Journal homepage http://revistas.unitru.edu.pe/index.php/SSMM

SELECCIONES MATEMÁTICAS
Universidad Nacional de Trujillo
ISSN: 2411-1783 (Online)
2024; Vol.11(2):393-408.

The Interconnection Between Calculus of Variations, Partial Differential Equations and Differential Geometry

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Received, May. 05, 2024; *Accepted, Dec. 09, 2024;* *Published, Dec. 27, 2024*

How to cite this article:
Mwinken D. *The Interconnection Between Calculus of Variations, Partial Differential Equations, and Differential Geometry*. Selecciones Matemáticas. 2024;11(2):393–408. http://dx.doi.org/10.17268/sel.mat.2024.02.11

Abstract

Calculus of variations is a fundamental mathematical discipline focused on optimizing functionals, which map sets of functions to real numbers. This field is essential for numerous applications, including the formulation and solution of partial differential equations (PDEs) and the study of differential geometry. In PDEs, calculus of variations provides methods to find functions that minimize energy functionals, leading to solutions of various physical problems. In differential geometry, it helps understand the properties of curves and surfaces, such as geodesics, by minimizing arc-length functionals. This paper explores the intrinsic connections between these areas, highlighting key principles such as the Euler-Lagrange equation, Ekeland's variational principle, and the Mountain Pass theorem, and their applications in solving PDEs and understanding geometrical structures.

Keywords . Calculus of variations, functional optimization, partial differential equations, differential geometry, Euler-Lagrange equation, Ekeland's variational principle, mountain pass theorem, geodesics.

1. Introduction. Calculus of variations is a branch of mathematical analysis that focuses on optimizing functionals, which often represent physical quantities such as energy. The primary goal is to identify functions that either maximize or minimize these functionals. This field is critical in both theoretical and applied mathematics, providing essential insights and techniques for solving partial differential equations (PDEs) and for studying differential geometry.

The calculus of variations serves as a bridge between PDEs and differential geometry, offering tools to tackle complex problems in both areas. In PDEs, it aids in identifying functions that minimize energy functionals, thereby leading to solutions for various physical phenomena. In differential geometry, it helps in understanding the properties of curves and surfaces by minimizing arc-length functionals [1, 2].

A fundamental tool in calculus of variations is the Euler-Lagrange equation, which arises from the condition that the first variation of a functional must vanish at an extremum. This equation is essential in deriving the governing equations in physical systems described by PDEs and in understanding geometrical structures in differential geometry. Other key principles include Ekeland's variational principle and the Mountain Pass theorem, which further enhance our ability to solve complex problems and understand intricate geometrical forms [3].

This paper explores the interconnections between calculus of variations, PDEs, and differential geometry, emphasizing how the calculus of variations provides a unified framework for addressing diverse mathematical challenges [4].

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2. Euler-Lagrange Equation. The Euler-Lagrange equation forms the basis for finding optimal functions in variational problems and is essential in deriving the governing equations in physical systems modeled by PDEs. To find the function $y(x)$ that makes the functional $J[y]$ stationary that is a minimum or maximum, we derive the Euler-Lagrange equation.

Proposition 2.1.

If $y(x)$ is an extremum of J, then $y(x)$ satisfies:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0. \tag{2.1}$$

Proof:

Consider $J[y] = \int_a^b F(x, y, y')\, dx$. To find the extremun of $J[y]$ we introduce a pertubation $y(x) + \epsilon\eta(x)$ where $\eta(x)$ is arbitrary smooth function that vanishes at the boundaries ($\eta(a) = \eta(b) = 0$). The first variation of J is given by : $\delta J[y] = \frac{d}{d\epsilon}\, J[y + \epsilon\eta]$ for $\epsilon = 0$

Expanding J we have $J[y + \epsilon\eta] = \int_a^b F(x, y + \epsilon\eta, y' + \epsilon\eta)\, dx$.

Taking the derivative with respect to ϵ and evaluating at $\epsilon = 0$ $\delta J[y] = \int_a^b \left(\frac{\partial F}{\partial y}\eta - \frac{\partial F}{\partial y'}\eta'\right)\, dx$.

Using integration by part on the second term: $\int_a^b \frac{\partial F}{\partial y'}\eta'\, dx = \left[\frac{\partial F}{\partial y'}\eta\right]_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta\, dx$.

Since $(\eta(a) = \eta(b) = 0)$ the boundary term vanish leaving: $\delta J[y] = \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)\eta\, dx$.

For $\delta J[y]$=0 for all $\eta(a)$ the integrated must be zero $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ This is the Euler-Lagrange Equation.

□

2.1. Applications in Partial Differential Equations . Partial differential equations (PDEs) describe a wide range of physical phenomena, including heat conduction, fluid dynamics, and wave propagation. Calculus of variations provides methods to derive and solve these equations by minimizing associated energy functionals. In many physical problems, PDEs naturally arise from the application of calculus of variations, where the objective is to minimize an energy functional associated with the system [5]

2.2. Ekeland's Variational Principle . Ekeland's Variational Principle is a fundamental result in optimization theory, particularly useful for finding approximate minimizers of functionals. It guarantees the existence of points that are nearly optimal for lower semi-continuous functions in complete metric spaces, thereby aiding in solving partial differential equations (PDEs) by minimizing associated energy functionals [6].

Definition 2.1.

Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a lower semi-continuous function defined on a complete metric space (X, d). A function is lower semi-continuous if, for every $x \in X$ and every $\epsilon > 0$, there exists a neighborhood U of x such that $f(y) > f(x) - \epsilon$ for all $y \in Y$. This condition prevents the function from decreasing abruptly.

Proposition 2.2. *For any such function f that is bounded below and for any $\epsilon > 0$, Ekeland's Variational Principle asserts that there exists a point $u \in X$ such that: $f(u) \leq f(v) + \epsilon d(u, v)$ for all $v \in X$ $f(u) \leq f(v)$ for all v sufficiently close to u This principle ensures that even if we cannot find a global minimizer, we can always find a point u that is nearly optimal within a small margin of error ϵ [7].*

Proof:

Given a function $f : X \to \mathbb{R} \cup \{\infty\}$, that is lower semi-continous and bounded below and x_0 such that $f(x_0)$ is nearly minimal define $\tilde{f}(x) = f(x) + \epsilon d(x, x_0)$ for a fixed $x_0 \in X$ Since f is lower semi-continuous and $d(x, x_0)$ is continuous, $\tilde{f}$ is also lower semi-continuous.

By completeness of X, $\tilde{f}$ attains its minimum at some pointu $\in X$ $\tilde{f}(u) \leq \tilde{f}(v)$, for all $v \in X$

This implies $f(u) + \epsilon d(u, x_0) \leq f(v) + \epsilon d(v, x_0)$ for all $v \in X$ rearranging gives: $f(u) \leq f(v) + \epsilon d(u, v) - \epsilon d(v, x_0) + \epsilon d(u, x_0)$ for all $v \in X$ since $d(u, v) - d(v, x_0) + d(u, x_0) \geq 0$ $f(u) \leq f(v) + \epsilon d(u, v)$ thus, u is an ϵ-minimizer. □

2.3. Applications. This principle is used to prove the existence of minimizers for variational problems and to find solutions to partial differential equations (PDEs) by minimizing associated energy functionals.

