\Psi_\nu''(t)=-\Psi_{\nu+1}(t)+t^2\Psi_{\nu+2}(t), \tag{3.4}$$

$$\Psi_\nu'''(t)=3t\Psi_{\nu+2}(t)-t^3\Psi_{\nu+3,}(t),$$

$$\Psi_\nu^{iv}(t)=3\Psi_{\nu+2}(t)-6t^2\Psi_{\nu+3}(t)+t^4\Psi_{\nu+4}(t), \tag{3.5}$$

$$\Psi_\nu^{v}(t)=-15t\Psi_{\nu+3}(t)+10t^3\Psi_{\nu+4}(t)-t^5\Psi_{\nu+5}(t),$$

and so on.

Thus, the following pattern emerges. If D denotes differentiation with repect to t, $D^k\Psi_\nu(t)$ is a polynomial of degree k in t with coefficients $c_j\,\Psi_{\nu+j}(t)$ for some scalars c_j with $0\le j\le k$. And, since as is readily seen from (3.1), $\Psi_\nu(t)$ is an even function, its derivatives of odd order are odd functions, and those of even order are even.

Moreover, for integers $k=1,2,\ldots$, $D^{2k}\Psi_\nu(t)$ is an even polynomial of degree $2k$ in t consisting of $k+1$ terms that can be written as

$$\begin{aligned}&D^{2k}\Psi_\nu(t)\\&\qquad=t^{2k}\Psi_{\nu+2k}(t)-c_{2k-1}t^{2(k-1)}\Psi_{\nu+2k-1}(t)+\cdots+(-1)^kc_k\Psi_{\nu+k}(t).\end{aligned} \tag{3.6}$$

In fact, it can be readily seen that if (3.6) holds for $2k$, then, on account of (3.4), it also holds for $2(k+1)$, and it is thus valid for all $2k$.

A similar argument applies to $D^{2k+1}\Psi_\nu(t)$ for integers $k=1,2,\ldots$ In this case $D^{2k+1}\Psi_\nu(t)$ is an odd polynomial of degree $2k+1$ in t consisting of $k+1$ terms that can be written as

$$\begin{aligned}&D^{2k+1}\Psi_\nu(t)\\&=-t^{2k+1}\Psi_{\nu+2k+1}(t)+d_{2k}t^{2k-1}\Psi_{\nu+2k}(t)+\cdots+(-1)^{k+1}d_k\,t\Psi_{\nu+k+1}(t),\end{aligned} \tag{3.7}$$

and which can be verified as in the even case.

We are particularly interested in the constant term when the polynomial is even, and in the term corresponding to t when the polynomial is odd. We adopt here the usual notation that $\varphi(\xi) = o(|\xi|^\eta)$, where $\eta > 0$, as $|\xi| \to 0$, provided that

$$\lim_{|\xi|\to 0} \frac{|\varphi(\xi)|}{|\xi|^\eta} = 0.$$

We begin by proving:

Proposition 3.1. *With $\nu > 0$, let*

$$\Psi_\nu(\xi) = \frac{J_\nu(|\xi|)}{|\xi|^\nu}, \quad \xi \in \mathbb{R}^n.$$

Then,

$$\lim_{|\xi|\to 0} \Psi_\nu(\xi) = 2^{-\nu} \frac{1}{\Gamma(\nu+1)}. \tag{3.8}$$

Moreover, for $\varepsilon > 0$, $1 \le j \le n$, and $k = 1, 2, \ldots$,

$$\frac{\partial^{2k}\Psi_\nu(\xi)}{\partial \xi_j^{2k}} = (-1)^k (2k-1)!!\, \Psi_{\nu+k}(\xi) + o(|\xi|^{2-\varepsilon}) \tag{3.9}$$

as $|\xi| \to 0$.

And

$$\frac{\partial^{2k+1}\Psi_\nu(\xi)}{\partial \xi_j^{2k+1}} = (-1)^{k+1} (2k+1)!!\, (\xi_j)\, \Psi_{\nu+k+1}(\xi) + o(|\xi|^{3-\varepsilon}) \tag{3.10}$$

as $|\xi| \to 0$.

