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Quadratic Fractionally Integrated Moving Average Processes with Long-Range Dependence

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Abstract

Stochastic processes with the long-range dependency (LRD) property are fundamental to modeling data that exhibit slow power decay of the covariance function. Such behavior often appears in the analysis of financial data, telecommunications, and various natural phenomena. Thus, introducing new stochastic models and statistical methods that take the LRD into account is of great interest. Based on previous work, we introduce a new stochastic process called quadratic fractionally integrated moving average, *that arises from the Quadratic Ornstein-Uhlenbeck Type (QOUT) process, proposed in the literature. We consider Lévy noises of finite second-order moments and use a construction based on a moving average stochastic process whose kernel is that of a QOUT process. Then, using Riemann-Liouville fractional integrals, we propose a fractionally integrated moving average process, for which we highlight some results, including the LRD. We also propose the estimation of the parameters for this process for the case of fractional Brownian noise, showing its efficiency through a Monte Carlo simulation. By an application based on Brazil's stock market prices, we illustrate how this process can be used in practice with the São Paulo's Stock Exchange Index data set, also known as the BOVESPA Index.*

Keywords . Fractionally integrated moving average processes, long-range dependence, quadratic Ornstein-Uhlenbeck type processes.

1. Introduction. It is well known that a Lévy process has an infinitely divisible distribution and that it can be written by a *Lévy-Khintchine representation* where the process distribution is generated by a unique triplet (γ, σ^2, ν), the so-called *Lévy process generating triplet* (see [1]). If $\{L(t)\}_{t\in\mathbb{R}}$ denotes a Lévy process in $\mathbb{R}$, for $\theta \in \mathbb{R}, \gamma \in \mathbb{R}$ and $\sigma \geqslant 0$, with zero mean and finite second-order moment, the characteristic exponent of $L(1)$ can be written as

$$\zeta(\theta) = \frac{1}{2}\theta^2\sigma^2 + \int_{\mathbb{R}} \left(1 - e^{i\theta x} + i\theta x \mathbf{1}_{(|x|<1)}(x)\right) \nu(dx) - i\gamma\theta, \tag{1.1}$$

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so that the characteristic function of the Lévy process is given by $\varphi(\theta) = e^{-\zeta(\theta)}$.

Lévy processes can appear in many forms, having been widely used in theoretical and applied works as it is one natural extension of the Brownian motion process. The work by [2] presents different methods for constructing fractional Lévy processes considering stochastic integrals driven by Lévy and fractional Lévy processes. The author introduces some conditions to be satisfied by these classes of processes to have LRD property. In [3], by using stochastic integration, the authors show the relevancy that Lévy processes can have in the study of the generalized Langevin equation and discuss its potential applications in anomalous diffusions observed in some physical systems. Magdziarz and Weron, [4] and Feltes and Lopes [5] consider Lévy processes with infinite second-order moments, hence capturing high variability behavior. These are considered a very rich class of processes. Feltes and Lopes [5] extended the construction of [2] using Riemann-Liouville fractional integrals to the case of non-Gaussian stable processes and proved the LRD using a generalized autocovariance function. Related to the biology field, [6] examined the effects of variations in the toggle switch bistable model by imposing a non-Gaussian Lévy stable motion. The newly specified model allowed the characterization of the appearance of large jumps and fundamentally changed the switch mechanism of the proposed system.

Another example where Lévy processes play an important role is in the modeling of financial data and fund indices, which in turn helps investors and managers measure the performance, risk, and gross returns of asset prices in the worldwide markets (see, e.g., [7] and [8]. Continuous-time models have successfully been used in finance and options pricing (see the review in [9]). In particular, the stationary continuous-time moving average has been considered a key model in various contexts, especially when driven by Lévy processes (see, e.g., [10], [2], [11] and [12]). In [11], the authors provide necessary and sufficient conditions on the kernel for the continuous-time moving average to be a semimartingale in the natural filtration of the Lévy process.

Based on [2], we consider a kernel induced from the QOUT process (see [13]), with LRD property. For the process based on this QOUT kernel, we study their second-order properties and their sample path generation as well. This specific kernel arises from the quadratic fractionally integrated moving average process considered in [13]. Besides giving a statistical analysis that includes the parameter estimation of the process, we also present a Monte Carlo simulation for the main parameter d, considering different scenarios, allowing for a better understanding of the LRD property. As an application, we consider the study of São Paulo's BOVESPA Index data set, showing its potential application in business and the financial market.

This work is organized as follows: Section 2 presents the main definitions and preliminary results for the moving average process with a quadratic kernel function showing that it is well-defined. Based on the Riemann-Liouville fractional integral, this section also presents the first version of the main process, denoted by $Y_{\lambda,d}(\cdot)$, which is in the fractionally integrated moving average process (FIMA) class. We show the properties related to its respective autocovariance and spectral density functions whose complex-valued expressions guarantee the LRD property for this class. Section 3 introduces a second version of the FIMA process based on the Riemann-Liouville derivatives. We give its autocovariance function and show that its rate of decay guarantees the LRD property. For the estimation of the parameters, presented in Section 4, we consider the particular case when the process is driven by the fractional Brownian motion noise. An approximated version of the process is given to generate its sample paths and to study the estimators $\hat{d}$ and $\hat{\lambda}$ for the main parameters of the process. Section 5 features a Monte Carlo simulation study for the estimation of parameters, while in Section 6 we present an application based on a data set of the Brazilian financial market (the BOVESPA Index), from March 13, 2008, to April 18, 2008. Section 7 concludes this work.

2. Definitions and Preliminary Results. A continuous-time *stationary moving average stochastic process* $\{Y(t)\}_{t\in\mathbb{R}}$ is defined in the literature as the process with the integral representation

$$Y(t) = \int_{\mathbb{R}} g(t-u)dL(u),\ t \in \mathbb{R}, \tag{2.1}$$

where the *kernel* $g : \mathbb{R} \to \mathbb{R}$ is a measurable function and the noise process $\{L(t)\}_{t\in\mathbb{R}}$ is a Lévy process in $\mathbb{R}$. The authors in [14] show that $\{Y(t)\}_{t\in\mathbb{R}}$ is *well-defined* when the kernel $g(\cdot)$ and the generating triplet (γ, σ^2, ν) of the Lévy process $\{L(t)\}_{t\in\mathbb{R}}$, given in (1.1), satisfy the condition

$$\int_{\mathbb{R}}\int_{\mathbb{R}_0}\left(|g(t-u)x|^2 \wedge |g(t-u)x|\right)\nu(dx)du < \infty, \tag{2.2}$$

where $\mathbb{R}_0 = \mathbb{R} - \{0\}$.