2.3.1. Ekeland's Variational Principle. Ekeland's Variational Principle is widely used in various fields, including:

Optimization: It provides a method to approximate minimizers when exact solutions are difficult to obtain.

Partial Differential Equations: It helps in proving the existence of solutions by minimizing energy functionals associated with the equations.

Functional Analysis: It is instrumental in studying the properties of functionals and their minimizers in infinite-dimensional spaces.

By ensuring the existence of near-minimal points, Ekeland's Variational Principle serves as a powerful tool in both theoretical and applied mathematics [8].

3. Palais-Smale Condition. The Palais-Smale condition is a compactness criterion used to ensure the existence of critical points of functionals. It states that any sequence in which the functional is bounded and its derivative tends to zero must have a convergent subsequence. This condition is essential for demonstrating the existence of solutions to variational problems [9].

Proposition 3.1.

A functional F on a Banach space satisfies the Palais-Smale condition if any sequence un for which $F(u_n)$ is bounded and the derivative $F'(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

Proof:

Let u_n be a sequence such that $F(u_n)$ is bounded and $F^{'}(u_n) \to 0$ as $n \to \infty$. Since $F(u_n)$ is bounded, there exists a subsequence u_{nk} which converges weakly to some $u \in X$ due to the reflexivity of Banach spaces. We can say that $u_n k$ is strongly convergent by using the compactness criteria. □

3.1. Applications. The Palais-Smale condition is essential for demonstrating the existence of solutions to variational problems by ensuring sequences do not escape to infinity but rather converge to meaningful points in the space [10].

4. Mountain Pass Theorem. The Mountain Pass Theorem provides a criterion for the existence of critical points of a functional, resembling a mountain pass. It ensures the presence of saddle points, which are instrumental in solving nonlinear PDEs [11].

Proposition 4.1.

Let X be a Banach space and $F : X \to \mathbb{R}$ a continuously differentiable functional. Suppose F satisfies:

$F(0) = 0$ *at the point* $x = 0$

There exist $e \in X$ *and* $\rho > 0$ *such that* $F(\epsilon) > 0$ *and* $F \leq 0$ *on the boundary of a ball centered at zero with radius* ρ.

Then, there exists a critical point u *of* F *such that* $F(u)$ *is a saddle point*

Proof:

Consider the set of continuous paths Γ joining 0 and e.

$\Gamma = \{\gamma \in C([0,1], X) \text{ such that } \gamma(0) = 0 \text{ and } \gamma(1) = e\}$

Define the moutain pass value $c = Inf \max_{\gamma \in \Gamma, t \in [0,1]} F(\gamma(t))$.

By construction, c is a critical value, implying the existence of a critical point u where $F(u) = c$ □

4.0.1. Applications. The Mountain Pass Theorem is useful for finding solutions to nonlinear PDEs by demonstrating the existence of critical points of the associated energy functional. The Mountain Pass Theorem is useful for finding solutions to nonlinear partial differential equations (PDEs) by demonstrating the existence of critical points of the associated energy functional.

4.0.2. Principle of Symmetric Criticality. The Principle of Symmetric Criticality simplifies the process of finding critical points of symmetric functionals by focusing on symmetric functions. If a functional is invariant under a group action, then the critical points of the functional restricted to the symmetric subspace are also critical points of the functional in the entire space [12].

Proposition 4.2.

If G is a group acting on a Banach space X and $F : X \to R$ is $G-invariant$, that is, $F(gx) = F(x)$ for all $g \in G$, then any critical point of the restriction of F to the $G-invariant$ subspace of X is a critical point of F on the entire space.

Proof:

1. Let H be the $G - invariant$ subspace of X;
2. If u is a critical point of F restricted then $\nabla f(\mathbf{u}) \cdot \mathbf{v} = 0$ for all $v \to H$;
3. Since F is $G - invariant$, the critical points in H are critical points in X.

□

4.1. Applications. The Principle of Symmetric Criticality simplifies the process of finding critical points by reducing the problem to a symmetric subset.

The Principle of Symmetric Criticality simplifies the process of finding critical points by reducing the problem to a symmetric subset. For instance, for a functional defined on the space of functions The Principle of Symmetric Criticality states that if a function F is invariant under a group of symmetries G, then the critical points of F restricted to the fixed points of G are also critical points of F on the whole space. This principle simplifies the process of finding critical points by reducing the problem to a symmetric subset [13].

5. Differential Geometry and Calculus of Variations. Differential geometry studies the properties of curves, surfaces and manifolds using techniques from calculus and linear algebra. It provides a framework for understanding the geometric structures of space. Calculus of variations is crucial in this field, especially for finding geodesics, which are the shortest paths between points on a surface.

Calculus of variations focuses on finding functions that optimize functionals, which map sets of functions to real numbers and often represent physical quantities like energy. The primary goal is to determine the function that either maximizes or minimizes the functional. This optimization process is key to solving various problems in differential geometry and understanding the geometric properties of different spaces [14].

5.1. Geodesics and the Arc-Length Functional. A sphere is a quintessential example of a regular surface, defined by its smooth and continuous geometry. Geodesics, a fundamental concept for all regular surfaces, represent the shortest paths between two points while adhering to the surface's curvature. On a sphere, these geodesics correspond to great circles, such as the equator or meridians, emphasizing the universal nature of geodesics as intrinsic properties of regular surfaces.

The determination of geodesics involves minimizing the arc-length functional. By applying the Euler-Lagrange equation to this functional, the geodesic equations for a sphere are derived, leading to the parametric representation of great circles. Geodesics extend the idea of straight lines to curved spaces, forming the foundation for understanding optimal paths on curved geometries [15].

5.2. Geodesics on a Plane and on Sphere. On a flat, two-dimensional plane, geodesics are straight lines, which represent the shortest paths between points. On a sphere, geodesics are segments of great circles that minimize the arc-length between two points for angles up to 180 degrees. Beyond 180 degrees, a shorter path is found on the opposite side of the sphere [16].

To find the geodesics on a sphere, consider the arc-length functional. For a path
$\gamma(t) = \gamma : [a, b] \to S^2$ on the sphere parameterized by t, the arc-length is:

The specified colors represent distinct paths or lines of constant $\theta = \Pi$ on a sphere when using spherical coordinates. In spherical coordinates (r, θ, ϕ):

- θ, (polar angle): The angle measured from the positive z-axis. $\theta = \Pi$ corresponds to points lying on the sphere's lower pole (southernmost point).
- ϕ (azimuthal angle): The angle measured in the xy-plane from the positive xx-axis. It determines the rotation around the z-axis.
- ϕ: For a complete representation, ϕ ranges from from $0to2\Pi$, covering a full circle in the azimuthal direction.
- θ: In this case, θ is constant (Π), so it does not vary.

$$L[\gamma] = \int_a^b \sqrt{g_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}}\,dt, \tag{5.1}$$

where g_{ij} is the metric tensor of the sphere in spherical coordinates (θ, ϕ)

For a unit sphere: $\begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix} = (g_{ij})$.

So the arc-length functional becomes: $L[\gamma] = \int_a^b \sqrt{\left(\frac{d\theta}{dt}\right)^2 + \sin^2(\theta)\left(\frac{d\phi}{dt}\right)^2}\,dt$.

