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A seasonal commensalism model with a weak Allee effect to describe climate-mediated shifts

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Abstract

Climate change is affecting the life cycle of tight interacting species. Commonly, the seasonal population dynamics of species is analyzed through models with periodic rates; however, assuming periodicity in seasonal phenomena which depend on environmental drivers is very restrictive. In this work, we analyze seasonal commensalism between two species in which the per capita growth rate of each species is affected by a weak Allee effect and the demographic and ecological rates are assumed almost periodic. To do this, we construct and analyze an almost periodic model to describe commensalism using a wide family of functions that describe weak Allee effects and the benefits granted by the interaction. We prove that the model admits a unique almost periodic global attractor for a wide family of functions. Numerical simulations of the solutions of the model shown the result proved in this work. We show that if periodic rates are used when the phenomenon is really almost periodic, underestimation or overestimation of the population size of both species can occur, which can lead to design wrong strategies by the decision makers.

Keywords . Almost periodic function, commensalism, climate change, Allee effect.

1. Introduction. In nature, species are involved in different ecological relationships as predator-prey, victim-exploiter, competition, mutualism, ammensalism and commensalism. When two species are either in a predator-prey or a victim-exploiter relationship, the interaction is beneficial for one species and detrimental for the other. When two species are in a competition relationship, the interaction is detrimental for both species. In contrast, when two species are in a mutualistic relationship, the interaction is beneficial for both species. There are other ecological relationships in which one species does not present any alteration by the presence of individuals of the other species. In this case, the interaction is said to be neutral for the first species. In this context, if two species are interacting and the interaction is detrimental for one species and neutral for the other, this relationship is called amensalism. In contrast, if two species are interacting and the interaction is beneficial for one species and neutral for the other species, this relationship is called commensalism. Among all ecological interactions, mutualism has been studied to a lesser extent compared to the others [1]. The same occurs with commensalism.

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Commensalism occurs in the plant and animal kingdoms. Commensalism can also be found among bacterial species. Although there are several variants of commensalism, this can be classified in three types of commensal relationships. We will briefly mention those cases. Phoresis is a type of commensalism in which one species is mechanically transported by another species (the host) for which the interaction is neutral. Phoresis occurs between different groups of arthropods. In particular, hippoboscid flies are in a phoretic association with skin mites [2]. Phoresy occurs between pseudoscorpions and other arthropods, including 44 families of insects and three families of arachnids [3]. Phoresy is frequently found among pseudoscorpions with mammals and even birds [4, 5]. Inquilinism is a type of commensalism in which one species uses a second species as a plattaform of cavity for the living circumstances of the beneficiary species. This type of commensalims can be found among barnacles that live on shells of mussels [6]. Finally, chemical commensalism occurs when bacteria produce a chemical that feeds another bacteria [7].

The ecological relationships are affected by changes in the population densities of the species interacting. Wolin analyzes mutualism through mathematical models with density-dependent rates [8]. In these models, it is assumed, in a first case, that the birth rate of each mutualistic species decreases while the death rate increases when the population size increases. Also, it is assumed that the presence of the partner species benefits the per capita growth rate of the recipient species. In that work, it is proved that mutualism can increase either the carrying capacity or the intrinsic growth rate or both. In this direction, there are mathematical models that describe other ecological mechanisms that affect the per capita growth rate of one species. For example, the called Allee effect. This effect has been recognized as a relevant phenomenon of positive density dependence in low-density population. An Allee effect can be either strong or weak. In particular, a strong Allee effect is related with a critical population level, below which the per capita growth rate is negative and a scenario of extinction can occur. A weak Allee effect occurs when the growth rate is reduced; however, the growth rate remains positive [9].

Commonly, the demographic rates of species are affected by environmental drivers. It is known that climate change can alter the phenology of tightly related species [10, 11, 12, 13, 14, 15, 16]. This occurs because environmental drivers can affect the life cycle of interacting species. It is known that some populations present a seasonal population behavior, for example, pseudoscorpions and arthropods present seasonal features of life cycle [17, 18]. To examplify, we mention that *Varroa destructor*, *Nosema sp* and *Apis mellifera* colonies present a seasonal dynamics [19]. When environmental drivers are modeled, seasonal rates are used. In many cases, the modeler uses periodic functions to describe seasonal effects [20, 21, 22]. However, using periodic functions can be very restrictive. A better choice is modelling such effects with almost periodic functions [23, 24, 25]. In this work, we propose a general model to describe a seasonal commensalism between two interacting species such that the per capita growth rate of each species is affected by a weak Allee effect. To do this, in Section 2, we present a general almost periodic model of commensalism. In section 3, some results about almost periodic functions and cooperative systems are shown. In Section 4, the existence and stability of a unique almost periodic solution is proved. In Section 5, numerical simulations of the solutions of the model are shown for different functions. Finally, in Section 6, the results obtained are discussed.