By a meticulous choice of the kernel in (2.1), the following definition specifies the stationary moving average process that shall be used throughout the present work.

Definition 2.1. *We denote $\{Y_\lambda(t)\}_{t\in\mathbb{R}}$ as the stationary moving average process with kernel given by*

$$g_\lambda(t) = e^{-\lambda t^2}\mathbf{1}_{(0,\infty)}(t), \ \text{for} \ \lambda > 0, \tag{2.3}$$

so that

$$Y_\lambda(t) = \int_{\mathbb{R}} e^{-\lambda(t-u)^2}\mathbf{1}_{(0,\infty)}(t-u)dL(u) = \int_{-\infty}^{t} e^{-\lambda(t-u)^2}dL(u), \quad t \in \mathbb{R}, \tag{2.4}$$

and the noise $\{L(t)\}_{t\in\mathbb{R}}$ is a Lévy process in $\mathbb{R}$ satisfying $\mathbb{E}[L(1)] = 0$ and $\mathbb{E}[L(1)^2] < \infty$.

From theorem 6.1 in [2], the process $\{Y_\lambda(t)\}_{t\in\mathbb{R}}$, defined in (2.4) is well-defined, that is, the kernel function $g_\lambda(\cdot)$, given in (2.3), satisfies the condition in (2.2).

The definition below gives the LRD property (see [2], [15], and [16]) for any process $\{Y(t)\}_{t\in\mathbb{R}}$.

Definition 2.2. *The stationary moving average process $\{Y(t)\}_{t\in\mathbb{R}}$ has the* long-range dependence *(LRD) property, with parameter $d \in (0, \frac{1}{2})$, when there exists a constant $c > 0$ such that*

$$\lim_{h\to\infty}\frac{\gamma_Y(h)}{c\,|h|^{2d-1}} = 1, \tag{2.5}$$

where $\gamma_Y(h)$ is the autocovariance function of order h for the process $\{Y(t)\}_{t\in\mathbb{R}}$.

Now, note that the kernel function given in (2.3) satisfies the two conditions: $g_\lambda(t) = 0$, for all $t < 0$ and $|g_\lambda(t)| \le Ce^{-ct}$, for some constants $C > 0$ and $c > 0$. The first condition is trivial. The kernel function $g_\lambda(\cdot)$ satisfies the second condition for $C = e^\lambda$ and $c = \lambda$. We also point out that the process in (2.1), with kernel function $g(\cdot)$, must have short memory property, otherwise one cannot use the results by [2] to construct a fractionally integrated stochastic process.

To construct a fractionally integrated stochastic process, we shall use the kernel function of a stationary moving average process. Together with the Riemann-Liouville integral on the right (see [17]), given in Definition 2.3, for the kernel function in (2.3), we obtain the desired process. For this, we define first the Riemann-Liouville fractional integrals and their corresponding derivatives.

Definition 2.3. *Let $0 < r < 1$, $1 \le p < \frac{1}{r}$, and $f \in L^p(\mathbb{R})$. The Riemann-Liouville fractional integrals to the left and the right of the function f, denoted by $(I^r_\pm f)(\cdot)$, are respectively given by*

$$\left(I^r_- f\right)(x) = \frac{1}{\Gamma(r)}\int_x^\infty f(t)(t-x)^{r-1}dt \ \ \text{and} \ \ \left(I^r_+ f\right)(x) = \frac{1}{\Gamma(r)}\int_{-\infty}^x f(t)(x-t)^{r-1}dt. \tag{2.6}$$

Now we define the *fractionally integrated kernel*, denoted by $g_d(\cdot)$, of a kernel function $g(\cdot)$ through its Riemann-Liouville fractional integral to the right. From the $g_d(\cdot)$ function, we shall obtain the main interesting process.

Definition 2.4. *The fractionally integrated kernel of a function $g(\cdot)$ is given by*

$$g_d(t) := \left(I^d_+ g\right)(t) = \int_0^t g(t-s)\frac{s^{d-1}}{\Gamma(d)}ds, \ \text{for} \ t \in \mathbb{R}, \tag{2.7}$$

where $0 < d < \frac{1}{2}$.

The next theorem presents the fractionally integrated kernel of the function $g_\lambda(\cdot)$ given in (2.3).

Theorem 2.1. *Let $g_\lambda(\cdot)$ be the kernel function given by (2.3). Then the fractionally integrated kernel of $g_\lambda(\cdot)$, defined by (2.7), is given by*

$$g_{\lambda,d}(t) = \frac{t^d}{\Gamma(d+1)} \times {}_2F_2\left(\frac{1}{2}, 1; \frac{d+1}{2}, \frac{d+2}{2}; -\lambda t^2\right), \tag{2.8}$$

where $d \in (0, \frac{1}{2})$ and ${}_2F_2(x, y; z, w; v)$ is the generalized hypergeometric function with parameters x, y, z, w and v.