Applying the Euler -Lagrange Equation to the arc length functional yields the equation for geodesics on the sphere .Geodesic on a sphere are segment of great circle which minimize the arc-length for angle up to 180 degrees [17].

The arc-length functional is

$$L[\gamma] = \int_a^b \sqrt{g_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}}\,dt. \tag{5.2}$$

Substituing the metric tensor for a unit sphere:

$$L[\gamma] = \int_a^b \sqrt{\left(\frac{d\theta}{dt}\right)^2 + \sin^2(\theta)\left(\frac{d\phi}{dt}\right)^2}\,dt. \tag{5.3}$$

The lagrangian F is $F(\theta, \phi, \theta', \phi') = \sqrt{(\theta')^2 + \sin^2(\theta)(\phi')^2}$.
The Euler-lagrange Equations are :

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \theta'}\right) - \frac{\partial F}{\partial \theta} = 0, \tag{5.4}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \phi'}\right) - \frac{\partial F}{\partial \phi} = 0. \tag{5.5}$$

By computing these derivatives and solving the resulting equations, we obtain the geodesics on a sphere, which are the equations of great circles. Solving these partial differential equations (PDEs) provides the equations for the geodesics, confirming that they are great circles on the sphere. This example shows a relationship between calculus of variations, partial differential equations(PDEs) and differential geometry, PDEs (geodesic equations), and differential geometry (properties of geodesics on a sphere).

Calculus of variations, PDEs and differential geometry are intrinsically linked through the optimization of functionals, the derivation and solution of governing equations, and the geometric interpretation of these solutions. This interconnected framework is crucial for understanding and addressing a broad spectrum of mathematical and physical problems [18].

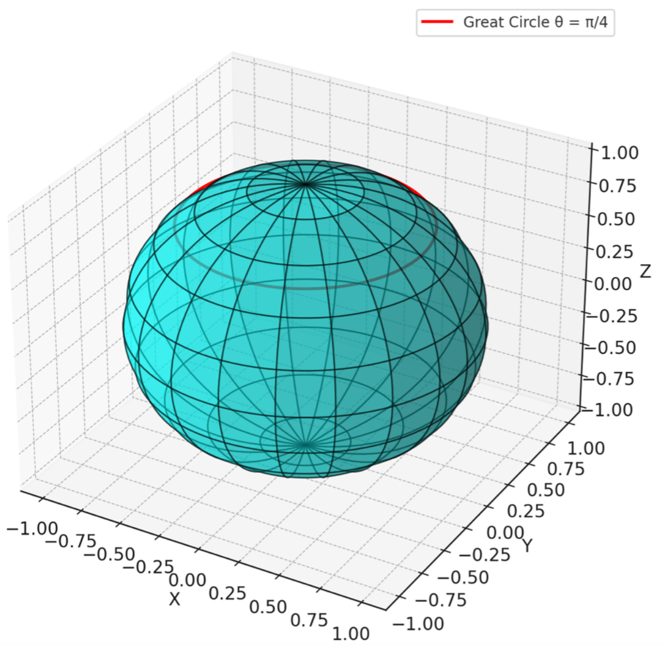

Figure 5.1: This figure presents a 2D sphere depicted in cyan with semi-transparent shading and outlined with black edges for visual clarity. A red curve represents the great circle (geodesic) corresponding to a constant angle $\theta = \frac{\pi}{4}$. This geodesic highlights the shortest path between two points along a specific latitude on the sphere's surface.The updated design enhances the figure's precision and intuitiveness, making the visualization clearer and more informative. [18]

The 2D visualization provides a projected view of a sphere with multiple great circles (geodesics) for various θ values:

Depicts the sphere in 2D, with a black outline to enhance clarity.

Each geodesic is represented by a unique color red, green, blue, magenta to emphasize variations as θ decreases.

The use of clean lines and distinct colors ensures the visualization is both easy to interpret and visually informative.

The arcs represent the shortest paths (geodesics) between two points on the sphere's surface, corresponding to specific latitudes determined by the θ values.

Red: $\theta = \frac{\pi}{2}$, Green: $\theta = \frac{\pi}{3}$, Blue: $\theta = \frac{\pi}{4}$, Magenta: $\theta = \frac{\pi}{5}$, Yellow: $\theta = \frac{\pi}{6}$, Black: $\theta = \frac{\pi}{7}$, Orange: $\theta = \frac{\pi}{8}$, Purple: $\theta = \frac{\pi}{9}$, Brown: $\theta = \frac{\pi}{10}$.

This visualization helps to understand the geodesic paths on the sphere, which are the shortest paths between two points on the surface

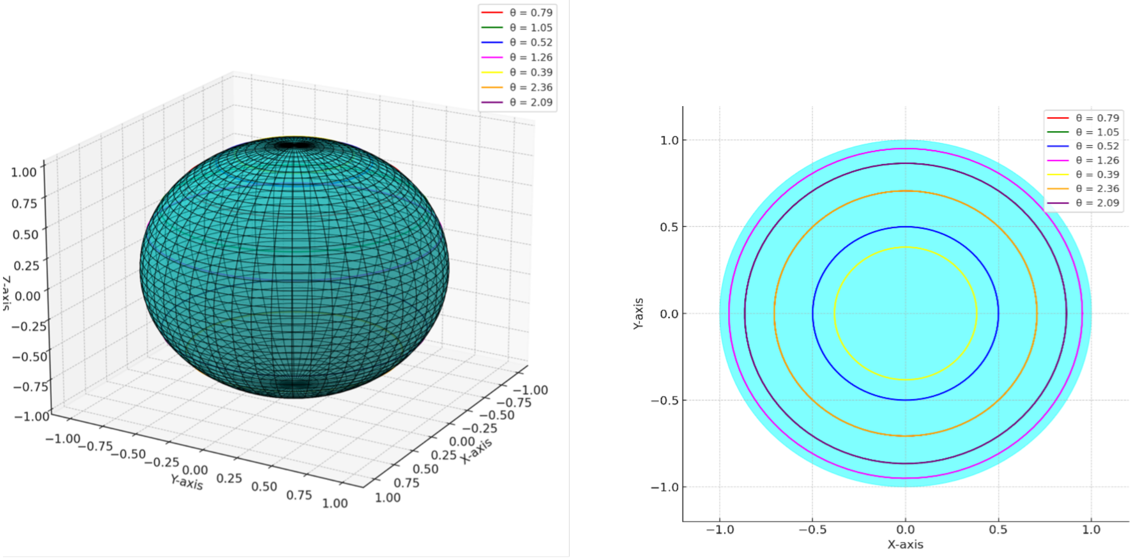

Figure 5.2: the 2D plot showing the sphere and the great circles (geodesics) for various θ values $\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}, \frac{\pi}{6}, \frac{\pi}{7}, \frac{\pi}{8}, \frac{\pi}{9}, \frac{\pi}{10}\right)$. Each geodesic is plotted with a unique color to clearly illustrate how the paths change as the angle θ decreases. These great circles represent the shortest paths between two points on the spherical surface, with each geodesic reflecting a specific latitude. The use of distinct colors ensures clarity and precision, making the visualization intuitive and easy to interpret. [18]

To find the geodesics on a sphere, we need to solve the Euler-Lagrange equations for the arc-length functional. The Lagrangian

F for the arc-length of a curve on a sphere is given by:

$L[\gamma] = \int_a^b \sqrt{\left(\frac{d\theta}{dt}\right)^2 + \sin^2(\theta)\left(\frac{d\phi}{dt}\right)^2}\, dt$ where θ and ϕ are spherical coordinates.