Proof: (3.8) is a restatement of (3.2). Now, since

$$\frac{\partial \Psi_\nu(\xi)}{\partial \xi_j} = \frac{\partial\Big(|\xi|^{-\nu}\, J_\nu(|\xi|)\Big)}{\partial \xi_j} = -\xi_j\, \frac{J_{\nu+1}(|\xi|)}{|\xi|^{\nu+1}} = -\xi_j\, \Psi_{\nu+1}(\xi),$$

and, similarly,

$$\frac{\partial^2 \Psi_\nu(\xi)}{\partial \xi_j^2} = -\Psi_{\nu+1}(\xi) + \xi_j^2 \Psi_{\nu+2}(\xi),$$

these expressions correspond to (3.3) and (3.4) with $\Psi_\nu(t)$ replaced by $\Psi_\nu(\xi)$ and t replaced by ξ_j there. The same applies to all other statements, so we will consider $\Psi_\nu(t)$ in what follows.

Now, in the even case we are interested in the constant term. Setting $t = 0$ in (3.6) it follows that

$$D^{2k}\Psi_\nu(0) = (-1)^k c_k \Psi_{\nu+k}(0). \tag{3.11}$$

Also, by (3.1) it readily follows that

$$D^{2k}\Psi_\nu(0) = 2^{-\nu}(-1)^k \frac{1}{k!} \frac{1}{\Gamma(\nu+k+1)2^{2k}}\, 2k!,$$

which, since

$$\frac{1}{2^{2k}}\, \frac{2k!}{k!} = 2^{-k}(2k-1)!!$$

can be restated as

$$D^{2k}\Psi_\nu(0) = (-1)^k 2^{-(\nu+k)} \frac{1}{\Gamma(\nu+k+1)} (2k-1)!!,$$

Thus, since by (3.8)

$$2^{-(\nu+k)}\, \frac{1}{\Gamma(\nu+k+1)} = \Psi_{\nu+k}(0),$$

it follows that

$$D^{2k}\Psi_\nu(0) = (-1)^k(2k-1)!!\,\Psi_{\nu+k}(0), \tag{3.12}$$

and by (3.11) and (3.12) we get

$$(-1)^k c_k \Psi_{\nu+k}(0) = (-1)^k(2k-1)!!\,\Psi_{\nu+k}(0),$$

and, therefore, $c_k = (2k-1)!!$

We are also interested in the coefficient of the term corresponding to t when k is odd. Note that, from (3.4) applied to (3.4) it follows that

$$D^{2k+2}\Psi_\nu(t) = (-1)^{k+1}2\,c_{k+1}\Psi_{\nu+k+1}(t) + (-1)^{k+1}c_k\Psi_{\nu+k+1}(t) + \Phi(t), \tag{3.13}$$

where $\Phi(t) = o(|t|^{4-\varepsilon})$, and $\Phi(0) = 0$. Hence,

$$D^{2k+2}\Psi_\nu(0) = (-1)^{k+1}\big(2c_{k+1} + c_k\big)\Psi_{\nu+k+1}(0). \tag{3.14}$$

And, since as in (3.12), it follows that

$$D^{2k+2}\Psi_\nu(0) = (-1)^{k+1}(2k+1)!!\,\Psi_{\nu+k+1}(0),$$

combining (3.13) and (3.14), we get

$$2c_{k+1} + c_k = (2k+1)!!, \tag{3.15}$$

which, since $c_k = (2k-1)!!$, implies that

$$c_{k+1} = \frac{1}{2}\big((2k+1)!! - (2k-1)!!\big) = k\,(2k-1)!!$$

To determine d_k now, observe that differentiating (3.5), from (3.3) it readily follows that

$$D^{2k+1}\Psi_\nu(t) = (-1)^{k+1}(c_k + 2c_{k+1})t\,\Psi_{\nu+k+1}(t) + \Phi(t),$$

where $\Phi(0) = 0$, and, therefore, comparing this expression with (3.6) it readily follows that $d_k = c_k + 2c_{k+1}$. Hence, from (3.15) we conclude that

$$d_k = (2k+1)!!$$

and the proof is finished. □

The above reasoning allows for the determination of all the desired coefficients in (3.6) and (3.7). For example, one may verify that c_{k+2} in (3.6) is equal to

$$c_{k+2} = \frac{1}{3!}\,k(k-1)(2k-1)!!$$

At this time, no obvious pattern to compute the various coefficients is apparent to us.