Proof: By definition, considering $g(t) = g_\lambda(t)$ in expression (2.7), we have that

$$g_{\lambda,d}(t) = \int_0^t e^{-\lambda(t-s)^2} \mathbf{1}_{(0,\infty)}(t-s) \frac{s^{d-1}}{\Gamma(d)} ds, \quad t \in \mathbb{R}. \tag{2.9}$$

Firstly, note that, for $s \in (0, t)$, we naturally have $(t - s) \in (0, \infty)$. Hence, by changing variables $x = t - s$, the expression given in (2.9) is reduced to

$$g_{\lambda,d}(t) = \int_0^t e^{-\lambda(t-s)^2} \frac{s^{d-1}}{\Gamma(d)} ds = \frac{1}{\Gamma(d)} \int_0^t e^{-\lambda x^2} (t-x)^{d-1} dx.$$

Secondly, we consider the formula 3.478(4) of [18], with $u = t, \nu = 1 > 0, \mu = d > 0, n = 2$ and $\beta = -\lambda$, obtaining that

$$g_{\lambda,d}(t) = \frac{t^d}{\Gamma(d)} \mathcal{B}(d, 1) \; {}_2F_2\left(\frac{1}{2}, 1; \frac{d+1}{2}, \frac{d+2}{2}; -\lambda t^2\right), \tag{2.10}$$

where $\mathcal{B}(a, b)$ is the Beta function $\mathcal{B}(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ with parameters a and b. Rearranging the terms in (2.10), for all $t \in \mathbb{R}$, we get (2.8), concluding the proof. □

Remark 2.1. *The generalized hypergeometric function with parameters x, y, z, w and v, is given by*

$${}_2F_2(x, y; z, w; v) = \sum_{n=0}^{\infty} \frac{(x)_n (y)_n}{(z)_n (w)_n} \times \frac{v^n}{n!}, \tag{2.11}$$

with $(\cdot)_n$ the Pochhammer symbol given by $(a)_n \equiv \frac{\Gamma(a+n)}{\Gamma(a)} = a \cdot (a+1) \cdot \ldots \cdot (a+n-1)$, for $n > 0$ and $(a)_0 \equiv 1$. The generalized hypergeometric function given in expression (2.11) is well-defined since, by definition, we have that

$$\begin{aligned} {}_2F_2\left(\frac{1}{2}, 1; \frac{d+1}{2}, \frac{d+2}{2}; -\lambda x^2\right) &= \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2}+n)\Gamma(1+n)}{\Gamma(\frac{1}{2})\Gamma(1)} . \frac{\Gamma(\frac{d+1}{2})\Gamma(\frac{d+2}{2})}{\Gamma(\frac{d+1}{2}+n)\Gamma(\frac{d+2}{2}+n)} . \frac{(-\lambda x^2)^n}{n!} \\ &= \frac{\Gamma(\frac{d+1}{2})\Gamma(\frac{d+2}{2})}{\sqrt{\pi}} \sum_{n=0}^{\infty} a_n, \end{aligned} \tag{2.12}$$

for $d \in (0, \frac{1}{2})$ and $x \in \mathbb{R}$ fixed, where

$$a_n = \frac{\Gamma(\frac{1}{2}+n)\Gamma(1+n)}{\Gamma(\frac{d+1}{2}+n)\Gamma(\frac{d+2}{2}+n)} \frac{(-\lambda x^2)^n}{n!}. \tag{2.13}$$

From (2.13), together with the Cauchy ratio test, where

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{2}+n\right)(1+n)}{(\frac{d+1}{2}+n)(\frac{d+2}{2}+n)} \frac{(-\lambda x^2)}{(n+1)} = 0,$$

the hypergeometric series given in (2.12) is absolutely convergent, for all $x \in \mathbb{R}$.

Proposition 2.1. *Let $g_{\lambda,d}(\cdot)$ be the fractionally integrated kernel given in expression (2.8). Then $g_{\lambda,d}(\cdot) \in L^2(\mathbb{R})$.*

Proof: Considering $p = 2$ in [19], it suffices to prove that

$$\int_{\mathbb{R}} |h(u) g_{\lambda,d}(u)|\, du \leq K ||h||_2,$$

for every function $h(\cdot) \in L^2(\mathbb{R})$ and K a positive constant. From expression (2.7), we have

$$\begin{aligned}\int_{\mathbb{R}} |h(u) g_{\lambda,d}(u)|\, du &= \int_0^{\infty}\int_0^{u} |h(u)|\, g(u-s) \frac{s^{d-1}}{\Gamma(d)}\, ds\, du \\ &= \frac{1}{\Gamma(d)} \int_0^{\infty}\int_0^{\infty} |h(u)|\, g(u-s) s^{d-1}\, ds\, du, \qquad (2.14)\end{aligned}$$

since $g(t) = 0$, for $t \leq 0$. Rewriting expression (2.14), we have that

$$\int_{\mathbb{R}} |h(u) g_{\lambda,d}(u)|\, du = \frac{1}{\Gamma(d)} \left(I_1 + I_2\right), \qquad (2.15)$$

with

$$I_1 = \int_0^{\infty}\int_0^{1} |h(u)|\, g(u-s) s^{d-1}\, ds\, du \quad \text{and} \quad I_2 = \int_0^{\infty}\int_1^{\infty} |h(u)|\, g(u-s) s^{d-1}\, ds\, du. \qquad (2.16)$$

By Fubini's theorem and Hölder's inequality, we obtain

$$I_1 = \int_0^{1} s^{d-1} \int_0^{\infty} |h(u)|\, g(u-s)\, du\, ds \leq \int_0^{1} s^{d-1} ||h||_2 ||g||_2\, ds = \frac{1}{d} ||h||_2 ||g||_2, \qquad (2.17)$$

since $g_\lambda(\cdot) \in L^2(\mathbb{R})$. By Fubini's theorem again in the integral I_2, given in (2.16), and changing variables $t = u - s$, it follows by Hölder's inequality that

$$I_2 = \int_0^{\infty} g(t) \int_1^{\infty} \left| h(t+s) s^{d-1} \right| ds\, dt \leq \int_0^{\infty} |g(t)|\, ||h||_2 \left(\int_1^{\infty} s^{2(d-1)} ds \right)^{\frac{1}{2}} dt. \qquad (2.18)$$

From expression (2.18), since $g_\lambda(\cdot) \in L^1(\mathbb{R})$, we have that

$$I_2 \leq \int_0^{\infty} |g(t)|\, ||h||_2 \frac{1}{\sqrt{1-2d}}\, dt \leq \frac{1}{\sqrt{1-2d}} ||g||_1 ||h||_2. \qquad (2.19)$$

Applying the results of (2.17) and (2.19) in (2.15), we conclude that

$$|h(u) g_{\lambda,d}(u)|\, du \leq \frac{1}{\Gamma(d)} \left(\frac{1}{d} ||g||_2 + \frac{1}{\sqrt{1-2d}} ||g||_1 \right) ||h||_2.$$

Therefore, $g_{\lambda,d}(\cdot) \in L^2(\mathbb{R})$. □

Next, we shall define the process with LRD by using the above results. This process originates from the class of *fractionally integrated moving average processes* (FIMA) (see [20]).