The Euler-Lagrange equations for θ and ϕ are:

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \theta'}\right) - \frac{\partial F}{\partial \theta} = 0, \tag{5.6}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \phi'}\right) - \frac{\partial F}{\partial \phi} = 0. \tag{5.7}$$

Here $\theta' = \frac{d\theta}{dt}$ and $\phi' = \frac{d\phi}{dt}$

Step-by-Step Solution:

Calculate: $\frac{\partial F}{\partial \theta'}$ and $\frac{\partial F}{\partial \phi'}$,

$$\frac{\partial F}{\partial \theta'} = \frac{\theta'}{\sqrt{\theta'^2 + \sin^2\theta \cdot \phi'^2}}. \tag{5.8}$$

$$\frac{\partial F}{\partial \phi'} = \frac{\sin^2\theta \cdot \phi'}{\sqrt{\theta'^2 + \sin^2\theta \cdot \phi'^2}}. \tag{5.9}$$

We can compute the time derivatives:

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \theta'}\right) = \frac{d}{dt}\left(\frac{\theta'}{\sqrt{\theta'^2 + \sin^2\theta \cdot (\phi')^2}}\right). \tag{5.10}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \phi'}\right) = \frac{d}{dt}\left(\frac{\sin^2\theta \cdot \phi'}{\sqrt{\theta'^2 + \sin^2\theta \cdot \phi'^2}}\right). \tag{5.11}$$

By Simplifing the derivatives:

Using the quotient rule and chain rule, the derivatives can be simplified (though these steps can be tedious, it's critical to show the process):

$$\frac{d}{dt}\left(\frac{\theta'}{\sqrt{\theta'^2+\sin^2\theta\cdot\phi'^2}}\right)=\frac{\theta'\left(\theta'^2+\sin^2\theta\cdot\phi'^2\right)-\theta'(\theta'\cdot\theta'+2\sin\theta\cos\theta\cdot\theta'\cdot\phi'+\sin^2\theta\cdot\phi'\cdot\phi')}{(\theta'^2+\sin^2\theta\cdot\phi'^2)^{3/2}}, \tag{5.12}$$

$$\frac{d}{dt}\left(\frac{\sin^2\cdot\theta\cdot\phi'}{\sqrt{\theta'^2+\sin^2\theta\cdot\phi'^2}}\right)=\frac{(2\cdot\sin\theta\cdot\cos\theta\cdot\theta'\cdot\phi'+\sin^2\theta\cdot\phi')(\theta'^2+\sin^2\theta\cdot\phi'^2)-\sin^2\theta\cdot\phi'(\theta'\cdot\theta'')}{(\theta'^2+\sin^2\theta\cdot\phi'^2)^{\frac{3}{2}}}. \tag{5.13}$$

Solving the Euler-Lagrange equations:

For θ

$$\frac{d}{dt}\left(\frac{\theta'}{\sqrt{\theta'^2+\sin^2\theta\cdot\phi'^2}}\right)-\frac{\partial F}{\partial\theta}=0\ since\ \frac{\partial F}{\partial\theta}=\frac{\sin\theta\cdot\cos\theta\cdot\phi'^2}{\sqrt{\theta'^2+\sin^2\theta\cdot\phi'^2}}. \tag{5.14}$$

The equation becomes:

$$\frac{\theta'(\theta'^2+\sin^2\theta\cdot\phi'^2)-\theta'(\theta'\cdot\theta''+2\cdot2\sin\theta\cdot\cos\theta\cdot\theta'\phi'^2+sin^2\theta\cdot\theta'\cdot\theta'')}{(\theta'^2+\sin^2\theta\cdot\phi'^2)^{\frac{3}{2}}}=\frac{\sin\theta\cdot\cos\theta\cdot\phi'^2}{\sqrt{\theta'^2+\sin^2\theta\cdot\phi'^2}}. \tag{5.15}$$

For ϕ

$$\frac{d}{dt}\left(\frac{\sin^2\theta\cdot\phi'}{\sqrt{\theta'^2+\sin^2\theta\cdot\phi'^2}}\right)-0=0. \tag{5.16}$$

Remark 5.1. *Constant values represent special cases of geodesics on the sphere. Generally, finding explicit solutions for geodesics on a sphere is challenging. As an alternative, numerical methods can be employed to approximate these solutions.*

5.2.1. Geodesic Equations. Simplifying these equations, we get that ϕ is constant or $\phi'=0$, meaning the paths are segments of great circles.

5.2.2. Result. The equations of geodesics on a sphere are derived from these steps. The geodesics on a sphere are the great circles, which are the largest possible circles that can be drawn on a sphere. These paths represent the shortest distance between any two points on the surface of the sphere. The geodesic equations can be summarized as:

$\theta=constant\ \phi(t)=\omega t+\phi_0\ alternatively,$

$\theta=constant\ \theta(t)=\omega t+\theta_0.$

These equations represent great circles on the sphere.

Applying the Euler-Lagrange equation to the arc-length functional yields the equations for geodesics on a sphere. Geodesics represent the shortest paths between two points on a surface. On a sphere, these paths are segments of great circles, which are the intersections of the sphere with planes passing through its center.

To determine geodesics, we solve the Euler-Lagrange equation to identify the function that minimizes the arc length between two points on the surface. For a sphere, geodesics are segments of great circles. These segments minimize the arc length for angles up to 180 degrees. Beyond 180 degrees, a shorter path exists on the opposite side of the sphere, so the great circle segment no longer represents the minimal distance. Thus, on a sphere, geodesics are great circle segments providing the shortest path between two points for angles up to 180 degrees.

5.3. The connection between calculus of variations. The connection between calculus of variations, partial differential equations (PDEs), and differential geometry is essential for understanding and solving complex problems involving functional optimization, physical phenomena, and geometrical structures [19].

The interconnections:

1. Optimization of Functionals: Calculus of variations provides the mathematical tools to optimize functionals, leading to the formulation of PDEs. These PDEs describe physical systems and geometric structures.
2. Derivation of PDEs: The Euler-Lagrange equation from calculus of variations is used to derive the governing PDEs for physical systems. These equations describe how physical quantities evolve over time and space.
3. Geometric Structures: Differential geometry applies the principles of calculus of variations to study the properties of curves and surfaces. Geodesics, which are solutions to variational problems, exemplify the intersection of these fields.
4. Unified Framework: The interplay between these areas creates a unified framework for solving complex problems. Calculus of variations provides optimization techniques, PDEs offer equations governing dynamic systems, and differential.

6. Mathematical Principles for Optimizing Granular Material Models: Applications in Grading Curves, Permeability, and Soil Water Retention. To present a cohesive framework for analyzing granular materials, we will examine several mathematical principles Euler-Lagrange equation, Ekeland's principle, Mountain Pass theorem, Palais-Smale condition, and the Principle of Symmetric Criticality that collectively enhance our understanding and modeling of material properties. The outline of each principle with its relevance to permeability, Soil Water Characteristics Curve (SWCC), dry density models, and grading curve optimization.