2. The model.

To analyze the population dynamics of two species in a seasonal commensalist relationship, in which each species is affected by a weak Allee effect, we propose the following model

$$\begin{aligned} x' &= x\left(b_1(t)-a_1(t)x\right)f_1(x)+a_2(t)f_2(y)x, \\ y' &= y\left(b_2(t)-a_3(t)y\right)f_3(y). \end{aligned} \tag{2.1}$$

Where the parameters $a_1(t), a_2(t), a_3(t), b_1(t)$ and $b_2(t)$ are almost periodic functions. Model (2.1) describes a scenario in which each species has a reduced per capita growth rate at low population densities; but for all population densities, the population effect is positive, unlike in a strong Allee effect. Such effects are modeled by the terms $x\left(b_1(t)-a_1(t)x\right)f_1(x)$, for species x, and $y\left(b_2(t)-a_3(t)y\right)f_3(y)$, for species y, in model (2.1). The benefits received for the species x from the interaction with the partner species y is modeled by $a_2(t)f_2(y)x$. Model (2.1) generalizes the model analyzed in [26] and [27].

To prove existence of a unique global attractor we need to make the following assumptions on the C^1-functions $f_i:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$, for $i=1,2,3$.

A1) $f_2(0)=0$ and $f_i'\geq 0,\ i=1,2.$

A2) $\lim_{z\to+\infty}(a_2(t)f_2(z)-a_{1*}zf_1(z))=-\infty$.

A3) $f_3(y)>0$ for $y>0$.

In the following, for an almost periodic function $v:\mathbb{R}\to\mathbb{R}$, we denote

$$v_*:=\inf_{t\in\mathbb{R}}v(t)\ \text{ and }\ v^*:=\sup_{t\in\mathbb{R}}v(t).$$

In the following section, we summarize some well known basic facts about almost periodic functions and cooperative systems.

3. Almost periodic functions and cooperative systems.

Almost periodic functions are nowadays a very active theme. Here, we only give a very basic introduction to the topic and refer the reader to [28, 29] for much more details.

Definition 3.1. *A function $\phi\in C^0(\mathbb{R})$ is* almost periodic *if, for all $\epsilon>0$ there exists a set of real numbers $T(\epsilon)\subseteq\mathbb{R}$ altogether with a length $l(\epsilon)>0$ such that for any interval of length $l(\epsilon)$, there is at least one point $\tau\in T(\epsilon)$ contained in that interval such that*

$$|\phi(x+\tau)-\phi(x)|<\epsilon$$

for each $x\in\mathbb{R}$. We will call numbers in $T(\epsilon)$ translation numbers *and* a length for $T(\epsilon)$ *will be a number $l(\epsilon)$.*

The above collection of all almost periodic functions will be denoted by $AP(\mathbb{R})$ which is a Banach space endowed with the usual $\sup-$norm. It is possible to associate to an almost periodic function φ its unique Fourier series:

$$\varphi\sim\sum_{n\in\mathbb{N}}a(\lambda_n)e^{i\lambda_n x}.$$

The exponents λ_n are called the frequencies of ϕ. Another well-known result in this area is that, for every almost periodic function there exists the *mean value*

$$M(\phi):=\lim_{T\to\infty}\frac{1}{T}\int_0^T\phi(x)dx,$$

this is a bounded linear function $M:AP(\mathbb{R})\to\mathbb{R}$ with the following properties:

1. $\phi\geq 0$ implies $M[\phi]\geq 0$.
2. The Parseval equality holds:

$$M[|\phi|^2]=\sum_{n\in\mathbb{N}}|a(\lambda_n)|^2.$$

Now we review some aspects about cooperative systems, for a brief introduction to cooperative systems see [30]. For two points $x,y\in\mathbb{R}^2$ denote the partial order $u\leq v$ if $u_i\leq v_i$ for each i, also denote $u<v$ if $u\leq v$ and $u\neq v$. Let $f,g:\mathbb{R}\times D\subseteq\mathbb{R}^3\to\mathbb{R}$ be a couple of differentiable and almost periodic functions on the first variable. We consider the system:

$$\begin{aligned}x'(t)&=f(t,x(t),y(t)),\\ y'(t)&=g(t,x(t),y(t)),\end{aligned}\tag{3.1}$$

where we suppose that $f(t,x,y),g(t,x,y)$ are both *uniformly almost periodic with respect to* $(x,y)\in C$ for every compact $C\subseteq D$, i.e., the set of translation numbers, $\tau(\epsilon)$, is independent of $(x,y)\in C$.

