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The SIRD epidemiological model using Caputo fractional derivatives applied to study the spread of the COVID-19 in the Peruvian region of Tacna

Edson A. Coayla-Teran, Angel J. Calsin-Cari and Guido Alvarez-Jauregui

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Abstract

Mathematical models are widely used to study the spreading dynamics of infectious diseases. In particular, the "Susceptibles-Infecteds-Recovereds-Deceases"(SIRD) model provides a framework that can be adapted to describe the core spreading dynamics of several human and wildlife infectious diseases. The present work uses a SIRD model using Caputo fractional derivative. In this investigation, the existence and uniqueness of solutions for the model were established. Numerical solutions were obtained using the Adams-Bashforth method. To illustrate the model's utility, we made forecasts for the spread of the virus SARS-Cov-2 in the region of Tacna in Perú. It is well known that these models can help to forecast the number of infected people, understand the disease dynamics and evaluate potential control strategies.

Keywords . Fractional ordinary differential equations; Fractional derivatives and integrals; Caputo fractional derivative; SIRD model for infectious diseases

1. Introduction. Infectious diseases have affected several times humankind, and it is worth citing here the second plague pandemic (The Black Death). For roughly six years, between 1346 and 1352, it annihilated some 60% of Europe's population [1].

Recently, infectious disease outbreaks such as SARS, Ebola and Zika have risen in different parts of the world, although these inflicted specific regions and received the attention of the World Health Organization (WHO), the last COVID-19 outbreak was the most sinister of the 21st century until the present. On January 30, 2020, the WHO issued an official statement declaring the COVID-19 epidemic a public health emergency of global importance. The pandemic classification indicates that the disease has spread to multiple countries, continents and even worldwide, affecting many people. The WHO expressed deep concern about the rapid spread of the virus and the severity of its effects, as well as the insufficiency of the actions taken up to that point.

Intending to understand the spreading dynamics of infectious diseases, some authors proposed mathematical models. It is important to cite the pioneering work of Kermack and McKendrick [2], which gave origin to the Susceptibles-Infecteds-Removeds (SIR) model, which is a system of ordinary differential equations on the variables S, I and R. Since then, several other works have risen, intending to improve that model. For example, the SIRD model incorporates the

*Department of Mathematics, Federal University of Bahia, Av. Milton Santos S/N, Salvador, Bahia, Brasil, 40170-110. **Correspondence author.** (coayla@ufba.br).

†Department of Basic Sciences, National University of Juliaca, Av. Nueva Zelandia, Juliaca, Puno, Perú. (ajcalsinc.doc@unaj.edu.pe).

‡Departmento Académico de Matemática, Universidad Nacional de San Antonio Abad del Cusco, Av. De la Cultura 733, Cusco, Perú. (guido.alvarez@unsaac.edu.pe).

class of deceased people to show the impact of the epidemic disease on the population. In the literature, the models SIR, SIRD or others that divide the population into compartments are called "compartmental models for infectious diseases".

On the other hand, some authors proposed SIRD models using delayed differential equations (see [3] for example). Authors such as [4] use fractional derivatives for the model with Caputo derivative for the case of COVID-19, [5] study a SEIR model (here E corresponds to the exposed people) for the case of COVID-19 using Caputo derivative, SIR epidemiological model proposes to analyse real case data in Pakistan in [6] with Caputo derivative.

In the present work, we study a fractional SIRD mathematical model with Caputo derivative of the variables S, I, R and D with respect to time. So, we consider the following SIRD model initial value problem

$$\begin{aligned}
{}^{C}D^{\alpha}_{t_0+}\left[S(t)\right] &= -\beta\frac{S(t)I(t)}{N},\\
{}^{C}D^{\alpha}_{t_0+}\left[I(t)\right] &= \beta\frac{I(t)S(t)}{N} - (\gamma+\kappa)I(t),\\
{}^{C}D^{\alpha}_{t_0+}\left[R(t)\right] &= \gamma I(t),\\
{}^{C}D^{\alpha}_{t_0+}\left[D(t)\right] &= \kappa I(t), \;\; t\in(t_0,T),
\end{aligned} \tag{1.1}$$

along with the initial conditions

$$S(t_0)=S_0,\; I(t_0)=I_0,\; R(t_0)=R_0,\; D(t_0)=D_0, \tag{1.2}$$

where $T>0$ is a real number and ${}^{C}D^{\alpha_i}_{t_0+}$ denotes the Caputo fractional derivative with $0<\alpha<1$.

Here, we need to remark that the constants β, γ and κ have the same meaning, so in the classical SIRD model

$$\begin{aligned}
\frac{dS(t)}{dt} &= -\beta\frac{S(t)\cdot I(t)}{N}\\
\frac{dI(t)}{dt} &= \beta\frac{I(t)\cdot S(t)}{N} - (\gamma+\kappa)\,I(t)\\
\frac{dR(t)}{dt} &= \gamma I(t)\\
\frac{dD(t)}{dt} &= \kappa I(t),
\end{aligned} \tag{1.3}$$

where the constants β, η, and μ are the mean infection rate, recovery rate and death rate, respectively. Moreover, N is the total population and satisfies the assumption $N=S+I+R+D$ for $0\le t\le T$.

Firstly, we demonstrate the existence and uniqueness of the solution to the initial value problem (1.1)-(1.2). To do this, we use a fixed point theorem and following the arguments from [7], our argument used here is slightly different than the one used in [4]. Afterwards, we used the Adams–Bashforth numerical method to obtain the approximate solution of the proposed model as in [4].

To illustrate the model's utility, we apply the numerical method to the case of the spread of COVID-19 in the region of Tacna in Perú during December 2021. Some authors studied the dynamics of COVID-19 in the same region ([3], [8] and [9]), but they used models involving ordinary derivatives.