Definition 2.5. *Let $d \in (0, \frac{1}{2})$ and $\lambda > 0$. The FIMA process $\{Y_{\lambda,d}(t)\}_{t\in\mathbb{R}}$* for the fractionally integrated kernel *$g_{\lambda,d}(\cdot)$, given in expression (2.8), is defined by*

$$Y_{\lambda,d}(t) = \int_{-\infty}^{t} \frac{(t-u)^d}{\Gamma(d+1)}\ {}_2F_2\left(\frac{1}{2}, 1; \frac{d+1}{2}, \frac{d+2}{2}; -\lambda(t-u)^2\right) dL(u), \quad t \in \mathbb{R}, \qquad (2.20)$$

where ${}_2F_2(x, y; z, w; v)$ is defined in (2.11) and $\{L(t)\}_{t\in\mathbb{R}}$ is a Lévy process with $\mathbb{E}[L(1)] = 0$ and $\mathbb{E}[L(1)^2] < \infty$.

Remark 2.2. *(a) The process given in (2.20) is well-defined, since* $g_{\lambda,d}(\cdot) \in L^2(\mathbb{R})$ *(see [14]).*
(b) We also point out that the process defined in expression (2.20) is stationary and, for all $t \in \mathbb{R}$*, the distribution of* $Y_{\lambda,d}(t)$ *is infinitely divisible (see [21]).*

The expressions for the autocovariance and spectral density functions of the process in (2.20), given in the lemma below, is a particular case of theorem 6.4 of [2].

Lemma 2.1. *Let* $\{Y_{\lambda,d}(t)\}_{t\in\mathbb{R}}$ *be the process given in (2.20). Then, the following statements are true.*

i) The autocovariance function of the process is given by

$$\gamma_{Y_{\lambda,d}}(h) = \frac{\mathbb{E}\left[L(1)^2\right]}{(\Gamma(d+1))^2} \int_0^\infty u^d (u+h)^d \psi_d(u)\, \psi_d(u+h)\, du, \tag{2.21}$$

for $h \geq 0$*, where* $\psi_d(x) \equiv {}_2F_2\left(\frac{1}{2}, 1; \frac{d+1}{2}, \frac{d+2}{2}; -\lambda x^2\right)$ *is the generalized hypergeometric function defined in (2.11).*

ii) The spectral density function $f_{Y_{\lambda,d}}(\cdot)$ *is given by*

$$f_{Y_{\lambda,d}}(\omega) = \frac{\mathbb{E}\left[L(1)^2\right]}{2\pi(\Gamma(d+1))^2} \left|F_{\lambda,d}(\omega)\right|^2, \tag{2.22}$$

for $\omega \in \mathbb{R}$*, where* $F_{\lambda,d}(\omega) = \int_0^\infty x^d e^{ix\omega} \psi_d(x)\, dx$*, is the Fourier transform of the function* $\psi_d(\cdot)$ *given in item i.*

Proof: To prove item *i.*, we only need to recall the definition of spectral autocovariance function when $\mathbb{E}[L(1)] = 0$. Hence, by changing variables $u = t - s$, we obtain

$$\gamma_{Y_{\lambda,d}}(h) = \frac{\mathbb{E}[L(1)^2]}{(\Gamma(d+1))^2} \int_0^\infty u^d (u+h)^d\, \psi_d(u)\, \psi_d(u+h)\, du.$$

To prove item *ii.*, we use the fact that the spectral density function of a stationary process is the inverse Fourier transform of the autocovariance function (see Herglotz's theorem in [22]) and we use item *i.* to find

$$f_{Y_{\lambda,d}}(\omega) = \frac{\mathbb{E}\left[L(1)^2\right]}{2\pi(\Gamma(d+1))^2} \int_0^\infty e^{-ih\omega} \int_0^\infty u^d (u+h)^d \psi_d(u) \psi_d(u+h)\, du\, dh, \tag{2.23}$$

for $\omega \in \mathbb{R}$ and $h \geq 0$. By applying Fubini's theorem and changing variables $v = u + h$ to the integral with respect to variable h, the expression (2.23) can be rewritten as

$$\begin{aligned} f_{Y_{\lambda,d}}(\omega) &= \frac{\mathbb{E}\left[L(1)^2\right]}{2\pi(\Gamma(d+1))^2} \int_0^\infty \int_0^\infty e^{-i(v-u)\omega} v^d u^d \psi_d(v) \psi_d(u) dv du \\ &= \frac{\mathbb{E}\left[L(1)^2\right]}{2\pi(\Gamma(d+1))^2} F_{\lambda,d}(-\omega)\, F_{\lambda,d}(\omega) = \frac{\mathbb{E}\left[L(1)^2\right]}{2\pi(\Gamma(d+1))^2} \left|F_{\lambda,d}(\omega)\right|^2, \end{aligned} \tag{2.24}$$

for $\omega \in \mathbb{R}$, proving item *ii.*. □

In the next section, the main properties of the process defined in (2.20) are studied.

3. The Long-Range Dependence Property.

By using a short memory integrated fractional kernel in the previous section theory, we can obtain the process defined in (2.20). We shall prove that this process has LRD. However, before this, it is necessary to prove some results. The following theorem shows an integral representation for the process given in expression (2.20) in terms of a fractional Lévy process integrator.

Theorem 3.1. *Consider the process* $\{Y_{\lambda,d}(t)\}_{t\in\mathbb{R}}$ *given in (2.20), following Definition 2.5. Then, it can be represented as*

$$Y_{\lambda,d}(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)^2} dM_d(s), \quad t \in \mathbb{R}, \tag{3.1}$$

where the integrator

$$M_d(s) = \frac{1}{\Gamma(d+1)} \int_{\mathbb{R}} \left[(t-s)_+^d - (-s)_+^d \right] L(dt) \tag{3.2}$$

is a fractional Lévy process, with $\{L(t)\}_{t\in\mathbb{R}}$ *a Lévy noise process such that* $\mathbb{E}[L(1)] = 0$, $\mathbb{E}[L(1)^2] < \infty$, *and* $d \in (0, \frac{1}{2})$.