More specifically, if f has generalized Fourier expansions,

$$f(t,x,y)\sim\overline{f}(x,y)+\sum_{n=0}^{\infty}a(f,\lambda_n)\cos(\lambda_n t)+b(f,\lambda_n)\sin(\lambda_n t),$$

f is uniformly almost periodic, whenever the coefficients $a(\cdot,\lambda_n),b(\cdot,\lambda_n)$ do not depend on (x,y), see [29] Chapter VI.

Definition 3.2. *System* (3.1) *is said to be of* cooperative type *if for all* $t \in \mathbb{R}$, $x \in (a(t), A(t))$, $y \in (b(t), B(t))$ *we have*

$$f_y(t, x, y) \geq 0, \; g_x(t, x, y) \geq 0.$$

We will say that $(a(t), b(t))$ *are a* subsolution pair *if*

$$\begin{aligned} a'(t) \leq & \quad f(t, a(t), b(t)), \\ b'(t) \leq & \quad g(t, a(t), b(t)). \end{aligned} \tag{3.2}$$

For every $t \in \mathbb{R}$. *A* super-solution $(A(t), B(t))$ *is defined similarly with the reversing inequalities. We will say that a sub-solution* $(a(t), b(t))$ *and a supersolution* $(A(t), B(t))$ *are* ordered *if* $a(t) \leq A(t)$ *and* $b(t) \leq B(t)$ *for all* $t \in \mathbb{R}$.

An important feature for cooperative systems, as given in (3.1), related to almost-periodic orbits, was established in [22] in Theorem 2. Explicitly, the following result holds.

Theorem 3.1. *Consider an ordered pair of a subsolution pair* $(a(t), b(t))$ *and a supersolution pair* $(A(t), B(t))$ *of the system* (3.1) *such that* $a(t) < A(t)$, *and* $b(t) < B(t)$. *Suppose that there is no equilibrium point* (x_0, y_0) *such that* $a(t) \leq x_0 \leq A(t)$ *and* $b(t) \leq y_0 \leq B(t)$. *If the system is cooperative type, then it has an almost periodic solution satisfying* $a(t) < x(t) < A(t)$ *and* $b(t) < x(t) < B(t)$ *for all* $t \in \mathbb{R}$. *Furthermore, if* $(\underline{x}(t), \underline{y}(t)), (\overline{x}(t), \overline{y}(t))$, *denote the minimal and maximal almost periodic solutions having initial data satisfying* $a(0) < x(0) < A(0)$ *and* $b(0) < y(0) < B(0)$. *Then any solution of* (3.1), *converges to the product of strips* $(\underline{x}(t), \overline{x}(t)) \times (\underline{y}(t), \overline{y}(t))$.

In the case that there is an equilibrium point we could have a stable equilibrium, instead of a genuine almost-periodic orbit.

4. Results.

In this section, we prove some results of the existence, uniqueness and stability of an almost periodic solution of model (2.1).

Theorem 4.1. *Assume* $b_i(t) \geq 0, a_i(t) \geq 0$ *are continuous almost-periodic functions (at least one of them not constant) with* $b_{i*} > 0, a_{i*} > 0$, *for* $i \neq 2$, *and* (2.1) *does not admit equilibrium points with positive coordinates. Suppose the conditions A1)-A3) are valid. Then there is at least one almost periodic solution* (N_1, N_2) *of* (2.1) *whose components are positive, the almost-periodic solution is unique in* $\mathbb{R}^2_{>0}$, *and it attracts all other positive solutions of* (2.1), *when* $t \to \infty$.