The rest of the paper is organised as follows: Section 2 is devoted to notation and definitions used throughout the text. In Section 3, we demonstrated the existence and uniqueness of the solution for the initial value problem (1.1)-(1.2). In Section 4, we developed the numerical solution used to do the forecasts. Section 5 is devoted to applying the tools developed in the previous section, which will make some forecasts for the Peruvian region of Tacna. Finally, section 6 is dedicated to conclusions.

2. Preliminaries. We need the following definitions of fractional integral and fractional derivative; more details can be found in [7]. Let $[a,b]$ be a finite interval of $\mathbb{R}$, we will denote by $C([a,b])$ the space of the continuous functions with norm $\|f\|_C = \max_{t\in[a,b]}|f(x)|$.

Definition 2.1. *Let $[a,b]$ be a finite interval of $\mathbb{R}$ and $f:[a,b]\to\mathbb{R}$ a function. The left-sided Riemann-Liouville fractional integral of f, $I^{\alpha}_{a+}f$, of order $\alpha\in\mathbb{C}$ $(\Re(\alpha)>0)$ is given by*

$$(I^{\alpha}_{a+}f)(t) := \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(p)\,dp}{(t-p)^{1-\alpha}},\; (t>a) \tag{2.1}$$

whenever the integral in (2.1) exists and is finite. Moreover, Γ is the Gamma function.

Definition 2.2. *Let $[a,b]$ be a finite interval of $\mathbb{R}$ and $f:[a,b]\to\mathbb{R}$ a function. The Riemann-Liouville fractional derivative of f, $D^{\alpha}_{a+}f$, of order $\alpha\in\mathbb{C}$ ($\Re(\alpha)>0$) is given by*

$$\begin{aligned}(D^{\alpha}_{a+}f)(t) &:= \left(\frac{d}{dt}\right)^{n}(I^{n-\alpha}_{a^+}f)(t)\\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_a^t \frac{f(p)\,dp}{(t-p)^{\alpha-n+1}},\ (n=[\![\Re(\alpha)]\!]+1; t>a).\end{aligned} \tag{2.2}$$

Definition 2.3. *Let $[a,b]$ be a finite interval of $\mathbb{R}$ and $f:[a,b]\to\mathbb{R}$ a function. The left-sided Caputo fractional derivative of f, ${}^{C}D^{\alpha}_{a+}f$, on $[a,b]$ of order $\alpha\in\mathbb{C}$ ($\Re(\alpha)>0$) is given by*

$$({}^{C}D^{\alpha}_{a+}f)(t) := \left(D^{\alpha}_{a^+}\left[y(x)-\sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(x-a)^k\right]\right)(t), \tag{2.3}$$

where $n=[\![\Re(\alpha)]\!]+1$ for $\alpha\notin\mathbb{N}_0$; $n=\alpha$ for $\alpha\in\mathbb{N}_0$.

Remark 2.1. *In particular, when $0<\Re(\alpha)<1$, we have*

$$({}^{C}D^{\alpha}_{a+}f)(t) = (D^{\alpha}_{a+}[f(x)-f(a)])(t).$$

The following consequence of Theorem 3.24 (page 199) from [7] will be important to demonstrate the existence of the solution of the SIRD model using the Caputo derivative.

Lemma 2.1. *Let α be a number such that $0<\alpha<1$. Let G be an open set in $\mathbb{R}$ and $f:(a,b]\times G\to\mathbb{R}$ such that, for any $y\in G$, $f[x,y]\in C[a,b]$.*

If $y(x)\in C[a,b]$, then $y(x)$ satisfies

$$\left({}^{C}D^{\alpha}_{a+}y\right)(x) = f[x,y(x)],\ y(a)=b_0\in\mathbb{R}$$

if and only if, $y(x)$ satisfies the Volterra integral equation

$$y(x) = b_0 + \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f[t,y(t)]}{(x-t)^{1-\alpha}}\,dt, \quad (a\le x\le b). \tag{2.4}$$

Proof: We will use the Theorem 3.24 from [7], $G\subset\mathbb{R}$ and for a real-valued function f. Also, we will take $\gamma=0$. Therefore, the space $C_\gamma[a,b]$ used in that theorem becomes $C[a,b]$. The conclusion follows from that theorem. □

The following Banach fixed point theorem will be used to demonstrate the existence of a solution for the initial value problem (1.1)-(1.2).

Theorem 2.1. *Let (U,d) be a nonempty complete metric space, let $0\le k<1$, and let $T:U\to U$ be the map such that, for every u, $v\in U$ is satisfied*

$$d(Tu,Tv)\le k\,d(u,v)$$

Then, the operator T has a unique fixed point $u^*\in U$.

Furthermore, if T^n ($n=1,2,\ldots$) is the sequence of operators defined by

$$T^1=T,\ \text{and}\ T^n=TT^{n-1},\ n=2,3,\ldots.$$

Then, for any $u_0\in U$, the sequence $\{T^n u_0\}_{k=1}^{\infty}$ converges to the above fixed point u^*.

3. Modelo SIRD with Caputo fractional derivative. This section is devoted to studying in more detail the SIRD model (1.1). In this model, the following assumptions are implicit:

- The number of susceptible (S) and contagious infective (I) individuals is very high. So, the random differences between individuals can be neglected.
- The disease spreads through contact from individual to individual.
- Some infected individuals can be deceased (denoted by (D)).
- Recovered individuals (R) become immune.
- The population is closed with no migration.

Lemma 3.1. *Let $T > 0$ be a fixed real number. Let $(S, I, R, D) \in C[t_0, T] \times C[t_0, T] \times C[t_0, T] \times C[t_0, T]$. Then, the initial value problem (1.1)-(1.2) is equivalent to the following system of integral Volterra equations*

$$\begin{aligned} S(t) &= S(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} -(t-p)^{\alpha-1} \beta \frac{S(p)\, I(p)}{N}\, dp, \\ I(t) &= I(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\gamma)^{\alpha-1} \beta \frac{S(p)\, I(p)}{N} - (\gamma+\kappa) I(p)\, dp, \\ R(t) &= R(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-p)^{\alpha-1} \gamma I(p)\, dp, \\ D(t) &= D(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-p)^{\alpha-1} \kappa I(p)\, dp, \end{aligned} \tag{3.1}$$

for $t \in [t_0, T]$.