Proof: From theorem 6.2 of [2], the process given in (2.20) can be rewritten as

$$Y_{\lambda,d}(t) = \int_{-\infty}^{t} g(t-s) dM_d(s), \quad t \in \mathbb{R}, \tag{3.3}$$

where

$$g(x) = \frac{1}{\Gamma(1-d)} \frac{d}{dx} \int_0^x g_d(s)(x-s)^{-d} ds, \quad x \in \mathbb{R}, \tag{3.4}$$

is the Riemann-Liouville derivative to the right of the fractional kernel $g_d(\cdot)$, since for any class of functions $f \in L^p(\mathbb{R})$, the Riemann-Liouville derivative to the right of f is given by

$$(\mathcal{D}_+^r f)(x) = \frac{1}{\Gamma(1-r)} \frac{d}{dx} \int_{-\infty}^x f(t)(x-t)^{-r} dt.$$

Hence, from the above expression and from expression (3.4), the Riemann-Liouville derivative to the right of the function $g_d(\cdot)$, of order d, is given by

$$\left(\mathcal{D}_+^d g_d\right)(x) = e^{-\lambda x^2} \mathbf{1}_{(0,\infty)}(x).$$

Applying this result to the expression in (3.3), we obtain that

$$Y_{\lambda,d}(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)^2} \mathbf{1}_{(0,\infty)}(t-s)\, dM_d(s) = \int_{-\infty}^{t} e^{-\lambda(t-s)^2}\, dM_d(s),$$

for all $t \in \mathbb{R}$, since $t - s > 0$, for all $s < t$. □

Remark 3.1. *the authors in [23] give conditions to completely characterize when the fractional Lévy process, defined in* (3.2)*, is a semimartingale or of finite variation.*

The next Proposition 3.1 and Theorem 3.2 present results on the first and second-order moments for the process given in (3.1).

Proposition 3.1. *Consider the stochastic process* $\{Y_{\lambda,d}(t)\}_{t\in\mathbb{R}}$ *given in* (3.1)*. Then its expected value is given by*

$$\mathbb{E}\left[Y_{\lambda,d}(t)\right] = 0, \;\; \textit{for all} \;\; t \in \mathbb{R}.$$

Proof: For a fixed $t \in \mathbb{R}$, we have

$$\begin{aligned} \mathbb{E}\left[Y_{\lambda,d}(t)\right] &= \mathbb{E}\left[\int_{-\infty}^{t} e^{-\lambda(t-s)^2} dM_d(s)\right] \\ &= \mathbb{E}\left[\int_{-\infty}^{t} \frac{(t-u)^d}{\Gamma(d+1)}\; {}_2F_2\left(\frac{1}{2}, 1; \frac{d+1}{2}, \frac{d+2}{2}; -\lambda(t-u)^2\right) dL(u)\right] = 0, \end{aligned} \tag{3.5}$$

where the second equality above is equivalent to the expression (2.20). One can write the expression (3.5) in the form $\mathbb{E}\left[\sum_{k=0}^{n-1} a_k \left(L(s_{k+1}) - L(s_k)\right)\right]$, for appropriated choices of $a_0, \ldots, a_{n-1} \in \mathbb{R}$, $n \in \mathbb{N}$, and $-\infty < s_0 < \cdots < s_n = t$. Besides, since the Lévy process $\{L(t)\}_{t\in\mathbb{R}}$ has stationary increments and $\mathbb{E}[L(1)] = 0$, we conclude that its mean is equal to 0. □

From proposition 5.6 of [2], we obtain the autocovariance function for the process given in (3.1).

Theorem 3.2. *Consider the stochastic process $\{Y_{\lambda,d}(t)\}_{t\in\mathbb{R}}$ given in (3.1). Then its autocovariance function is given by*

$$\gamma_{Y_{\lambda,d}}(k) \equiv Cov\,[Y_{\lambda,d}(t+k), Y_{\lambda,d}(t)] \\ = \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \times \mathbb{E}[L(1)^2] \int_0^\infty \int_0^\infty e^{-\lambda(x^2+y^2)} |k-x+y|^{2d-1} dx\, dy, \quad (3.6)$$

for $k > 0$, with $d \in (0, \frac{1}{2})$.

Proof: From Proposition 3.1 and Theorem 3.1, it follows that

$$\gamma_{Y_{\lambda,d}}(k) \equiv \text{Cov}\,[Y_{\lambda,d}(t+k), Y_{\lambda,d}(t)] = \mathbb{E}[Y_{\lambda,d}(t+k)\, Y_{\lambda,d}(t)] \\ = \mathbb{E}\left[\int_{-\infty}^{t+k} e^{-\lambda(t+k-s)^2} dM_d(s) \int_{-\infty}^{t} e^{-\lambda(t-u)^2} dM_d(u)\right]. \quad (3.7)$$

Now, from proposition 5.6 of [2], expression (3.7) can be rewritten as

$$\mathbb{E}[Y_{\lambda,d}(t+k)\, Y_{\lambda,d}(t)] = C \int_{-\infty}^{t} \int_{-\infty}^{t+k} e^{-\lambda(t+k-s)^2}\, e^{-\lambda(t-u)^2}\, |s-u|^{2d-1}\; ds\, du, \quad (3.8)$$

where $C = \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \times \mathbb{E}[L(1)^2]$. The result in (3.6) is attained by changing variables $x = t+k-s$ and $y = t-u$ in expression (3.8). □

We recall that the goal of this section is to obtain the LRD property (see Definition 2.2) for the process given in (2.20). This goal is attained by Theorem 3.3 below, by using theorem 6.3 of [2].