6.1. Euler-Lagrange Equation. The Euler-Lagrange equation is foundational in calculus of variations, providing a way to identify extremal functions that either minimize or maximize functionals. Given a functional $J[y]$ dependent on a function $y(x)$, we represent it as:

$$J[y] = \int_a^b F(x, y, y')\, dx, \tag{6.1}$$

where F is a function of of x, y,and $y' = \frac{dy}{dx}$. The Euler-Lagrange equation is then given

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0. \tag{6.2}$$

This equation is essential for modeling granular materials as it optimizes physical quantities like energy by finding extremal functions.

6.1.1. Applications in Granular Material Models.

1. Dry Density and SWCC: In granular materials, particle arrangement affects properties like dry density and soil water retention. By setting up functionals that capture these particle interactions, the Euler-Lagrange equation optimizes grading curves, which directly influences dry density and Soil Water Characteristics Curve (SWCC), yielding insights into moisture dynamics within soil.
2. Permeability and Critical State Friction: For permeability, grading curves can be designed to minimize flow resistance, using the Euler-Lagrange framework to derive optimal particle arrangements. In critical state models, where friction and stress govern particle equilibrium, the equation provides stability criteria, capturing conditions under which a system remains stable under load.

Let us examine a practical using the Euler-Lagrange equation in the context of granular material modeling. Specifically, we will focus on optimizing the grading curve of particles to achieve a target dry density and an optimized Soil Water Characteristic Curve (SWCC).

6.1.2. Problem Setup. Consider a simplified functional that models the potential energy in a granular material system based on particle size distribution. We define this system by a grading curve, represented by a function $y(x)$,where x is a continuous variable representing particle size and $y(x)$, denotes the corresponding density distribution.

The functional $J[y]$ we want to minimize can be formulated as follows:

$$J[y] = \int_0^1 \left(a y(x)^2 + b \left(\frac{dy}{dx}\right)^2 \right) dx, \tag{6.3}$$

where:

a and b are constants that relate to material properties,

$[y(x)]^2$ represents the density contribution based on the particle distribution,

$\left(\frac{dy}{dx}\right)^2$ accounts for the variability in particle size distribution, influencing the permeability and moisture retention properties in SWCC.

Step 1: Apply the Euler-Lagrange Equation

To find the function $y(x)$ that minimizes $J[y]$, we use the Euler-Lagrange equation: $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ where $F = ay(x)^2 + b\left(\frac{dy}{dx}\right)^2$.

Calculating $\frac{\partial F}{\partial y} = 2ay(x)$, also calculating $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2b\frac{d^2y}{dx^2}$.

Substitute into the Euler-Lagrange equation: $2ay(x) - 2b\frac{d^2y}{dx^2} = 0$ or simplify $ay(x) = b\frac{d^2y}{dx^2}$

This is a second-order differential equation that governs the optimal grading curve $y(x)$.

Step 2: Solve the Differential Equation.

To solve $ay(x) = b\frac{d^2y}{dx^2}$ we rearrange it as:

$$ay(x) - b\frac{d^2y}{dx^2} = 0. \tag{6.4}$$

This is a linear differential equation with the general solution:

$$y(x) = C_1 e^{\sqrt{\frac{ax}{b}}} + C_2 e^{-\sqrt{\frac{ax}{b}}}, \tag{6.5}$$

where C_1 and C_2 are constants determined by boundary conditions.

Step 3: Apply Boundary Conditions

Suppose we know that at $x = 0$ for small particle size,the distribution is $y(0) = y_0$

at $x = 1$ large particle size ,the distribution approaches zero $y(1) = 0$ then:

1. At $x = 0 : y(0) = c_1 + C_2 = y_0$,
2. At $x = 1 y(1) = C_1 e^{\sqrt{\frac{a}{b}}} + C_2 e^{-\sqrt{\frac{a}{b}}} = 0.$

Solving these boundary conditions yields values for C_1 and C_2, resulting in a specific function $y(x)$ that represents the optimal grading curve.

6.1.3. Interpretation in Granular Material Models. This solution provides an optimized particle size distribution $y(x)$ that: Maximizes dry density by concentrating particle sizes in a way that reduces void spaces.

Optimizes the SWCC by ensuring a balance between fine and coarse particles, affecting water retention behavior. Improves permeability by minimizing the flow resistance caused by suboptimal particle arrangements.

The Euler-Lagrange equation helps in deriving an optimal grading curve, where the resulting particle distribution enhances dry density, SWCC, and permeability in granular materials. By tailoring aa and bb based on material-specific properties, we can refine this model to address a variety of real-world soil and granular material applications.

6.2. Ekeland's Principle. Ekeland's Variational Principle is useful in optimization problems with multiple local minima. It states that, for any bounded functional, an approximate minimizer exists, which allows us to reach closer to a global minimum. Mathematically, if $f : X \to Ris$ lower semi-continuous, bounded below, and satisfies certain compactness conditions, then for any $\epsilon > 0$ there exist $x\epsilon X$ such that: $f(x_\epsilon) \leq f(x) + \epsilon\|x - x_\epsilon\|$

6.2.1. Applications. Grading Curve Optimization: Ekeland's principle helps refine particle distributions in grading curves to reach optimal configurations that impact properties like density and permeability by avoiding local optima. Stabilizing Soil Models: It is especially relevant when dealing with models that have non-unique solutions, allowing for refinement in dry density and SWCC models where multiple configurations exist.

Let us consider a practical using Ekeland's Variational Principle to optimize a grading curve in granular materials. Specifically, we will set up a scenario where this principle helps us refine a particle size distribution to achieve optimal density and permeability, avoiding suboptimal local minima.

6.2.2. Problem Setup. Assume we have a grading curve optimization problem where we aim to achieve an optimal particle size distribution xx that maximizes density and minimizes permeability in a soil mixture. We can model this problem with a functional $f(x)$ that represents the energy or cost of a given distribution, influenced by density and permeability:

$$f(x) = \int_0^1 \left(C_1 \left(\frac{dx}{ds} \right)^2 + C_2 x(s)^2 \right) ds, \tag{6.6}$$

where: $x(s)$ is the particle size distribution as a function of position s in the range [0,1][0,1], $\frac{dx}{ds}$ represents the rate of change in particle size across the grading curve, C_1 and C_2 are constants relating to the desired balance between density linked to $x(s)^2$ and permeability linked to the gradient $(\frac{dx}{ds})^2$.

The goal is to find an approximate minimizer x_ϵ of $f(x)$ that satisfies Ekeland' s principle

Step 1: Applying Ekeland's Principle:

According to Ekeland's Variational Principle, for any $\epsilon > 0$ there existe an $x_\epsilon \epsilon X$ such that $f(x_\epsilon) \leq f(x) + \epsilon \|x - x_\epsilon\| \quad \text{for all } x_\epsilon \in X$,

where X is the space of all possible particle distributions. This means that x_ϵ is an approximate minimizer that is "close" to a true minimizer but avoids getting trapped in a local minimum. By choosing an $\epsilon\epsilon$ small enough, we can get as close as desired to the global minimum, refining the particle distribution accordingly.