Proof: Since $(S, I, R, D) \in C[t_0, T] \times C[t_0, T] \times C[t_0, T] \times C[t_0, T]$, we can apply the Lemma 2.1 to each equation in (1.1). For example, the first equation in (3.1) was obtained by taking $a = t_0$, $b = T$, $G = (0, N)$ and $f(t, S) = -\beta \dfrac{S\, I}{N}$, analogously for each of the equations in (3.1) using the same a, b and G. But the function f changes for each equation. Thus, we conclude the proof. □

The demonstration of the existence of a solution for the initial value problem (1.1)-(1.2) is given in the following theorem. The proof uses the arguments shown in the proof of Theorem 3.25 [7] (page 202).

We will use the following notation $X = C([t_0, T] \times C([t_0, T] \times C([t_0, T] \times C([t_0, T])$, with norm $\|(S, I, R, D)\|_X = \|S\|_C + \|I\|_C + \|R\|_C + \|D\|_C$.

Theorem 3.1. *There is a unique solution for the initial value problem* (1.1)-(1.2) *in the space* X.

Proof: First, we need to remark that from the Lemma 3.1, to find a solution for the initial value problem (1.1)-(1.2) it is equivalent to finding maps S, I, R and D and satisfying (3.1). So, to find S, I, R and D, let us consider the space $X_1 = C([t_0, x_1] \times C([t_0, x_1] \times C([t_0, x_1] \times C([t_0, x_1])$, with norm $\|(S, I, R, D)\|_{X_1} = \|S\|_C + \|I\|_C + \|R\|_C + \|D\|_C$, where x_1 is such that $t_0 < x_1 \leq T$ and

$$\frac{2\beta(x_1 - t_0)^{\alpha}}{\Gamma(\alpha+1)} + \frac{2(\gamma+\kappa)(x_1 - t_0)^{\alpha}}{\Gamma(\alpha+1)} < 1. \tag{3.2}$$

Let $F : X_1 \to X_1$ be a map such that for $\varphi \in X_1$, $F(\varphi) = (F_1(\varphi), F_2(\varphi), F_3(\varphi), F_4(\varphi))$, where

$$\begin{aligned} F_1(S, I, R, D)(t) &= S_0 + \tfrac{1}{\Gamma(\alpha)} \int_{t_0}^{t} -(t-p)^{\alpha-1} \beta \frac{S(p)\, I(p)}{N}\, dp \\ F_2(S, I, R, D)(t) &= I_0 + \tfrac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\gamma)^{\alpha-1} \beta \frac{S(p)\, I(p)}{N} - (\gamma+\kappa) I(p)\, dp \\ F_3(S, I, R, D)(t) &= R_0 + \tfrac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-p)^{\alpha-1} \gamma I(p)\, dp \\ F_4(S, I, R, D)(t) &= D_0 + \tfrac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-p)^{\alpha-1} \kappa I(p)\, dp. \end{aligned} \tag{3.3}$$

By using the Lemma 2.8 (a) from [7], the map F is well defined.

Furthermore, we will demonstrate that the map F is a contraction under the condition (3.2).

$$\begin{aligned} \|F(S_1, I_1, R_1, D_1) &- F(S_2, I_2, R_2, D_2)\|_{X_1} = \\ & \|F_1(S_1, I_1, R_1, D_1) - F_1(S_2, I_2, R_2, D_2)\|_C \\ & + \|F_2(S_1, I_1, R_1, D_1) - F_2(S_2, I_2, R_2, D_2)\|_C \\ & + \|F_3(S_1, I_1, R_1, D_1) - F_3(S_2, I_2, R_2, D_2)\|_C \\ & + \|F_4(S_1, I_1, R_1, D_1) - F_4(S_2, I_2, R_2, D_2)\|_C. \end{aligned}$$

We need to calculate each term above. Observe that because S, I, R and D belong to $C([t_0, x_1])$ we get

$$\begin{aligned}\|F_1(S_1, I_1, R_1, D_1) \quad &-F_1(S_2, I_2, R_2, D_2)\|_C \leq \\ &\sup_{t\in[t_0,x_1]} \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} (t-p)^{\alpha-1} \left| \frac{-I_1(p)S_1(p) + I_2(p)S_2(p)}{N} \right| dp \leq \\ &\frac{\beta(x_1 - t_0)^\alpha}{\Gamma(\alpha+1)} (\|I_1 - I_2\|_C + \|S_1 - S_2\|_C),\end{aligned}$$

$$\begin{aligned}\|F_2(S_1, I_1, R_1, D_1) \quad &-F_2(S_2, I_2, R_2, D_2)\|_C \leq \\ &\frac{\beta(x_1 - t_0)^\alpha}{\Gamma(\alpha+1)} (\|I_1 - I_2\|_C + \|S_1 - S_2\|_C) + \\ &\frac{(\gamma+\kappa)(x_1 - t_0)^\alpha}{\Gamma(\alpha_2+1)} \|I_1 - I_2\|_C,\end{aligned}$$

$$\begin{aligned}\|F_3(S_1, I_1, R_1, D_1) \quad &-F_3(S_2, I_2, R_2, D_2)\|_C \leq \\ &\frac{\gamma(x_1 - t_0)^\alpha}{\Gamma(\alpha+1)} \|I_1 - I_2\|_C,\end{aligned}$$

and

$$\begin{aligned}\|F_4(S_1, I_1, R_1, D_1) \quad &-F_4(S_2, I_2, R_2, D_2)\|_C \leq \\ &\frac{\kappa(x_1 - t_0)^\alpha}{\Gamma(\alpha+1)} \|I_1 - I_2\|_C.\end{aligned}$$