Proof: For i). Since $f_2' \geq 0$, the system (2.1) is cooperative. We construct sub- and super-solution pairs. For a super-solution pair; we consider

$$(A(t), B(t)) = (N, N), \; N > 0.$$

These functions satisfy the inequalities

$$\begin{aligned} A'(t) = 0 \quad & \geq \quad (b_1^* + f_2(N) - a_{1*} N f_1(N)) \, N \\ & \geq \quad (b_1(t) + f_2(N) - a_1(t) N f_1(N)) \, N, \\ B'(t) = 0 \quad & \geq \quad (b_2^* - a_{3*} N) \, f_3(N) N \\ & \geq \quad (b_2(t) - a_3(t) N) \, f_3(N) N. \end{aligned}$$

Using A3) and taking N big enough, the right side is positive, then they constitute a super-solution pair.

For constructing a sub-solution pair, we consider

$$(a(t), b(t)) = (\epsilon, \; \epsilon) \, , \; \epsilon > 0.$$

These functions satisfy the inequalities in (3.2). Then,

$$\begin{aligned} a'(t) = 0 \quad & \leq \quad [b_{1*} + f_2(\epsilon) - a_1^* f_1(\epsilon) \epsilon] \, \epsilon \leq [b_1(t) + f_2(\epsilon) - a_1(t) \epsilon f_1(\epsilon)] \, \epsilon, \\ b'(t) = 0 \quad & \leq \quad [b_{2*} - a_3^* \epsilon] \, f_3(\epsilon) \epsilon \leq [b_2(t) - a_3(t) \epsilon] \, f_3(\epsilon) \epsilon. \end{aligned}$$

Since f_i are continuous and $f_2(0) = 0$ the right side is positive for $\epsilon > 0$ small enough. Thus we have a sub-solution pair. Therefore, by Theorem 3.1 there exists at least one almost periodic solution for system (2.1). This finishes the proof of the existence of an almost periodic solution.

ii) For uniqueness, we consider a maximal pair $(\hat{x}, \hat{y})$ and minimal pair $(\check{x}, \check{y})$ of almost periodic solutions. We just need to prove that $\hat{x}(t) = \check{x}(t)$ and $\hat{y}(t) = \check{y}(t)$, to do this, we consider the following result

Claim 1. *Let $\hat{\phi}, \check{\phi}$ be almost periodic functions such that*

$$\hat{\phi}(t) \geq \check{\phi}(t) \geq 0, \qquad \mathcal{M}\left[\hat{\phi}\right] = \mathcal{M}\left[\check{\phi}\right].$$

Then $\hat{\phi}(t) = \check{\phi}(t)$ for every $t \in \mathbb{R}$.

We continue with the proof. Note that the mean $M[(\ln \hat{x})'] = M[(\ln \check{x})'] = 0$, then

$$M[b_1 + a_2 f_2(\hat{y})] = M\left[a_1 \hat{x} f_1(\hat{x})\right]. \tag{4.1}$$

On the other hand, by part i) we have that $\epsilon \leq \check{y} \leq \hat{y} \leq N$ for $\epsilon > 0$ and $N > 0$ suitable. Also by A3) the function $\frac{y'}{y f_3(y)}$ has a continuous primitive $F(y)$ for $y > 0$, which is bounded in $[\epsilon, N]$, then $M[F(\hat{y})'] = M[F(\check{y})'] = 0$, thus

$$M[b_2] = M[a_3 \hat{y}]. \tag{4.2}$$

We have similar relationships to $\check{x}$ and $\check{y}$. From the equation (4.1) we get

$$0 \leq a_{3*} M[(\hat{y} - \check{y})] \leq M[a_3(t)\,(\hat{y} - \check{y})] = 0,$$

by the Claim we obtain $\hat{y} = \check{y}$, substituting in (4.1) and since f_1 is increasing, we have

$$0 \leq a_{1*} f(\epsilon) M[(\hat{x} - \check{x})] \leq M[a_1(t) f_1(\check{x})\,(\hat{x} - \check{x})] \leq M[a_1(t)\,(\hat{x} f_1(\hat{x}) - \check{x} f_1(\check{x}))] = 0.$$

Then $\hat{x} = \check{x}$ by the above Claim.

Now we prove the Claim 1.