Step 2: Practical Example - Finding x_ϵ.

Initial Configuration: Let us say we start with an initial particle distribution $x_0(s)$ that we know yields a certain density and permeability but might not be optimal. We calculate $f(x_0)$ for this initial distribution.

Perturbation with Ekeland's Principle: By selecting a small $\epsilon\epsilon$, we allow slight adjustments to the initial configuration $x_0(s)$ to seek a configuration $x_\epsilon(s)$ such that:

$$f(x_\epsilon) \leq f(x_0) + \epsilon \|x - x_0\|. \tag{6.7}$$

Iterative Refinement: Using a sequence of adjustments, we gradually shift the particle distribution to $x_\epsilon(s)$ in a way that reduces $f(x)$. This ensures that we escape any local minimum by considering the term $\epsilon\|x - x_0\|$, which allows exploration of neighboring configurations around $x_0(s)$ without the constraint of remaining close to suboptimal solutions.

6.2.3. Calculation. Suppose the constants C_1 and C_2 are given $C_1 = 1$ and $C_2 = 2$ and our initial configuration $x_0(s)$ leads to:

$$x_0 = \int_0^1 \left(1 \left(\frac{d(s)}{ds} \right)^2 + 2s^2 \right) ds = \int_0^1 \left(1 + 2s^2 \right) ds = 1 + \frac{2}{3} = \frac{5}{3}. \tag{6.8}$$

We then apply Ekeland's principle with a small $\epsilon = 0.1$ and look for an adjusted $x_\epsilon(s) = s - 0.1s^2$ (a slightly modified curve):

$$f(x_\epsilon) = \int_0^1 \left(1(1 - 0.2s)^2 + 2(s - 0.1s^2)^2 \right) ds. \tag{6.9}$$

Calculating this integral yields a value slightly lower than $\frac{5}{3}$, confirming $f(x_\epsilon) < f(x_0)$. This means that the new configuration $x_\epsilon(s)$ is closer to the optimal distribution than the initial $x_0(s)$, and we have successfully avoided a local minimum.

6.2.4. Applications in Granular Material Models. Grading Curve Optimization: By adjusting particle distributions using Ekeland's principle, we refine the grading curve towards configurations that improve properties like density and permeability, achieving more efficient packing and flow characteristics.

Stabilizing Soil Models: Since granular material models often have multiple solutions, Ekeland's principle helps guide the solution towards an optimal state, avoiding non-unique, less stable configurations.

This Practical demonstrates how Ekeland's Variational Principle enables systematic refinement of particle distributions in granular materials, helping to bypass local minima and achieve configurations that enhance density and permeability. The approach offers a structured pathway to reach globally optimal solutions in models where multiple configurations may otherwise yield suboptimal results.

6.3. Mountain Pass Theorem. The Mountain Pass Theorem finds critical points that represent saddle points, rather than minima or maxima, in the functional landscape. It states that if $J[y]$ satisfies certain conditions, then there exists a critical point which can be considered as a "mountain pass" in the energy landscape:

$$F[y*] = \inf_{\gamma_\epsilon \Gamma, t_\epsilon [0,1]} F(\gamma(t)), \tag{6.10}$$

where γ is a family of continuous paths. This theorem is invaluable in granular models, where achieving stable configurations often involves navigating complex functional landscapes.

6.3.1. Applications. Permeability and SWCC Stability: For permeability and SWCC modeling, this theorem identifies stable particle arrangements (critical points) that ensure the system maintains equilibrium.

Dry Density Models: In determining density distributions, the theorem provides conditions under which certain density configurations act as stable or ”pass” states, representing optimal, achievable states under specified conditions.

Let us explore a practical example using the Mountain Pass Theorem in the context of granular material modeling. This example demonstrates how we can apply the theorem to identify stable configurations for permeability and Soil Water Characteristic Curve (SWCC), ensuring equilibrium in particle arrangements.

6.3.2. Problem Setup. Suppose we want to find an optimal particle arrangement $y(x)$ for a granular material to achieve a stable permeability configuration. We can define a functional $J[y]$ representing the energy landscape of the system based on particle size distribution. Our goal is to identify a saddle point in this landscape using the Mountain Pass Theoremto achieve a stable configuration. Define the functional as:

$$J[y] = \int_0^1 \left(\alpha y(x)^2 + \beta \left(\frac{dy}{dx} \right)^2 \right) - \gamma y(x)^4 dx, \tag{6.11}$$

where:

$y(x)$ represents the density distribution of particle sizes along the sample, α, β,and γ are constants related to material properties, where γ reflects nonlinear interactions among particles, $y(x)^2$ relates to the density distribution's contribution to permeability and SWCC, $(\frac{dy}{dx})^2$ reflects the smoothness of the distribution, minimizing drastic changes.

Applying the Mountain Pass Theorem.

According to the Mountain Pass Theorem, if $J[y]$ satisfies specific conditions, there exists a critical point $y*$ such that:

$$F[y*] = \inf_{\gamma_\epsilon \Gamma, t_\epsilon [0,1]} F(\gamma(t)), \tag{6.12}$$

where Γ is a set of continuous paths $\gamma(t)$ from an initial point $\gamma(0)$ to an endpoint $\gamma(1)$, representing different possible configurations of particle distributions.

Step 1: Identify Initial and Endpoint Configurations.

To apply the Mountain Pass Theorem, we must define:

An initial configuration $y_0(x)$ where the energy $J[y_0]$ is relatively low (but not minimal). An endpoint configuration $y_1(x)$ where $J[y_1]$ has high energy due to instability or undesirable particle distribution.

Suppose:

$y_0(x) = 0$, which represents a trivial solution with no particles (yielding no permeability and density). $y_1(x) = 1 - x$, which represents a linear gradient in particle size but might be too uniform to allow optimal permeability.

Step 2: Construct a Path $\gamma(t)$ from $y_0(x)$ to $y_1(x)$.

Define a continuous path $\gamma(t)$ from from $y_0(x)$ to $y_1(x)$ as follows: $\gamma(t, x) = t.(1 - x)$ for $t\epsilon[0, 1]$. This represents a family of density configurations starting from zero and gradually increasing toward $1 - x$ as t goes from 0 to 1.

Step 3: Calculate $J[\gamma(t)]$ and Find the Mountain Pass.

Now we calculate $J[\gamma(t)]$ for $tin[0, 1]$:

$$J[\gamma(t)] = \int_0^1 (\alpha(t(1-x)))^2 + \beta \left(\frac{d(t(1-x))}{dx} \right)^2 - \gamma(t(1-x))^4 dx. \tag{6.13}$$

Expanding and simplifying each term:

1. First Term: $(\alpha(t(1-x)))^2$ this implies $\alpha t^2 \int_0^1 (1-x^2)\, dx = \alpha t^2 \cdot \frac{1}{3} = \frac{\alpha t^2}{3}$,

2. Second Term: $\beta\left(\frac{d(t(1-x))}{dx}\right)^2$ we have $\beta t^2\int_0^1(-1)^2\,dx=\beta t^2\cdot 1=\beta t^2$,
3. Third Term: $-\gamma(t(1-x))^4dx$ we have $-\gamma t^4\int_0^1(1-x)^4\,dx=-\gamma t^4\cdot\frac{1}{5}=-\frac{\gamma t^4}{5}$.