From the four inequalities above, we obtain

$$\begin{aligned}\|F(S_1 \quad &, I_1, R_1, D_1) - F\left(S_2, I_2, R_2, D_2\right)\|_{X_1} = \\ &\Big(\frac{\beta(x_1 - t_0)^\alpha}{\Gamma(\alpha+1)} + \frac{\beta(x_1 - t_0)^\alpha}{\Gamma(\alpha+1)} + \frac{(\gamma+\kappa)(x_1 - t_0)^\alpha}{\Gamma(\alpha+1)} \\ &+ \frac{\gamma(x_1 - t_0)^\alpha}{\Gamma(\alpha+1)} + \frac{\kappa(x_1 - t_0)^\alpha}{\Gamma(\alpha+1)}\Big)(\|I_1 - I_2\|_C + \|S_1 - S_2\|_C \\ &+ \|R_1 - R_2\|_C + \|D_1 - D_2\|_C).\end{aligned}$$

Hence and from the Theorem (2.1), there is a unique solution y for the initial value problem (1.1)-(1.2) on the interval $[t_0, x_1]$. The rest of the demonstration is done by using the arguments from the demonstration of Theorem 3.3 from [7] (page 148). Replacing x_1 by x_2 in the condition (3.2) with $x_1 < x_2 \leq T$ and using the same procedure, we can find a unique solution y on the interval $[x_1, x_2]$ for the following system

$$\begin{aligned} S(t) &= S(x_1) + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{t} -(t-p)^{\alpha-1} \beta \frac{S(p)\, I(p)}{N}\, dp, \\ I(t) &= I(x_1) + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{t} (t-\gamma)^{\alpha-1} \beta \frac{S(p)\, I(p)}{N} - (\gamma+\kappa) I(p)\, dp, \\ R(t) &= R(x_1) + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{t} (t-p)^{\alpha-1} \gamma I(p)\, dp, \\ D(t) &= D(x_2) + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{t} (t-p)^{\alpha-1} \kappa I(p)\, dp, \end{aligned} \tag{3.4}$$

for $t \in [x_1, x_2]$. Repeating this procedure, we can conclude that there is a unique solution $(S, I, R, D) \in X$ for the equation (3.1). This completes the demonstration of Theorem 3.1. □

4. Numerical solution. Now, we will develop the numerical solution for the initial value problem (1.1)-(1.2). We will use the Adams-Bashforth method [4].

Let us define $t_r := rh$, $r = 0, 1, 2, ..., J$, where $h = \dfrac{T}{J}$ is the step size and $J > 0$ is a integer number.

Replacing $t = t_{r+1}$ and $t = t_r$ in the first equation of (3.1) we get

$$S(t_{r+1}) - S_0 = \frac{1}{\Gamma(\alpha)} \int_0^{t_{r+1}} -(t_{r+1} - p)^{\alpha-1} \beta \frac{S(p)\, I(p)}{N}\, dp, \tag{4.1}$$

and

$$S(t_r) - S_0 = \frac{1}{\Gamma(\alpha)} \int_0^{t_r} -(t_r - p)^{\alpha-1} \beta \frac{S(p)\, I(p)}{N}\, dp, \tag{4.2}$$

respectively. Doing the equation (4.2) minus the equation (4.1) we obtain

$$\begin{aligned} S(t_{r+1}) = S(t_r) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{r+1}} -(t_{r+1} - p)^{\alpha-1} \beta \frac{S(p)\, I(p)}{N}\, dp, \\ - \frac{1}{\Gamma(\alpha)} \int_0^{t_r} -(t_r - p)^{\alpha-1} \beta \frac{S(p)\, I(p)}{N}\, dp. \end{aligned} \tag{4.3}$$

We will use the Lagrange polynomial $P(t)$ to approximate it to the function $-\beta \frac{S(p)\, I(p)}{N}$, given by

$$P(t) = \frac{t - t_{r-1}}{t_r - t_{r-1}} (-\beta \frac{S(t_r)\, I(t_r)}{N}) + \frac{t - t_r}{t_{r-1} - t_r} (-\beta \frac{S(t_{r-1})\, I(t_{r-1})}{N}).$$

Replacing $t_r - t_{r-1} = rh - (r-1)h = h$ in $P(t)$ we obtain

$$P(t) = \frac{-\beta \frac{S(t_r)\, I(t_r)}{N}}{h} (t - t_{r-1}) - \frac{-\beta \frac{S(t_{r-1})\, I(t_{r-1})}{N}}{h} (t - t_r). \tag{4.4}$$

Replacing (4.4) by (4.3) we obtain an approximation of the integral terms. Calculating the integrals and after replacing $t_{r+1} = (r+1)h$, $t_r = rh$, we get

$$\begin{aligned} S(t_{s+1}) \quad &\simeq S\,(t_s) - \frac{\beta S(t_s) I(t_s)}{N\Gamma(\alpha)} (h^\alpha) \Bigg(\frac{2(s+1)^\alpha - s^\alpha}{\alpha} \\ &- \frac{(s+1)^{\alpha+1} - s^{\alpha+1}}{\alpha+1} \Bigg) \\ &- \frac{\beta S(t_{s-1}) I(t_{s-1})}{N\Gamma(\alpha)} (h^\alpha) \left(\frac{(s+1)^{\alpha+1} - s^{\alpha+1}}{\alpha+1} - \frac{(s+1)^\alpha}{\alpha_1} \right) \end{aligned} . \tag{4.5}$$