Proof: Since $\hat{\phi}(t), \check{\phi}(t)$ are almost periodic, then they are bounded. Hence,

$$\begin{aligned} 0 \leq \mathcal{M}\left[\hat{\phi}^2 - \check{\phi}^2\right] &\leq \mathcal{M}\left[(\hat{\phi} - \check{\phi})(\hat{\phi} + \check{\phi})\right] \\ &\leq (2 \sup\{\hat{\phi}(t)\}) \cdot \mathcal{M}\left[\hat{\phi} - \check{\phi}\right] = 0. \end{aligned}$$

Therefore, $\mathcal{M}\left[\hat{\phi}^2\right] = \mathcal{M}\left[\check{\phi}^2\right]$. Thus,

$$0 \leq \mathcal{M}\left[(\hat{\phi} - \check{\phi})^2\right] \leq 2\left(\mathcal{M}\left[\hat{\phi}^2\right] - \mathcal{M}\left[\hat{\phi}\check{\phi}\right]\right) \leq 2\left(\mathcal{M}\left[\hat{\phi}^2\right] - \mathcal{M}\left[\check{\phi}^2\right]\right) = 0.$$

If we apply Parseval's Theorem on the sum of the squares of the Fourier coefficients of $\hat{\phi} - \check{\phi}$ we get $\hat{\phi} \equiv \check{\phi}$. ☐

Finally, with our construction, we can make the super-solutions arbitrarily large, and the sub-solutions arbitrarily small. We have a single attractor almost-periodic orbit in the set $\mathbb{R}^2_{>0}$ for the above, then the almost-periodic orbit is attractor at $\mathbb{R}^2$, this concludes ii) and therefore ends the proof of the theorem. ☐

5. Examples. In this section, we show numerical simulations of the solutions of model (2.1) using different functions $f_1(x)$, $f_2(y)$ and $f_3(y)$ to show the results proved. To do this, we use the following almost periodic functions.

$$\begin{aligned} b_1(t) &= b_{11}\left(b_{12} + b_{13}\sin(b_{14}t) + b_{15}\sin(b_{16}t)\right), \\ b_2(t) &= b_{21}\left(b_{22} + b_{23}\sin(b_{24}t) + b_{25}\sin(b_{26}t)\right), \\ a_1(t) &= a_{11}\left(a_{12} + a_{13}\sin(a_{14}t) + a_{15}\sin(a_{16}t)\right), \\ a_2(t) &= a_{21}\left(a_{22} + a_{23}\sin(a_{24}t) + a_{25}\sin(a_{26}t)\right), \\ a_3(t) &= a_{31}\left(a_{32} + a_{33}\sin(a_{34}t) + a_{35}\sin(a_{36}t)\right). \end{aligned} \tag{5.1}$$

With the values of the parameters given by $b_{11} = 80, b_{12} = 1, b_{13} = 0.4, b_{14} = \sqrt{2}, b_{15} = 0.2, b_{16} = \sqrt{3}, b_{21} = 0.1, b_{22} = 1.25, b_{23} = 0.4, b_{24} = \sqrt{2}, b_{25} = 0.5, b_{26} = \sqrt{3}, a_{11} = 0.6, a_{12} = 0.8, a_{13} = 0.25, a_{14} = \sqrt{2}, a_{15} = 0.01, a_{16} = \sqrt{3}, a_{21} = 70, a_{22} = 0.9, a_{23} = 0.35, a_{24} = \sqrt{2}, a_{25} = 0.1, a_{26} = \sqrt{3}, a_{31} = 0.3, a_{32} = 1.25, a_{33} = 0.1, a_{34} = \sqrt{2}, a_{35} =$

$0.1, a_{36} = \sqrt{3}$. The initial conditions used in the numerical simulations are given by $(0.2, 0.1)$, $(1000, 300)$ and $(100, 130)$.

Example 1. Here, we use the functions $f_1(x) = \frac{x}{10+x}$, $f_2(y) = \frac{y}{10+y}$ and $f_3(y) = \frac{y}{20+y}$. Notice that, the functions f_i, for $i = 1, 2, 3$, are non negative functions. Particularly, $f_2(0) = 0$. Since $\lim\limits_{z\to\infty} f_i(z) = 1$, $\lim_{z\to+\infty}(a_{12}(t)f_2(z) - a_{11*}zf_1(z)) = -\infty$. Therefore, these functions satisfy the conditions A1)-A3).

The model with the functions chosen is given by

$$\begin{aligned} x' &= x\,(b_1(t) - a_1(t)x)\,\tfrac{x}{10+x} + a_2(t)\tfrac{y}{10+y}x, \\ y' &= y\,(b_2(t) - a_3(t)y)\,\tfrac{y}{20+y}. \end{aligned} \tag{5.2}$$

Figure 5.1 shows that all solutions of the model go to a unique almost periodic solution.