4. The Fourier Transform of Monomials Restricted to the Unit Sphere. The theory of polynomials on the unit sphere of $\mathbb{R}^n$ is developed in [18, Chapter 5], where in particular a refined version of Weyl's result (1.2) for harmonic polynomials is given. Here we complement those results by computing the Fourier transform of monomials restricted to the unit sphere of $\mathbb{R}^n$ and discussing their behaviour at the origin..

Recall that when $n \ge 3$, the Fourier transform of the surface area measure carried on the sphere $\partial B(0,1)$ centered at the origin of radius 1 in $\mathbb{R}^n$, is given by [19, p. 154],

$$\int_{\partial B(0,1)} e^{-ix\cdot\xi}\,d\sigma(x) = (2\pi)^{n/2}\,\Psi_{(n-2)/2}(\xi). \tag{4.1}$$

To fix ideas we consider first the Fourier transform of the monomial x_1^{2k} restricted to the unit sphere. Differentiating (4.1) with respect to ξ_1 it follows that

$$\begin{aligned}\frac{1}{(2\pi)^{n/2}}&\int_{\partial B(0,1)} x_1^{2k}\,e^{-ix\cdot\xi}\,d\sigma(x)\\ &= (2k-1)!!\;\Psi_{(n/2)+k-1}(\xi) - k(2k-1)!!\,\xi_1^2\,\Psi_{(n/2)+k}(\xi)\\ &+\frac{1}{3!}k(k-1)(2k-1)!!\,\xi_1^4\,\Psi_{(n/2)+k+1}(\xi) + o(|\xi|),\end{aligned}$$

and we are done in this case.

For the applications we have in mind it suffices to invoke Proposition 3.1 with $\nu = (n-2)/2$ there, and observe that

$$\int_{\partial B(0,1)} x_1^{2k} e^{-ix\cdot\xi}\, d\sigma(x) = (2\pi)^{n/2}\,(2k-1)!!\,\Psi_{(n/2)+k-1}(\xi) + o(|\xi|). \tag{4.2}$$

It is important to note that all the summands that appear in the $o(|\xi|)$ term have a positive power of ξ_1 in them. The reason being that, when derivatives are taken with respect to variables other than ξ_1, the expression remains an $o(|\xi|)$ term.

Next, concerning the monomial x_1^{2k+1}, differentiating (4.2) one more time with respect to ξ_1, by (3.10) in Proposition 3.1 with $\nu = (n-2)/2$ there it follows that

$$\begin{aligned}\int_{\partial B(0,1)} & x_1^{2k+1} e^{-ix\cdot\xi}\, d\sigma(x)\\ &= (2\pi)^{n/2}(2k+1)!!\,(i\,\xi_1)\,\Psi_{(n/2)+k}(\xi) + o(|\xi|),\end{aligned} \tag{4.3}$$

which gives the desired result in this case.

We introduce some notations to address the general case. Given a multi-index $\alpha = (\alpha_1,\dots,\alpha_n)$, let $\alpha^o = (\alpha_1^o,\dots,\alpha_n^o)$ where $\alpha_j^o = \alpha_j$ if α_j is an odd integer and $= 0$ otherwise, $\mu(\alpha^o)$ the number of indices with $\alpha_j^o \neq 0$, and $\alpha^e = (\alpha_1^e,\dots,\alpha_n^e)$, where $\alpha_j^e = \alpha_j - \alpha_j^o$.

Let

$$A_\alpha(\xi) = (2\pi)^{n/2}\Big(\prod_{\alpha_m^e\neq 0, m=1} n(\alpha_m^e-1)!!\Big)\Big(\prod_{\alpha_m^o\neq 0,m=1}^{n} \alpha_m^o!!\,(i\xi_m)\Big).$$

We consider now the general case, which follows essentially by iterating (4.2) and (4.3):

Theorem 4.1. *Let $n \geq 3$. Then, for a multi-index α we have*

$$\begin{aligned}\int_{\partial B(0,1)} & x^\alpha\, e^{-ix\cdot\xi}\, d\sigma(x)\\ &= A_\alpha(\xi)\,\Psi_{(n+|\alpha|+\mu(o)-2)/2}(\xi) + o(|\xi|).\end{aligned} \tag{4.4}$$

Proof:

Given a multi-index α, let $\alpha = \alpha^e + \alpha^o$. Then, iterating the relation (4.3) for those ξ_m with $\alpha_m^o \neq 0$ it readily follows that

$$\begin{aligned}\int_{\partial B(0,1)} & x^{\alpha^o} e^{-ix\cdot\xi}\, d\sigma(x)\\ &= (2\pi)^{n/2}\Big(\prod_{\alpha_m^o\neq 0,m=1}^{n} (\alpha_m^o)!!\,(i\xi_m)\Big)\,\Psi_{(n+|\alpha^o|+\mu(o)-2)/2}(\xi) + o(|\xi|).\end{aligned} \tag{4.5}$$

Now, iterating the relation (4.2) for those ξ_m with $\alpha_m^e \neq 0$, since $(n+|\alpha^o|+\mu(o)-2)/2 + |\alpha^e|/2 = (n+|\alpha|+\mu(o)-2)/2$, from (4.5) it readily follows that

$$\begin{aligned}\int_{\partial B(0,1)} x^\alpha e^{-ix\cdot\xi}\, d\sigma(x) &= \int_{\partial B(0,1)} x^{\alpha^o} x^{\alpha^e} e^{-ix\cdot\xi}\, d\sigma(x)\\ &= A_\alpha(\xi)\,\Psi_{(n+|\alpha|+\mu(o)-2)/2}(\xi) + o(|\xi|).\end{aligned}$$

and we have finished. □

Wallis n–dimensional formula follows letting $\xi \to 0$ in (4.4). Indeed, the limit is equal to

$$A_\alpha(0)\,\Psi_{(n+|\alpha|+\mu(o)-2)/2}(0),$$

which in turn is 0 if $\mu(o) \neq 0$. Thus, we may assume that all the α_m are even, and then on account of (3.8) we have that the limit is equal to

$$(2\pi)^{n/2} \Big(\prod_{\alpha_m \neq 0, m=1}^{n} (\alpha_m - 1)!! \Big) \, \Psi_{(n+|\alpha|-2)/2}(0)$$
$$= (2\pi)^{n/2} 2^{-(n+|\alpha|-2)/2} \frac{1}{\Gamma((n+|\alpha|)/2)} \Big(\prod_{\alpha_m \neq 0, m=1}^{n} (\alpha_m - 1)!! \Big),$$

and the conclusion follows by a computational argument left to the reader.

5. Applications. We close the note with the integration of circular and hyperbolic trigonometric functions over the unit sphere and the unit ball in $\mathbb{R}^n$. The results follow from the known fact that a power series converges uniformly on compact subsets of its disc of convergence, and, therefore, it can be integrated termwise there. As for the uniform convergence of the series, one may invoke the known fact that if $f(t)$ is analytic in the disc $|t| < R$, it is the sum of its Taylor series there, or else establish the convergence invoking the asymptotics for the Bernoulli numbers, B_k, or Euler numbers, E_k, [20, 21].

Recall that

$$t \coth(t) = \sum_{k=0}^{\infty} \frac{2^{2k}}{(2k)!} \, B_{2k} \, t^{2k},$$

where the series, by either criteria described above, converges uniformly and absolutely for $|t| < \pi$, and so for a multi-index α, since $B_0 = 1$, the series

$$x^\alpha \coth(x^\alpha) = 1 + \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} \, B_{2k} \, x^{2k\alpha} \tag{5.1}$$

converges uniformly and absolutely for $|x| < R$ for some $R > 1$. We may then integrate (5.1) termwise, and by (1.2) or (2.1) conclude that

$$\frac{1}{\omega_n} \int_{\partial B(0,1)} x^\alpha \, \coth(x^\alpha) \, d\sigma(x)$$
$$= 1 + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} \frac{\prod_{\alpha_m \neq 0, m=1}^{n} (2k\alpha_m - 1)!!}{(n + 2|\alpha| \, k - 2) \cdots (n+2)n}.$$

Similarly, for the integral over the unit ball, by (1.3) or (2.7) it follows that

$$\frac{1}{v_n} \int_{B(0,1)} x^\alpha \coth(x^\alpha) \, dx$$
$$= 1 + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} \, \frac{\prod_{\alpha_m \neq 0, m=1}^{n} (2k\alpha_m - 1)!!}{(n + 2|\alpha| k)(n + 2|\alpha| k - 2) \cdots (n+2)}.$$

As for circular trigonometric functions, consider, for instance, the series expansion for $t/\sin(t)$ given by

$$\frac{t}{\sin(t)} = 1 + 2 \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(2^{2k-1} - 1)}{(2k)!} \, B_{2k} \, t^{2k},$$

which converges uniformly and absolutely for $|t| < \pi$. Hence, for a multi-index α the series

$$\frac{x^\alpha}{\sin(x^\alpha)} = 1 + 2 \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(2^{2k-1} - 1)}{(2k)!} \, B_{2k} \, x^{2k\alpha}, \tag{5.2}$$

converges uniformly and absolutely for $|x| < R$ for some $R > 1$, and we may evaluate the integral of (5.2) over the unit sphere and the unit ball as above.