Using the same procedure for the second, third and fourth equations in (3.1), we get

$$\begin{aligned} I\,(t_{s+1}) \quad &\simeq I\,(t_s) + \left(\beta \frac{I(t_s) S(t_s)}{N} - (\gamma + \kappa) I(t_s) \right) \left(\frac{1}{\Gamma(\alpha)} \right) (h^\alpha) \times \\ &\left(\frac{2(s+1)^\alpha - s^\alpha}{\alpha} - \frac{(s+1)^{\alpha+1} - s^{\alpha+1}}{\alpha+1} \right) \\ &+ \left(\beta \frac{I(t_{s-1}) S(t_{s-1})}{N} - (\gamma + \kappa) I(t_{s-1}) \right) \left(\frac{1}{\Gamma(\alpha_2)} \right) (h^\alpha) \times \\ &\left(\frac{(s+1)^{\alpha_2+1} - s^{\alpha+1}}{\alpha+1} - \frac{(s+1)^\alpha}{\alpha} \right), \end{aligned} \tag{4.6}$$

$$\begin{aligned} R(t_{s+1}) \simeq R(t_s) + \frac{\gamma I(t_s)}{\Gamma(\alpha)} (h^\alpha) \left(\frac{2(s+1)^\alpha - s^\alpha}{\alpha} - \frac{(s+1)^{\alpha+1} - s^{\alpha+1}}{\alpha+1} \right) \\ + \frac{\gamma I(t_{s-1})}{\Gamma(\alpha)} (h^\alpha) \left(\frac{(s+1)^{\alpha+1} - s^{\alpha+1}}{\alpha+1} - \frac{(s+1)^\alpha}{\alpha} \right), \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} D(t_{s+1}) \quad &\simeq D(t_s) + \tfrac{\kappa I(t_s)}{\Gamma(\alpha)} (h^\alpha) \left(\tfrac{2(s+1)^\alpha - s^\alpha}{\alpha} - \tfrac{(s+1)^{\alpha+1} - s^{\alpha+1}}{\alpha+1} \right) \\ &+ \tfrac{\kappa I(t_{s-1})}{\Gamma(\alpha)} (h^\alpha) \left(\tfrac{(s+1)^{\alpha+1} - s^{\alpha+1}}{\alpha+1} - \tfrac{(s+1)^\alpha}{\alpha} \right) \end{aligned} \tag{4.8}$$

respectively.

5. Rates estimation. Set

$$\begin{aligned} A &= -S(t_r)I(t_r)\left(h^\alpha\right)\left(\frac{2(r+1)^\alpha - r^\alpha}{\alpha} - \frac{(r+1)^{\alpha+1} - r^{\alpha+1}}{\alpha+1}\right) \\ &\quad -S(t_{r-1})I(t_{r-1})\left(h^\alpha\right)\left(\frac{(r+1)^{\alpha+1} - r^{\alpha+1}}{\alpha+1} - \frac{(r+1)^\alpha}{\alpha}\right) \end{aligned}$$

Thus, from (4.5) we obtain

$$\beta_r \simeq \frac{\left[S(t_{r+1}) - S(t_r)\right] N\Gamma(\alpha)}{A}$$

Let us denote β_i for $i = 1, 2, ..., n$; we will use the following *infection rate*

$$x = \frac{\sum_{i=1}^{n} \beta_i}{n}, \quad i = 1, 2, \ldots, n. \tag{5.1}$$

Set

$$\begin{aligned} B &= I(t_r)\left(h^\alpha\right)\left(\frac{2(r+1)^\alpha - r^\alpha}{\alpha} - \frac{(r+1)^{\alpha+1} - r^{\alpha+1}}{\alpha+1}\right) \\ &\quad +I(t_{r-1})\left(h^\alpha\right)\left(\frac{(r+1)^{\alpha+1} - r^{\alpha+1}}{\alpha+1} - \frac{(r+1)^\alpha}{\alpha}\right) \end{aligned}.$$

From (4.7), we get

$$\gamma_r \simeq \frac{\left[R(t_{r+1}) - R(t_r)\right]\Gamma(\alpha)}{B}.$$

For $i = 1, 2, ..., n$; we will use the following *recovered rate*

$$y = \frac{\sum_{i=1}^{n} \gamma_i}{n}, \quad i = 1, 2, \ldots, n. \tag{5.2}$$

Set

$$\begin{aligned} C &= I(t_r)\left(h^\alpha\right)\left(\frac{2(r+1)^\alpha - r^\alpha}{\alpha} - \frac{(r+1)^{\alpha+1} - r^{\alpha+1}}{\alpha+1}\right) \\ &\quad +I(t_{r-1})\left(h^\alpha\right)\left(\frac{(r+1)^{\alpha+1} - r^{\alpha+1}}{\alpha+1} - \frac{(r+1)^\alpha}{\alpha}\right) \end{aligned}.$$

From (4.8), we obtain

$$\kappa_r \simeq \frac{\left[D(t_{r+1}) - D(t_r)\right]\Gamma(\alpha)}{C}$$

For $i = 1, 2, ..., n$; we will use the following *death rate*

$$z = \frac{\sum_{i=1}^{n} \kappa_i}{n}, \quad i = 1, 2, \ldots, n. \tag{5.3}$$

6. Forecasts for the region of Tacna. In this section, we forecast the COVID-19 trend using the numerical scheme in section 4 and the rates in section 5. To use the approximations (4.6), (4.7), (4.8) and calculate the rates given by (5.1), (5.2) and (5.3) we take the step size $h = 1$. To do this, we will use the Software Scilab 2024.1.0.

6.1. Dataset. The data was obtained from September 01 to December 31 of 2021 from the official Facebook page of the Tacna region Health Directory (DIRESA) [10]. We chose this interval because the data is complete during this period.

Table 6.1: Data for the period between September 1 and 30.

Day	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
I	28	4	12	4	4	3	5	6	5	6	2	4	5	3	8	13
R	26	8	10	12	8	6	9	11	10	8	6	6	5	7	5	8
D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1

Day	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
I	4	2	2	8	9	8	6	4	2	3	3	4	10	8
R	11	6	4	5	6	7	8	5	7	7	7	6	5	8
D	0	0	0	0	0	0	0	0	0	1	1	1	0	0

Table 6.2: Data for the period between October 1 and 31.

Day	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
I	17	2	9	4	7	7	5	11	11	17	2	7	14	14	22	17
R	7	10	6	5	7	6	5	6	6	6	9	8	10	8	10	9
D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Day	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**
I	12	21	13	21	26	15	13	9	9	19	28	13	37	29	21
R	8	10	9	11	9	9	10	8	7	8	9	10	9	14	18
D	0	1	0	0	0	0	0	0	0	0	0	1	1	1	0

Table 6.3: Data for the period between November 1 and 30.