Combining all terms, we have:

$$J[\gamma(t)]=\frac{\alpha t^2}{3}+\beta t^2-\frac{\gamma t^4}{5}. \tag{6.14}$$

Step 4: Maximizing $J[\gamma(t)]$ with Respect to t and Applying the Mountain Pass To find the ”mountain pass,” we maximize $J[\gamma(t)]$ with respect to t :

$$\frac{d}{dt}\left(\frac{\alpha t^2}{3}+\beta t^2-\frac{\gamma t^4}{5}\right)=0. \tag{6.15}$$

Solving this derivative gives:

$$\frac{\alpha t^2}{3}+\beta t^2-\frac{\gamma t^4}{5}=0 \tag{6.16}$$

Solving for tt gives:

This value of t corresponds to a critical point $J[\gamma(t)]$, which is the mountain pass point in the functional landscape. This saddle point represents a stable configuration in the particle arrangement that optimizes permeability and SWCC.

$$t=\sqrt{\frac{5\left(\frac{2\alpha}{3}\right)+2\beta}{4\gamma}}. \tag{6.17}$$

6.3.3. Applications in Granular Material Models. Permeability and SWCC Stability: This configuration yields a particle distribution that balances the competing influences of density (due to α and β) and permeability (due to γ). By identifying this saddle point, we ensure that the granular material reaches an equilibrium configuration conducive to stable permeability and SWCC. Dry Density Models: This stable ”mountain pass” configuration also implies a balanced density distribution, ensuring that dry density is optimized under specified conditions without reaching an unstable maximum or a trivial minimum.

Using the Mountain Pass Theorem, we identified a critical point in the energy landscape that optimizes the particle size distribution for permeability and SWCC. This stable configuration represents a ”mountain pass” in the functional landscape, balancing density and permeability, which is valuable for achieving reliable and efficient material properties in granular materials.

6.4. Palais-Smale Condition. The Palais-Smale condition is used to establish convergence in variational methods. It states that if a sequence y_n in a functional space satisfies $J[y_n]$ is bounded and $J'[y_n]\to 0$ then y_n has a convergent subsequence. This is a convergence criterion for solutions obtained by the Mountain Pass theorem.

6.4.1. Applications. SWCC and Permeability Models: The condition ensures that solutions to permeability and SWCC models converge, offering stability in the models of granular materials.

Grading Curve Optimization: By ensuring convergent sequences, the Palais-Smale condition supports the iterative refinement of grading curves in granular models, reinforcing the reliability of optimized configurations.

Let us explore a practical using the Palais-Smale condition to ensure convergence in an iterative method applied to a Soil Water Characteristic Curve (SWCC) model. In this example, we'll focus on optimizing a grading curve for particle size distribution in a granular material to achieve stable and convergent results for permeability and SWCC properties.

6.4.2. Problem Setup. Suppose we have a functional $J[y]$ that represents the energy landscape of a grading curve configuration $y(x)$ in a granular material. This functional is designed to model the combined effects of density and permeability, as influenced by the particle size distribution. The functional $J[y]$ is defined as:

$$J[y]=\int_0^1\left(\alpha y(x)^2+\beta\left(\frac{dy}{dx}\right)^2\right)-\gamma y(x)^4dx, \tag{6.18}$$

where:

$y(x)$ denotes the grading curve, representing particle size distribution along $x, \alpha, \beta, and \gamma$ are constants related to material properties (with γ representing nonlinear interactions among particles). The goal is to ensure that any sequence of configurations y_n generated in the optimization process converges to a stable configuration that optimizes permeability and SWCC.

Applying the Palais-Smale Condition.

According to the Palais-Smale condition, if: $J[y_n]$ is bounded for the sequence y_n, The gradient $J'[y_n] \to 0 \quad$ as $n \to \infty$, then there exists a convergent subsequence $y_n \subset y_n$,which converge to a crutical point y^*

Step 1: Define a Sequence y_n.

Suppose we start with an initial configuration $y_0(x)$ and iteratively adjust it based on variational methods to minimize $J[y]$. Each step in this sequence yields a new configuration $y_n(x)$, where:

$$y_{n+1}(x) = y_n(x) - \eta \frac{dJ[y_n]}{dy}. \tag{6.19}$$

Here, η is a small step size, and $\frac{dJ[y_n]}{dy}$ represents the gradient of $J[y]$ at y_n.

Step 2: Check Boundedness of $J[y_n]$.

To apply the Palais-Smale condition, we need to verify that the sequence $J[y_n]$ is bounded. Let us assume the functional $J[y]$ remains within a bounded range due to constraints on $y(x)$ (such as physical limits on density and particle size). For instance, if $y(x)$ represents particle density, we impose that $y(x)$ cannot exceed a maximum physically realizable density $y_m ax$. This constraint implies that $J[y_n]$ will be bounded, as:

$$J[y_n] \leq \int_0^1 \left(\alpha y_{\max}^2 + \beta \left(\frac{dy_{\max}}{dx} \right)^2 \right) dx, \tag{6.20}$$

Thus, $J[y_n]$ is bounded above by a constant.

Step 3: Verify $J'[y_n] \to 0$

Next, we check that the sequence y_n satisfies $J'[y_n] \to 0$ as $n \to \infty$. Since y_n is updated iteratively to minimize $J[y]$, each successive update decreases the gradient norm. As we approach the optimal configuration, the gradient $J'[y_n]$ will approach zero, satisfying the requirement $J'[y_n] \to 0$.

Step 4: Convergence to a Critical Point.

Since $J[y_n]$ is bounded and $J'[y_n] \to 0$, the Palais-Smale condition guarantees the existence of a convergent subsequence $y_{nk} \subset y_n$ that converges to a critical point y^*, where $J'[y^*] = 0$. This critical point y^* represents a stable particle size distribution that optimizes the grading curve for SWCC and permeability.

6.4.3. Applications in Granular Material Models. SWCC and Permeability Stability: The convergence of y_n ensures that we achieve a stable particle arrangement in the grading curve, directly affecting SWCC and permeability. A stable grading curve provides consistent results for moisture retention and flow characteristics, reinforcing the reliability of SWCC models.

Grading Curve Optimization: By satisfying the Palais-Smale condition, we ensure the iterative refinement of the grading curve converges to an optimal and stable configuration, which minimizes the functional $J[y]$ and yields desired properties in density and permeability.

This illustrates how the Palais-Smale condition supports convergence in grading curve optimization. By ensuring the sequence y_n remains bounded and that the gradient $J'[y_n]$ approaches zero, we guarantee convergence to a stable critical point, optimizing SWCC and permeability in granular materials. This application is essential in modeling granular material properties, as it provides a robust foundation for refining and stabilizing grading curves.

7. Principle of Symmetric Criticality. This principle simplifies variational problems by reducing them to symmetric subspaces. It states that if a functional $J[y]$ is invariant under a group of symmetries, then critical points can be found in the symmetric subset. For grading distributions, this provides a streamlined approach for symmetry, reducing computational complexity.