A slight variant works when odd powers are involved in the expansion. For instance, consider the tangent, which can be expanded as

$$\tan(t) = \sum_{k=1}^{\infty} \tau_{2k-1}\, B_{2k}\, t^{2k-1},$$

where

$$\tau_{2k-1} = (-1)^k 2^{2k} \frac{(2^{2k-1}-1)}{(2k)!}, \quad k = 1, 2, \ldots$$

which converges absolutely and uniformly for $|t| < \pi/2$.

Thus, for a multi-index α the series

$$\tan(x^\alpha) = \sum_{k=1}^{\infty} \tau_{2k-1}\, B_{2k}\, x^{(2k-1)\alpha} \tag{5.3}$$

converges absolutely and uniformly for $|x| < R$, for some $R > 1$, and therefore (5.3) may be integrated termwise over the unit sphere and the unit ball in $\mathbb{R}^n$.

Now, the monomials in (5.3) integrate to 0 unless all the coordinates of the $(2k-1)\,\alpha$ are even integers, which is the case when all the α_j are even. Let then $\alpha = 2\alpha'$, where the α'_j are nonnegative integers.

Then, on account of (1.2) for $k = 1, 2, \ldots$, we have

$$\begin{aligned}\frac{1}{\omega_n}\int_{\partial B(0,1)} x^{(2k-1)\alpha}\, d\sigma(x) &= \frac{1}{\omega_n}\int_{\partial B(0,1)} x^{2(2k-1)\alpha'}\, d\sigma(x)\\ &= \frac{\prod_{\alpha'_m\neq 0, m=1}^{n}(2(2k-1)\alpha'_m-1)!!}{(n+2(2k-1)|\alpha'|-2)\cdots(n+2)n}\\ &= \frac{\prod_{\alpha_m\neq 0, m=1}^{n}((2k-1)\alpha_m-1)!!}{(n+(2k-1)|\alpha|-2)\cdots(n+2)n},\end{aligned}$$

and the answer follows readily from this by integrating (5.3) termwise.

As for the integral over the unit ball, observe that, similarly, by (1.3),

$$\begin{aligned}&\frac{1}{v_n}\int_{B(0,1)} x^{(2k-1)\alpha}\, dx\\ &\qquad = \frac{\prod_{\alpha_m\neq 0, m=1}^{n}((2k-1)\alpha_m-1)!!}{(n+(2k-1)|\alpha|)(n+(2k-1)|\alpha|-2)\cdots(n+2)},\end{aligned}$$

and the integral can be readily obtained by integrating (5.3) termwise.

Which brings us to the closing remarks. G. S. Ely considered the question of expanding powers of trigonometric functions given the expansions of the functions themselves [22]. In the case of the tangent one derives

$$\tan^3(t) = \sum_{k=1}^{\infty} \big(\, k\,(2k+1)\,\tau_{2k+1}\, B_{2k+2} - \tau_{2k-1}\, B_{2k}\,\big)\, t^{2k-1}.$$

We leave to the reader the evaluation of the integral

$$\int_{\partial B(0,1)} \tan^3(x^\alpha)\, d\sigma(x)\,.$$

Finally, unlike the functions considered above, the expansion of the secant involves the *Euler numbers*, E_k, and is given by

$$\sec(t) = \sum_{k=0}^{\infty} (-1)^k\, \frac{1}{(2k)!} E_{2k}\, t^{2k}.$$

Simple algebraic manipulations give that the expansion for $\sec^3(t)$ is given by

$$\sec^3(t) = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}\big(E_{2k} - E_{2(k+1)}\big)t^{2k}.$$

Then, for an appropriate real multi-index β the reader is invited to evaluate

$$\int_{B(0,1)} \sec^3(x^\beta)\, dx.$$

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