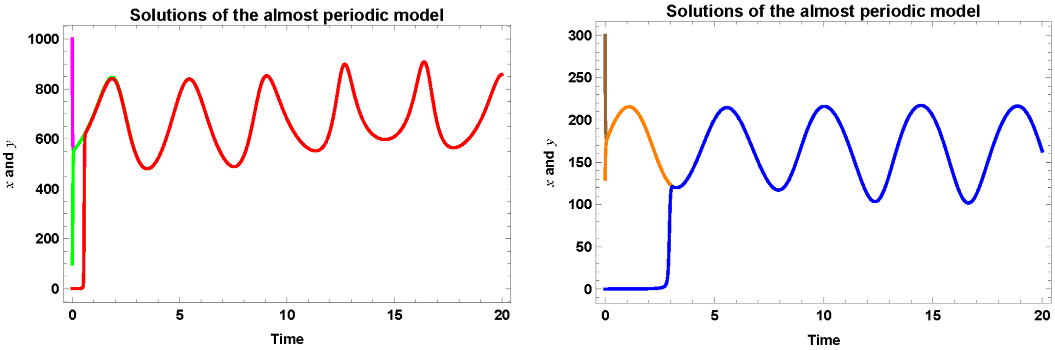

Figure 5.1: Cases (a) and (b) show the population dynamics of the species x and y, respectively. Notice that all solutions tend to the unique almost periodic attractor, for different initial conditions.

Example 2. Now, we use the functions $f_1(x) = \frac{x}{10+x}$, $f_2(y) = \frac{y^2}{10+y^2}$ and $f_3(y) = \frac{y^3}{2+y^3}$. These functions satisfy the conditions A1)-A3). Model (2.1) with these particular functions is shown as follows.

$$\begin{aligned} x' &= x\,(b_1(t) - a_1(t)x)\,\tfrac{x}{10+x} + a_2(t)\tfrac{y^2}{10+y^2}x, \\ y' &= y\,(b_2(t) - a_3(t)y)\,\tfrac{y^3}{2+y^3}. \end{aligned} \tag{5.3}$$

Figure 5.2 shows the behavior of the solutions of the model in this scenario.

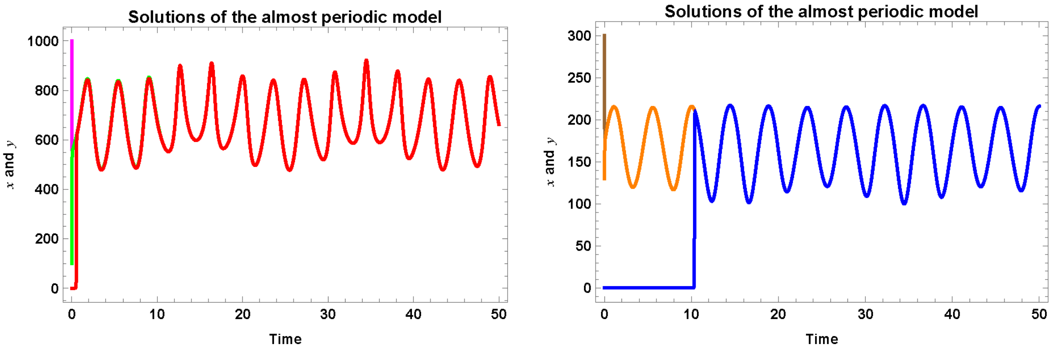

Figure 5.2: For different initial conditions, solutions of the model go to a unique global attractor.

Example 3. In this scenario, we use the functions $f_1(x) = \frac{x^3}{10+x^3}$ and $f_2(y)$ and $f_3(y)$ are the same used in the Example 1. We can prove that these functions satisfy the conditions A1)-A3). In this case, the model is

$$\begin{aligned} x' &= x\left(b_1(t) - a_1(t)x\right)\frac{x^3}{10+x^3} + a_2(t)\frac{y}{10+y}x, \\ y' &= y\left(b_2(t) - a_3(t)y\right)\frac{y}{20+y}. \end{aligned} \tag{5.4}$$

Figure 5.3 shows the numerical simulations of the solutions of the model in this case.