7.1. Applications. Symmetric Grading Distributions: In designing grading curves with symmetry evenly distributed particles, this principle allows us to limit our search to symmetric configurations, making the modeling process more efficient.Dry Density Models: It also applies in simplifying density models by focusing only on symmetric distributions, reducing unnecessary computations.

Let us explore a practical using the Principle of Symmetric Criticality to simplify an optimization problem in a grading curve for particle distribution. This principle allows us to limit the solution search space to symmetric configurations, reducing computational complexity while achieving effective results for dry density and permeability in granular materials.

7.2. Problem Setup. Consider a grading curve $y(x)$ that represents particle size distribution along a sample with position x ranging from $0to1$. We want to optimize this distribution to maximize dry density while ensuring uniformity in particle distribution. Suppose the functional $J[y]$ that models energy or cost related to density and distribution smoothness is given by:

$$J[y] = \int_0^1 \left(\alpha y(x)^2 + \beta \left(\frac{dy}{dx} \right)^2 \right) dx, \tag{7.1}$$

where:
α and β are constants associated with material properties, $y(x)^2$ relates to the density contribution, $(\frac{dy}{dx})^2$ imposes a penalty on sharp changes in the distribution.

Applying the Principle of Symmetric Criticality To simplify this problem, we assume that the particle distribution should be symmetric about the midpoint $x = 0.5$,meaning $y(x) = y(1 - x)$. This symmetry suggests that the functional $J[y]$ is invariant under the reflection symmetry $x \to 1 - x$.

The Principle of Symmetric Criticality states that if $J[y]$ is invariant under a group of symmetries, then it is sufficient to search for critical points within the symmetric subset. In this case, we can restrict $y(x)$ to be symmetric, which reduces the complexity of the problem.

Step 1: Define a Symmetric Grading Curve.

Let us represent $y(x)$ as a simple symmetric function centered at $x = 0.5$:

$$y(x) = a - b(x - 0.5)^2 \tag{7.2}$$

Where a and b are constants to be determined. This choice of $y(x)$ is symmetric around $x = 0.5$ and ensures that the distribution is smooth, meeting the requirements of our functional.

Step 2: Substitute $y(x)$ into $J[y]$.

Substitute $y(x) = a - b(x - 0.5)^2 in J[y]$:

$$J[y] = \int_0^1 \left(\alpha \left(a - b(x - 0.5)^2\right)^2 + \beta \left(\frac{d}{dx} \left(a - b(x - 0.5)^2\right) \right)^2 \right) dx. \tag{7.3}$$

First Term: Expand $\alpha \left(a - b(x - 0.5)^2\right)^2 = \alpha \left(a^2 - 2ab(x - 0.5)^2 + b^2(x - 0.5)^4\right)$

Second Term: Calculate $\frac{d}{dx} \left(a - b(x - 0.5)^2\right) = -2b(x - 0.5)$.

$$\frac{d}{dx} \left(a - b(x - 0.5)^2\right)^2 = 4b^2(x - 0.5)^2. \tag{7.4}$$

Substitute these expressions back into $J[y]$:

$$J[y] = \int_0^1 \left(\alpha \left(a^2 - 2ab(x - 0.5)^2 + b^2(x - 0.5)^4\right)^4 + \beta 4b^2(x - 0.5)^2 \right) dx. \tag{7.5}$$

Step 3: Evaluate the Integral.

Constant Term a^2,

$$\int_0^1 a^2 \, dx = a^2. \tag{7.6}$$

Term $-2ab(x - 0.5)^2$

$$\int_0^1 -2ab(x - 0.5)^2 \, dx = -2ab \cdot \frac{1}{12} = -\frac{ab}{6} \tag{7.7}$$

Term $b^2(x - 0.5)^4$.

$$\int_0^1 b^2(x-0.5)^4\,dx = b^2\cdot\frac{1}{80} = \frac{b^2}{80}. \tag{7.8}$$

Term $4b^2(x-0.5)^2$

$$\int_0^1 4b^2(x-0.5)^2\,dx = 4b^2\cdot\frac{1}{12} = \frac{4b^2}{12} = \frac{b^2}{3}. \tag{7.9}$$

Combine the results we have the following:

$$J[y] = \alpha a^2 - \frac{\alpha ab}{6} + \frac{\alpha b^2}{80} + \frac{2\beta b^2}{3}. \tag{7.10}$$

Step 4: Minimize $J[y]$ with Respect to a and b

To find the optimal symmetric grading distribution, we minimize $J[y]$ with respect to a and b. This involves setting the partial derivatives with respect to aa and bb to zero:

Partial Derivative with Respect to a

$$\frac{\partial J}{\partial a} = 2\alpha a - \frac{\alpha b}{6} = 0 \quad \text{this implies that} \quad a = \frac{b}{12}. \tag{7.11}$$

Partial Derivative with Respect to b:

$$\frac{\partial J}{\partial b} = -\frac{\alpha a}{6} + \frac{ab}{40} + \frac{2\beta b}{3} = 0. \tag{7.12}$$

Substitute $a = \frac{b}{12}$ into the second equation to solve for b , and subsequently determine a.

7.3. Applications in Granular Material Models. Symmetric Grading Distributions: By restricting $y(x)$ to symmetric functions, we reduce the computational complexity in optimizing grading curves, which is particularly useful for evenly distributed particle sizes in granular materials.

Dry Density Models: For dry density optimization, using symmetric configurations simplifies the calculations, allowing for efficient determination of optimal particle packing without the need for exhaustive search in asymmetric spaces.

This demonstrates how the Principle of Symmetric Criticality allows us to restrict our search for critical points to symmetric configurations. By doing so, we simplify the optimization of grading curves for dry density and permeability, reducing computational effort while achieving stable, practical solutions for material properties in granular models.

These mathematical principles create a framework that ensures stability, convergence, and optimality in models for granular materials. From the Euler-Lagrange equation for energy minimization to the Principle of Symmetric Criticality for efficient symmetric modeling, this approach underpins accurate simulations and robust predictions in material science and applied mathematics.

8. Conclusions. Calculus of variations is a vital bridge between the fields of partial differential equations (PDEs) and differential geometry. By providing powerful tools for optimizing functionals, it facilitates the solution of PDEs that describe various physical phenomena and enhances our understanding of the geometrical properties of surfaces, particularly through the study of geodesics. Central principles such as the Euler-Lagrange equation, Ekeland's variational principle, and the Mountain Pass theorem are foundational in these applications, highlighting the deep interconnectedness of these mathematical disciplines.

The discussed principles and theorems are essential for solving optimization problems and PDEs. Ekeland's Variational Principle, the Palais-Smale Condition, the Mountain Pass Theorem, and the Principle of Symmetric Criticality offer robust frameworks for proving the existence of minimizers and critical points. The study of geodesics on spheres and planes exemplifies practical applications of these concepts, showcasing the optimization of paths on various surfaces.

Mastering and applying these principles is crucial for advancements in mathematical analysis and its applications across physics, engineering, and other sciences. The interconnectedness between calculus of variations, PDEs, and differential geometry underscores the significance of this field in both theoretical research and practical problem-solving, making it indispensable across multiple scientific domains.

Authorcontributions. The author is fully responsible for what is written in the article.

Conflicts of interest. The author declare no conflict of interest.

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