Theorem 3.3. *Consider the stochastic process $\{Y_{\lambda,d}(t)\}_{t\in\mathbb{R}}$ given in expression (3.1). Then, its autocovariance function $\gamma_{Y_{\lambda,d}}(\cdot)$, given in (3.6), has a decay rate given by*

$$\gamma_{Y_{\lambda,d}}(k) \sim \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)}\, \mathbb{E}[L(1)^2] \left(\frac{\pi}{4\lambda}\right) |k|^{2d-1}, \quad (3.9)$$

when $k \to \infty$. Therefore, the process has LRD with constant

$$c_\gamma = \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \times \mathbb{E}\left[L(1)^2\right] \left(\frac{\pi}{4\lambda}\right). \quad (3.10)$$

Proof: Initially, we need to prove that,

$$\int_0^\infty \int_0^\infty e^{-\lambda(x^2+y^2)}\, |k-x+y|^{2d-1}\, dx\, dy \sim \left(\int_0^\infty e^{-\lambda x^2} dx\right)^2 |k|^{2d-1}, \quad (3.11)$$

when $k \to \infty$. First, notice that it is possible to write

$$\int_0^\infty \int_0^\infty \frac{e^{-\lambda(x^2+y^2)}\, |k-x+y|^{2d-1}}{|k|^{2d-1}}\, dx\, dy = \int_0^\infty \int_0^\infty e^{-\lambda(x^2+y^2)} \left|1 - \frac{x}{k} + \frac{y}{k}\right|^{2d-1} dx\, dy \\ = \int_0^\infty \int_0^\infty e^{-\lambda(x^2+y^2)} dx\, dy + I, \quad (3.12)$$

where the integral I is given by

$$I = \int_0^\infty \int_0^\infty e^{-\lambda(x^2+y^2)} \left(\left|1 - \frac{x}{k} + \frac{y}{k}\right|^{2d-1} - 1\right) dx\, dy. \quad (3.13)$$

From Lemma 2.1(item *ii.*), we have, for a given $\epsilon > 0$, that

$$|I| \leq \int_0^\infty \int_0^\infty e^{-\lambda(x^2+y^2)} \left|\left|1 - \frac{x}{k} + \frac{y}{k}\right|^{2d-1} - 1\right| \mathbb{1}_{\{|y-x|\leq\epsilon k\}}\, dx\, dy \\ + \int_0^\infty \int_0^\infty e^{\lambda(1-x)}\, e^{\lambda(1-y)} \left|\left|1 - \frac{x}{k} + \frac{y}{k}\right|^{2d-1} - 1\right| \mathbb{1}_{\{|y-x|>\epsilon k\}}\, dx\, dy. \quad (3.14)$$

For $|y-x| \leq \epsilon k$, we have

$$\left|\left|1-\frac{x}{k}+\frac{y}{k}\right|^{2d-1}-1\right| \leq \max\{(1-\epsilon)^{2d-1}-1, 1-(1+\epsilon)^{2d-1}\} \leq (1-\epsilon)^{2d-1}-1. \quad (3.15)$$

Applying the inequality (3.15) into (3.14), we obtain

$$|I| \leq \left((1-\epsilon)^{2d-1}-1\right) \int_0^\infty \int_0^\infty e^{-\lambda(x^2+y^2)}\, dx\, dy + R(k), \quad (3.16)$$

where

$$R(k) = \int_0^\infty \int_0^\infty e^{\lambda(2-(x+y))} \left|\left|1-\frac{(x-y)}{k}\right|^{2d-1}-1\right| \mathbf{1}_{\{|y-x|>\epsilon k\}}\, dx\, dy. \quad (3.17)$$

By making the change of variables $u = x+y$ and $v = x-y$, we get $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$, where the Jacobian of this transformation is equal to $-\frac{1}{2}$. Since $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$, we have $u \geq |v|$. Then, (3.17) can be written as

$$\begin{aligned} R(k) &= \frac{1}{2}\int_{\mathbb{R}}\int_{|v|}^\infty e^{\lambda(2-u)}du \left|\left|1-\frac{v}{k}\right|^{2d-1}-1\right| \mathbf{1}_{\{|v|>\epsilon k\}}\, dv \\ &= \frac{1}{2}\left(\int_{-\infty}^{-\epsilon k}\int_{-v}^\infty e^{\lambda(2-u)}du \left|\left|1-\frac{v}{k}\right|^{2d-1}-1\right| dv + \int_{\epsilon k}^\infty \int_v^\infty e^{\lambda(2-u)}du \left|\left|1-\frac{v}{k}\right|^{2d-1}-1\right| dv\right) \\ &\leq \frac{1}{2}\left(\int_{-\infty}^{-\epsilon k} \frac{e^{\lambda(2+v)}}{\lambda}\left(\left|1-\frac{v}{k}\right|^{2d-1}+1\right) dv + \int_{\epsilon k}^\infty \frac{e^{\lambda(2-v)}}{\lambda}\left(\left|1-\frac{v}{k}\right|^{2d-1}+1\right) dv\right). \end{aligned} \quad (3.18)$$

Now, notice that for $v < -\epsilon k$, we get $|1-vk^{-1}|^{2d-1} \leq 1$, since $2d-1 < 0$. Using this fact into (3.18), we obtain that

$$R(k) \leq \int_{-\infty}^{-\epsilon k} \frac{e^{\lambda(2+v)}}{\lambda}\, dv + \frac{1}{2}\int_{\epsilon k}^\infty \frac{e^{\lambda(2-v)}}{\lambda}\left(\left|1-\frac{v}{k}\right|^{2d-1}+1\right) dv. \quad (3.19)$$

By computing the first integral in expression (3.19), we have

$$R(k) \leq \frac{e^{\lambda(2-\epsilon k)}}{\lambda^2} + \frac{I_1(k)}{2}, \quad (3.20)$$

where

$$I_1(k) = \int_{\epsilon k}^\infty \frac{e^{\lambda(2-v)}}{\lambda}\left(\left|1-\frac{v}{k}\right|^{2d-1}+1\right) dv = I_2(k) + I_3(k), \quad (3.21)$$

with $I_2(k) = \int_{\epsilon k}^{2k} \frac{e^{\lambda(2-v)}}{\lambda}\left(\left|1-\frac{v}{k}\right|^{2d-1}+1\right) dv$ and $I_3(k) = \int_{2k}^\infty \frac{e^{\lambda(2-v)}}{\lambda}\left(\left|1-\frac{v}{k}\right|^{2d-1}+1\right) dv$.