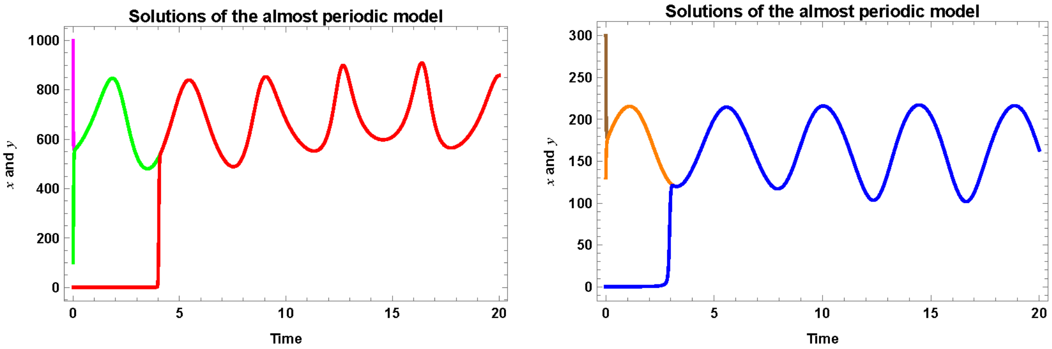

Figure 5.3: The model presents a unique almost periodic attractor. Notice that solutions with different initial conditions go to a unique almost periodic solution.

Model (2.1) also admits a unique periodic solution which is globally asymptotically stable. To compare the periodic case with the almost periodic case we give the following example.

Example 4. Model (2.1) also admits a unique periodic solutions which is asymptotically stable. To compare the solutions in the periodic case against the solutions in the almost periodic case, we use the functions given in (5.1) with the following values of the parameters $b_{14} = b_{24} = a_{14} = a_{24} = a_{34} = 2$, $b_{16} = b_{26} = a_{16} = a_{26} = a_{36} = 3$ to simulate the periodic scenario. All other parameters of the model are the same used in the Example 1. Therefore, the periodic model is given by

$$\begin{aligned} x' &= x\left(b_1(t) - a_1(t)x\right)\frac{x}{10+x} + a_2(t)\frac{y}{10+y}x, \\ y' &= y\left(b_2(t) - a_3(t)y\right)\frac{y}{20+y}. \end{aligned} \tag{5.5}$$

To model the almost periodic case we add an almost periodic term to the seasonal birth rates $b_1(t)$ and $b_2(t)$ given in (5.1). In this case, the almost periodic model is given by

$$\begin{aligned} x' &= x\left(b_1(t) + 5\sin(\sqrt{5}t) - a_1(t)x\right)\frac{x}{10+x} + a_2(t)\frac{y}{10+y}x, \\ y' &= y\left(b_2(t) + 6\sin(\sqrt{3}t) - a_3(t)y\right)\frac{y}{20+y}. \end{aligned} \tag{5.6}$$

Figure 5.4 shows that the periodic rates and the almost periodic rates used in the example remain very close in different intervals of time but they are separated in other intervals of time. In such a situation, the periodic and the almost periodic solutions are close in different intervals and they are separated in other intervals of time.