Day	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
I	27	23	23	18	19	24	12	30	50	41	32	24	23	22	20
R	19	20	20	21	27	35	17	23	39	42	25	25	21	16	25
D	0	0	0	0	2	2	0	0	0	0	0	2	1	0	1

Day	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
I	16	31	45	81	27	11	5	24	51	31	36	22	14	6	18
R	12	35	58	47	13	8	7	18	32	24	29	16	11	5	11
D	1	1	1	0	2	0	1	0	0	0	1	0	0	0	0

Table 6.4: Data for the period between December 1 and 31.

Day	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
I	35	38	49	32	30	6	28	40	14	38	25	19	19	52	39	66
R	13	21	67	45	44	14	34	12	22	26	13	14	17	41	49	66
D	1	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0

Day	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**
I	21	53	25	18	43	46	38	35	33	13	11	36	82	74	24
R	18	36	19	11	56	37	46	41	38	9	9	42	53	47	24
D	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0

6.2. Parameter setup. For the total population, we use $N = 346,000$ inhabitants; this number is the estimate for June 30, 2020, made by the National Institute of Statistics and Computing of Perú (INEI)[11].

The rate of infection, rate of recovered and rate of deceased were obtained by using the equations (5.1), (5.2) and (5.3) respectively and are shown in the table (6.5) according to the order of derivative α indicated.

α	x	y	z
0.87	0.1291815	6.791E-08	0.0006076
0.96	0.0042911	0.0000001	0.0009955
0.67	0.1631146	5.619E-08	0.0004528
0.48	0.48805695	0.00000007	0.00046927
0.39	0.89879506	0.00000008	0.00050098
0.50	0.42513222	0.00000007	0.00046406

Table 6.5: Different values for rates estimates.

6.3. Forecasts. With the parameters obtained in the last section, we made forecasts showing data in the figures (6.1) and (6.3). It is important to recall that the results showed in these figures are the rounded numbers obtained from approximations (4.6), (4.7), (4.8). Moreover, we made the graphics (6.2) and (6.4) to illustrate the disease's trend for the date from figure (6.1), and the graphics (6.5) and (6.6) to illustrate the disease's trend for the date from figure (6.3).

$\alpha = 0.87$

S	I	R	D
345943	35	21	1
345959	28	13	0
345957	30	13	0
345953	34	13	0
345950	37	13	0
345946	41	13	0
345941	45	13	0
345937	50	13	0
345931	55	13	0
345926	61	13	0
345919	67	13	0
345912	74	13	0
345905	82	13	0
345896	91	13	0
345887	100	13	0
345876	110	13	0
345865	122	13	0
345852	135	13	1
345838	149	13	1
345822	165	13	1
345804	182	13	1
345785	202	13	1
345763	223	13	1
345739	247	13	1
345711	274	13	1
345681	304	13	1
345647	338	13	1
345610	376	13	2
345567	418	13	2
345520	465	13	2
345467	518	13	2

$\alpha = 0.96$

S	I	R	D
345943	35	21	1
345959	28	13	0
345959	28	13	0
345959	28	13	0
345959	28	13	0
345959	28	13	0
345958	28	13	0
345958	28	13	0
345958	29	13	0
345958	29	13	0
345958	29	13	0
345958	29	13	0
345958	29	13	0
345958	29	13	0
345958	29	13	0
345957	29	13	0
345957	29	13	0
345957	29	13	0
345957	29	13	0
345957	30	13	0
345957	30	13	0
345957	30	13	1
345957	30	13	1
345957	30	13	1
345956	30	13	1
345956	30	13	1
345956	30	13	1
345956	30	13	1
345956	30	13	1
345956	30	13	1
345956	30	13	1

$\alpha = 0.67$

S	I	R	D
345943	35	21	1
345959	28	13	0
345958	29	13	0
345955	32	13	0
345952	35	13	0
345949	38	13	0
345946	41	13	0
345942	45	13	0
345939	48	13	0
345935	52	13	0
345931	56	13	0
345926	60	13	0
345922	65	13	0
345917	70	13	0
345911	76	13	0
345905	81	13	0
345899	88	13	0
345892	95	13	0
345885	102	13	0
345877	110	13	0
345868	119	13	0
345859	128	13	0
345849	138	13	0
345837	149	13	0
345825	161	13	0
345812	175	13	0
345797	189	13	0
345782	205	13	0
345764	222	13	1
345745	241	13	1
345724	262	13	1

Figure 6.1: Simulations for $\alpha = 0.87, 0.96, 0.67$

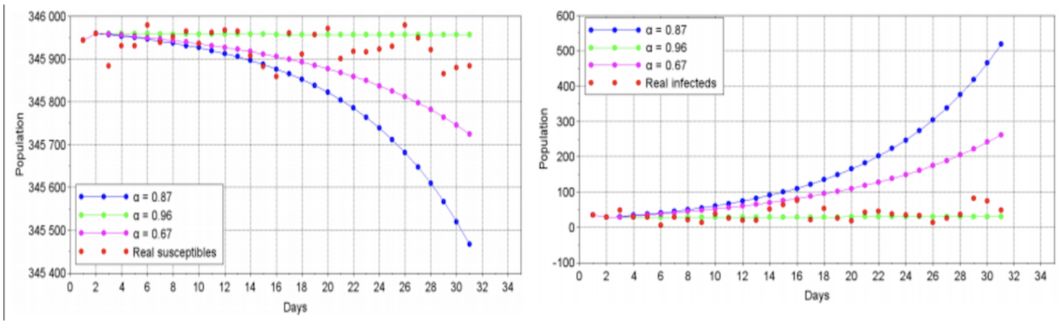

Figure 6.2: Simulations vs. real data

$\alpha = 0.50$

S	I	R	D
345943	35	21	1
345959	28	13	0
345958	29	13	0
345954	33	13	0
345947	40	13	0
345940	47	13	0
345930	57	13	0
345918	69	13	0
345904	83	13	0
345886	101	13	0
345865	122	13	0
345837	149	13	0
345804	183	13	0
345761	226	13	0
345706	281	13	0
345636	351	13	0
345545	442	13	0
345426	561	13	1
345269	717	13	1
345061	925	13	1
344782	1204	13	1
344404	1581	13	2
343890	2095	13	2
343183	2801	13	3
342206	3777	13	4
340846	5136	13	6
338945	7034	13	8
336284	9692	13	11
332566	13406	13	15
327406	18560	13	21
320346	25611	13	29