Since $|1-vk^{-1}|^{2d-1} \leq 1$, for $v > 2k$, we get

$$I_3(k) \leq 2\int_{2k}^\infty \frac{e^{\lambda(2-v)}}{\lambda}\, dv = \left(-\frac{2e^{\lambda(2-v)}}{\lambda^2}\right)\Bigg|_{2k}^\infty = \frac{2e^{2\lambda(1-k)}}{\lambda^2}.$$

Therefore, from (3.21), we have

$$I_1(k) \leq I_2(k) + \frac{2e^{2\lambda(1-k)}}{\lambda^2}. \quad (3.22)$$

Finally, to determine an upper bound for $I_2(\cdot)$, note that the integral of this expression can be rewritten as

$$\begin{aligned}
I_2(k) &= \int_{\epsilon k}^{2k} \frac{e^{\lambda(2-v)}}{\lambda}\left|1-\frac{v}{k}\right|^{2d-1} dv + \int_{\epsilon k}^{2k} \frac{e^{\lambda(2-v)}}{\lambda} dv \\
&= \int_{\epsilon k}^{k} \frac{e^{\lambda(2-v)}}{\lambda}\left(1-\frac{v}{k}\right)^{2d-1} dv + \int_{k}^{2k} \frac{e^{\lambda(2-v)}}{\lambda}\left(\frac{v}{k}-1\right)^{2d-1} dv + \int_{\epsilon k}^{2k} \frac{e^{\lambda(2-v)}}{\lambda} dv \\
&\leq \int_{\epsilon k}^{k} \frac{e^{\lambda(2-\epsilon k)}}{\lambda}\left(1-\frac{v}{k}\right)^{2d-1} dv + \int_{k}^{2k} \frac{e^{\lambda(2-k)}}{\lambda}\left(\frac{v}{k}-1\right)^{2d-1} dv + \int_{\epsilon k}^{2k} \frac{e^{\lambda(2-v)}}{\lambda} dv \\
&= \frac{e^{\lambda(2-\epsilon k)}}{\lambda}\left(\frac{k}{2d}\right)(1-\epsilon)^{2d} + \frac{e^{\lambda(2-k}}{\lambda}\left(\frac{k}{2d}\right) - \frac{e^{\lambda(2-2k)}}{\lambda^2} + \frac{e^{\lambda(2-\epsilon k)}}{\lambda^2}. \qquad (3.23)
\end{aligned}$$

Substituting the result of (3.23) into (3.22), we conclude from (3.20) that

$$\begin{aligned}
R(k) &\leq \frac{e^{\lambda(2-\epsilon k)}}{\lambda^2} + \frac{1}{2}\left(\frac{2e^{2\lambda(1-k)}}{\lambda^2} + \frac{e^{\lambda(2-\epsilon k)}}{\lambda}\left(\frac{k}{2d}\right)(1-\epsilon)^{2d}\right) \\
&\quad + \frac{1}{2}\left(\frac{e^{\lambda(2-k)}}{\lambda}\left(\frac{k}{2d}\right) - \frac{e^{2\lambda(1-k)}}{\lambda^2} + \frac{e^{\lambda(2-\epsilon k)}}{\lambda^2}\right) \\
&\leq \frac{e^{\lambda(2-\epsilon k)}}{2\lambda^2}\left(3 + \left(\frac{\lambda k}{2d}\right)(1-\epsilon)^{2d}\right) + \frac{e^{2\lambda(1-k)}}{2\lambda^2} + \frac{ke^{\lambda(2-k)}}{4\lambda d}. \qquad (3.24)
\end{aligned}$$

We may choose $\epsilon > 0$ such that

$$((1-\epsilon)^{2d} - 1)\int_0^\infty \int_0^\infty e^{-\lambda(x^2+y^2)} dx\, dy \leq \frac{K}{2},$$

for some $K > 0$. For $\epsilon > 0$ fixed, given in expression (3.24), $R(k) \to 0$, when $k \to \infty$. This implies that we can consider $|I|$ as small as one wants in (3.12). Therefore, the approximation given in (3.11) is true. Hence, from the autocovariance function given in expression (3.6) and from the result in (3.11), we obtain

$$\begin{aligned}
\gamma_{Y_d}(k) &\sim \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \times \mathbb{E}\left[L(1)^2\right]\left(\int_0^\infty e^{-\lambda u^2} du\right)^2 |k|^{2d-1} \\
&= \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \times \mathbb{E}\left[L(1)^2\right]\left(\frac{\pi}{4\lambda}\right)|k|^{2d-1}.
\end{aligned}$$

From the expression (3.9) and considering the constant $c_\gamma > 0$, given in expression (3.10), Definition 2.2 guarantees the desired result. ☐

Remark 3.2. *From Theorem 3.3, we can consider the properties of the Gamma function, when $\mathbb{E}[L(1)^2] = 1$, and rewrite the expression (3.9) as*

$$\gamma_{Y_{\lambda,d}}(k) = \frac{\pi}{8\lambda\Gamma(2d)\cos(\pi d)}|k|^{2d-1}. \qquad (3.25)$$

4. Specific Case and Parameter Estimation. After the introduction of a new stochastic process, we present in this section the specific case of the process $\{Y_{\lambda,d}(t)\}$, given in (3.1), when the noise process $M_d(\cdot)$ is a fractional Brownian motion. Next, we present the study of process parameter estimation. For the sake of completeness, below we define the fractional Brownian motion process.

Definition 4.1. *The* fractional Brownian motion *with Hurst parameter $H \in (0,1)$ is a Gaussian process $B_H = \{B^H(t)\}_{t\in\mathbb{R}}$ with the following properties:*

i. $B^H(0) = 0$;

ii. $\mathbb{E}\left[B^H(t)\right] = 0$, *for all $t \in \mathbb{R}$;*

iii. $\mathbb{E}\left[B^H(t).B^H(s)\right] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$, *for* $s, t \in \mathbb{R}$.

Remark 4.1. *(a) By Definition 4.1, we have* $B^H(t) \sim \mathcal{N}(0, t^{2H})$.
(b) The process $\{B^H(t)\}_{t\in\mathbb{R}}$ *is a fractional Lévy process.*
(c) The Hurst parameter H *is directly related to the parameter* d*, that is,* $H = d + \frac{1}{2}$.

For the fractional Brownian motion, it is important to note that this process has a long-range dependence property whenever $\frac{1}{2} < H < 1$ (see [15]), that is, for $0 < d < \frac{1}{2}$. Therefore, the study performed here will consider this case.