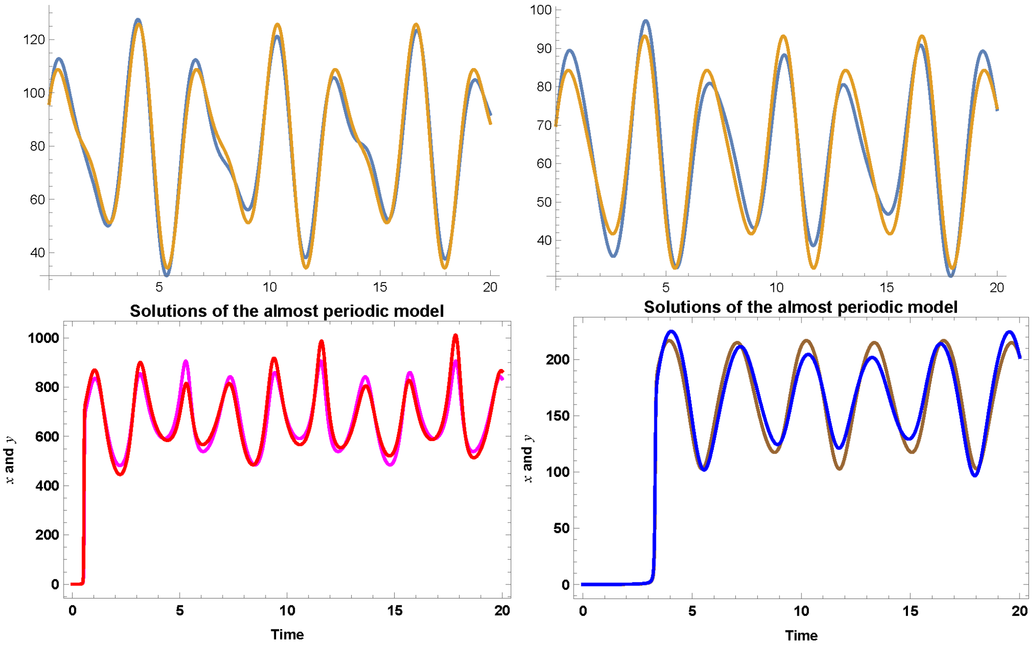

Figure 5.4: Case (a) shows $b_1(t)$ and $b_1(t) + 5\sin(\sqrt{5}t)$ in yellow and blue color respectively. Case (b) shows $b_2(t)$ and $b_2(t) + 6\sin(\sqrt{3}t)$ in yellow and blue color respectively. In each case, both functions are close. Case (c) shows the solution x for the almost periodic case (red color) and for the periodic case (magenta color) respectively. Case (d) shows an analogous scenario for the solution y in the almost and the periodic case in blue and brown color respectively.

6. Discussion. Climate-mediated shifts have affected the synchrony between species interacting in different ecological relationships since the life cycle of some species is altered when some biotic or abiotic factors change. Modeling the population dynamics of species when they are affected by changes in environmental factors such as climate and temperature could be useful when designing public strategies to maintain biodiversity in nature.

Seasonal models are useful tools to describe the population dynamics when environmental factors affect either ecological or demographic rates of species interacting. Commonly, the seasonal rates used in some mathematical models are described through periodic functions. However, although these rates can have a periodic behavior in normal conditions, the climate change is leading to loss of periodicity in these rates. In such a situation, modelling seasonal dynamics through almost periodic models is a better alternative.

In this work, we propose a general almost periodic model to describe commensalism between two species when the per capita growth rate of each one is affected by a weak Allee effect which is described by a wide family of functions f_1 and f_3. We also prove that for a wide family of almost periodic functions, which describe ecological and demographic rates, model (2.1) has a unique almost periodic global attractor when the conditions over the parameters given in theorem 4.1 are satisfied. Numerical simulations of the solutions of the model show that the solutions converge to a unique almost periodic solution, when different functions f_i, for $i = 1, 2, 3$, are used; see Figures 5.1, 5.2 and 5.3. Observe that, although species x is affected by an weak Allee effect, the benefits received from the interaction with the species y lead the species x to increase its number of individuals and the population quickly reaches the almost periodic solution. Notice that, the species y reaches this almost periodic solution after a time.

Figure 5.4 shows a scenario of particular interest. Figure 5.4 cases (a) and (b) show the growth rates $b_1(t)$ and $b_2(t)$ in the periodic and $b_1(t) + 5\sin(\sqrt{5}t)$ and $b_2(t) + 6\sin(\sqrt{3}t)$ in the almost periodic case. By comparison, in each case, we observe that both functions are very close in different intervals of time and they are separated in other intervals of time. Figure 5.4 cases (c) and (d) show that these characteristics are inherited such that the solutions of the model in the periodic and the almost periodic scenario have a similar behavior. To wit, solutions are very close in different intervals of time and there are intervals of time in which they are separated. In this case, the almost periodic solution gives bigger population values than the population values given by the periodic solution. In contrast, in some intervals of time, the almost periodic solution gives smaller population values than the population values given by the periodic solution. Therefore, if periodic rates are used in the modeling process when almost periodic rates must be considered, the size of the population can be underestimated or overestimated, which can lead to design wrong sustainable strategies.

In summary, we proved that the model proposed has a unique almost periodic global attractor for a wide family of functions describing weak Allee effects. Numerical simulations of the solutions of the model show that climate-mediated shifts in environmental drivers must be modeled by almost periodic functions when it is possible. Since, if periodic rates are used, when they are really almost periodic, underestimation or overestimation of the sizes of the populations may occur, which can lead to the design of wrong sustainable strategies. Therefore, from an ecological and biodiversity perspective, it is relevant the use of mathematical models to forecasting population changes in interacting species when climate-mediated shifts occur.

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