$\alpha = 0.48$

S	I	R	D
345943	35	21	1
345959	28	13	0
345958	29	13	0
345953	34	13	0
345946	41	13	0
345937	50	13	0
345925	62	13	0
345910	77	13	0
345890	97	13	0
345865	122	13	0
345832	155	13	0
345789	198	13	0
345732	255	13	0
345656	331	13	0
345553	433	13	0
345413	573	13	1
345220	766	13	1
344951	1035	13	1
344573	1413	13	1
344036	1950	13	2
343265	2719	13	3
342152	3831	13	4
340531	5450	13	5
338162	7817	13	8
334694	11282	13	11
329641	16330	13	16
322364	23600	13	24
312134	33818	13	35
298334	47602	13	51
280904	65010	13	72
260939	84948	13	100

$\alpha = 0.39$

S	I	R	D
345943	35	21	1
345959	28	13	0
345960	27	13	0
345955	32	13	0
345945	42	13	0
345926	61	13	0
345893	94	13	0
345835	152	13	0
345734	253	13	0
345554	433	13	0
345227	760	13	0
344624	1363	13	1
343493	2492	13	1
341346	4639	13	3
337228	8754	13	5
329335	16642	13	10
314478	31491	13	18
287970	57982	13	36
246272	99648	13	67
196386	149484	13	117
162730	183078	13	178
154887	190876	13	223
151227	194521	13	239
147248	198490	13	249
143702	202024	13	261
140360	205356	13	271
137250	208456	13	281
134328	211368	13	291
131582	214104	13	301
128991	216685	13	311
126542	219125	13	320

Figure 6.3: Simulations for $\alpha = 0.50, 0.48, 0.39$

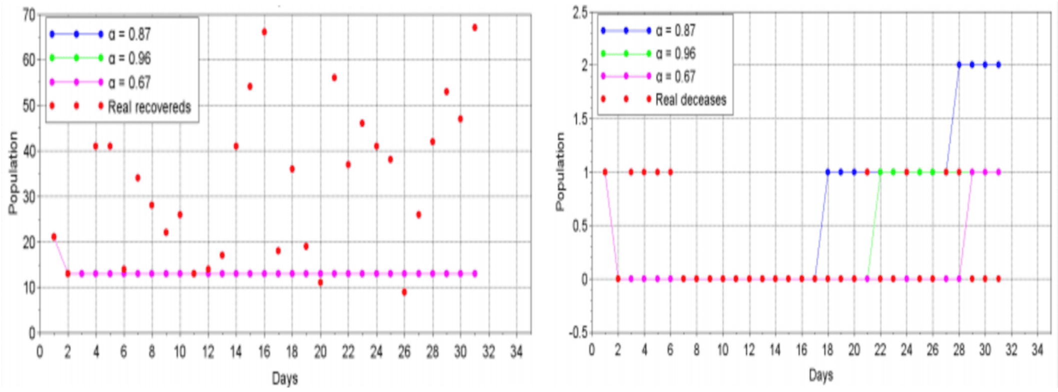

Figure 6.4: Simulations vr. real data

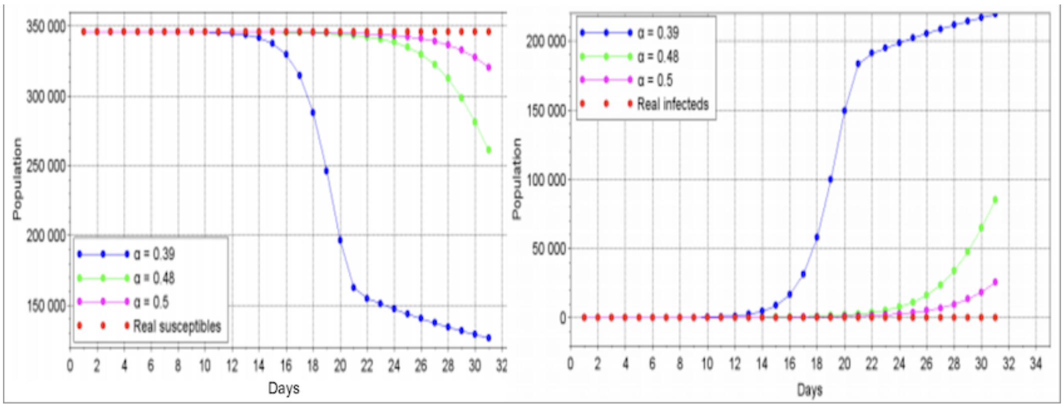

Figure 6.5: Simulations vs. real data

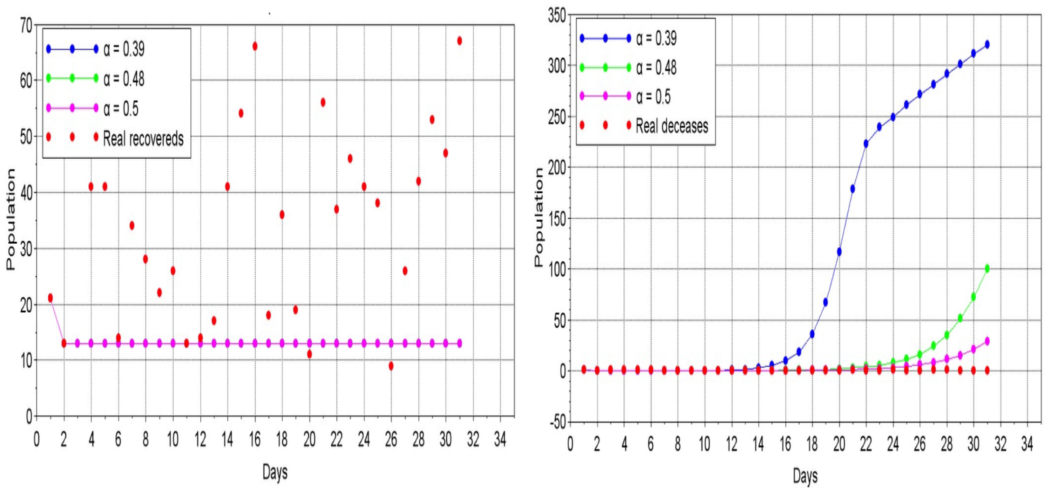

Figure 6.6: Simulations vs. real data

7. Comparison. To compare the data obtained by using the numerical approximation given by the equations (4.6), (4.7) and (4.8) with the real data, we chose the month of December 2021.

To measure the approximation obtained for the forecast, we will use the following number

$$E = \sqrt{\frac{\sum_{i=1}^{31}(Y_i - R_i)^2}{31}}.$$

where Y_i and R_i denote the value of the simulations (from the figures (6.1) and (6.3)) and the real data, respectively. The table (7.1) shows the number E calculated for each variable S, I, R, D and for different values according to the order of integration α.

Table 7.1: Error (E)

α	S	I	R	D
0.87	167.04924906	182.86069085	27.18336778	0.87988269
0.96	43.64408839	19.78350567	27.18336778	0.62217102
0.67	77.16070369	90.86004836	27.18336778	0.59568340
0.48	22651.25392598	22641.35863643	27.18336778	25.37016282
0.39	126621.99988154	126479.30018833	27.18336778	161.98416888
0.50	6646.47608063	6654.72761130	27.18336778	7.46389157

8. Discussion. Although we studied a simplified mathematical model and the dynamics of infectious diseases is complex, the use of a simple mathematical model is useful to get a perception of the spread of the diseases [12]. It allows us to try strategies to control the spread and to identify the general patterns of the disease for the period studied.

Another aspect is that the rates calculated for each α are estimates based on the model studied and, therefore, are affected by the limitations of the underlying model. When considering other factors, it is possible to obtain different results.

To apply the model studied, we made simulations for the spread of COVID-19 in the Peruvian region of Tacna for December 2021. These simulations were obtained by choosing values according to the order of integration α and observing their effect on the simulations compared with the real data. This procedure permitted to obtain some values for α such that the respective simulations obtained approximated better the real data. Thus, from the table (7.1), we can infer that the simulations obtained for $\alpha = 0.96$ approach better the real data.

9. Conclusions.

i. We demonstrated the existence and uniqueness of a solution for a SIRD mathematical model with Caputo fractional derivative and developed numerical solutions.

ii. We studied and applied a mathematical model SIRD with Caputo fractional derivative for the spread of COVID-19 in the Peruvian region of Tacna for December 2021.

iii. From the simulations, the graphics and the (7.1), we can conclude that some values α reproduce in a better manner the dynamics of the spread of COVID-19 in the mentioned region for that period; for $0.5 < \alpha < 1$ the simulations are close to real data.

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Conflicts of interest. The authors declare no conflict of interest.

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ORCID and License
Edson A. Coayla-Teran https://orcid.org/0000-0002-6329-4929
Angel J. Calsin-Cari https://orcid.org/0009-0004-6661-2787
Guido Alvarez-Jauregui https://orcid.org/0000-0002-2669-9847

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References

[1] Benedictow OJ. The complete history of the Black Death. Woodbridge: Boydell & Brewer; 2021.

[2] Kermack WO, McKendrick AG. A contribution to the mathematical theory of epidemics. Proceedings of the Royal Society of London Series A, Containing papers of a mathematical and physical character. 1927;115(772):700 -721.

[3] Coayla-Teran EA. A COVID-19 time-dependent SIRD model using functional differential equations. International Journal of Ecology and Development. 2021;36(3):1-11.

[4] Nisar KS, Ahmad S, Ullah A, Shah K, Alrabaiah H, Arfan M. Mathematical analysis of SIRD model of COVID-19 with Caputo fractional derivative based on real data. Results in Physics. 2021;21:103772. Available from: https://www.sciencedirect.com/science/article/pii/S2211379720321823.

[5] Rezapour S, Mohammadi H, Samei ME. SEIR epidemic model for COVID-19 transmission by Caputo derivative of fractional order. Advances in difference equations. 2020;2020:1-19.

[6] Alshomrani AS, Ullah MZ, Baleanu D. Caputo SIR model for COVID-19 under optimized fractional order. Advances in Difference Equations. 2021;2021(1):185.

[7] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. vol. 204. Amsterdam: Elsevier; 2006.

[8] Vargas-Pichón HB, Coayla-Teran EA, Tejada-Vásquez E. The SIRD epidemiological model applied to spread of the COVID-19 in the Peruvian region of Tacna. Selecciones Matemáticas. 2022;9(01):137 -144.

[9] Vergara-Moreno E, et al. Modelo básico epidemiológico SIR para el COVID-19: caso las Regiones del Perú. Selecciones Matemáticas. 2020;7(01):151 -161.

[10] DIRESA. Health directory of Tacna region in Peru;. May 2023. Available from: https://www.facebook.com/drstacna.

[11] INEI. National Institute of Statistics and Computing of Peru;. June 2020. Available from: https://www.inei.gob.pe/media/MenuRecursivo/publicacionesdigitales/Est/Lib1715/libro.pdf.

[12] Brauer F, Castillo-Chavez C. Mathematical models in population biology and epidemiology. vol. 2. New York: Springer; 2012.