We shall consider the process $\{Y_{\lambda,d}(t)\}$ given in (3.1) where the noise process $M_d(\cdot) := B^H(\cdot)$. Hence, the interesting process in this section is given by

$$Y_{\lambda,d}(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)^2} dB^H(s), \quad t \in \mathbb{R}, \tag{4.1}$$

where $\lambda > 0$, $0 < d < \frac{1}{2}$ and $H = d + \frac{1}{2}$. The main parameters are λ and d. An approximation for the process in (4.1) is given by

$$Y^m_{\lambda,d}(t) = \sum_{j=-m^2}^{mt} e^{-\lambda(t-\frac{j}{m})^2}\left[B^H\left(\frac{j+1}{m}\right) - B^H\left(\frac{j}{m}\right)\right]. \tag{4.2}$$

Theorem 4.1 below ensures that this approximation is convergent.

Theorem 4.1. *Let* $\phi_m(\cdot) : \mathbb{R} \to \mathbb{R}$ *be a sequence of real functions, for* $m \in \mathbb{N}$*, and let* $g_\lambda(\cdot) : \mathbb{R} \to \mathbb{R}$ *be a real function, both defined by*

$$\phi_m(s) = \sum_{j=-m^2}^{mt} e^{-\lambda(t-\frac{j}{m})^2} \mathbf{1}_{(\frac{j}{m}, \frac{j+1}{m})}(s) \tag{4.3}$$

and

$$g_\lambda(s) = e^{-\lambda(t-s)^2}\mathbf{1}_{(-\infty,t)}(s), \tag{4.4}$$

such that, for fixed t *and* $m \in \mathbb{N}$*,*

$$Y^m_{\lambda,d}(t) = \int_{\mathbb{R}} \phi_m(s)\, dB^H(s) \qquad \text{and} \qquad Y_{\lambda,d}(t) = \int_{\mathbb{R}} g_\lambda(s)\, dB^H(s).$$

Then, $||\phi_m - g_\lambda||_H \to 0$*, when* $m \to \infty$*, with*

$$||g||_H = \left(\mathbb{E}[B(1)^2] \int_{\mathbb{R}} (I^d_-(g(u)))^2(u)du\right)^{\frac{1}{2}} = ||I^d_- g||_2,$$

where $I^d_-(f)$ *is the Riemann-Liouville fractional integral to the left of a function* f*, defined in (2.6), and* $H = d + \frac{1}{2}$*. Furthermore,* $Y^m_{\lambda,d}$ *converges to* $Y_{\lambda,d}$*, in* $L^2(\Omega, \mathbb{P})$*, as* $m \to \infty$*, since by theorem 5.3 in [2],*

$$||Y^m_{\lambda,d} - Y_{\lambda,d}||_{L^2(\Omega,\mathbb{P})} = ||\phi_m - g_\lambda||_H. \tag{4.5}$$

Proof: We know, from [24], that

$$||\phi_m - g_\lambda||_H \leq C\left[||\phi_m - g_\lambda||_1 + ||\phi_m - g_\lambda||_2\right]. \tag{4.6}$$

We need to prove that

$$||\phi_m - g_\lambda||_1 + ||\phi_m - g_\lambda||_2 \to 0,$$

when $m \to \infty$. First, we note that

$$g_\lambda(s) = \sum_{j=-\infty}^{mt-1} e^{-\lambda(t-s)^2}\mathbf{1}_{(\frac{j}{m}, \frac{j+1}{m}]}(s).$$

Then, for all $s \in (-\infty, t)$ fixed, we observe that

$$|\phi_m(s) - g_\lambda(s)| \leq$$
$$\left|\sum_{j=-\infty}^{-m^2-1} e^{-\lambda(t-s)^2} \mathbf{1}_{(\frac{j}{m},\frac{j+1}{m}]}(s)\right| + \left|\sum_{j=-m^2}^{mt-1} \left[e^{-\lambda(t-\frac{j}{m})^2} - e^{-\lambda(t-s)^2}\right] \mathbf{1}_{(\frac{j}{m},\frac{j+1}{m}]}(s)\right| + \mathbf{1}_{(t,t+\frac{1}{m}]}(s).$$

The first and third terms in the above expression go to zero when $m \to \infty$, while the second term has the following upper bound

$$\left|\sum_{j=-m^2}^{mt-1} \left[e^{-\lambda(t-\frac{j}{m})^2} - e^{-\lambda(t-s)^2}\right] \mathbf{1}_{(\frac{j}{m},\frac{j+1}{m}]}(s)\right| \leq \left|\sum_{j=-m^2}^{mt-1} \left[e^{-\lambda(t-\frac{j}{m})^2} - e^{-\lambda(t-\frac{j+1}{m})^2}\right] \mathbf{1}_{(\frac{j}{m},\frac{j+1}{m}]}(s)\right|, \tag{4.7}$$

whose sum has at most one non-zero term of the form

$$|e^{-\lambda(t-\frac{j}{m})^2} - e^{-\lambda(t-\frac{j}{m}-\frac{1}{m})^2}| \to 0, \quad \text{when } m \to \infty.$$

We also have an upper bound for the function $\phi_m(\cdot)$. In fact, for every fixed $m \in \mathbb{N}$, consider

$$|\phi_m(s)| = \sum_{j=-m^2}^{mt} e^{-\lambda(t-\frac{j}{m})^2} \mathbf{1}_{(\frac{j}{m},\frac{j+1}{m}]}(s),$$

for every fixed $s \in \mathbb{R}$. There exists only one j^* such that $\frac{j^*}{m} < s < \frac{j^*+1}{m}$. Hence,

$$|\phi_m(s)| = e^{-\lambda(t-\frac{j^*}{m})} \leq e^{-\lambda(t-s)^2},$$

since $\frac{j^*}{m} < s \Rightarrow (t - \frac{j^*}{m})^2 > (t-s)^2 \Rightarrow -\lambda(t - \frac{j^*}{m})^2 < -\lambda(t-s)^2$. From item *ii)* of Lemma 2.1, we get $|\phi_m(s)| \leq e^{-\lambda(s-1)}(s+1) =: h(s)$, that is in $L^p(\mathbb{R})$, for all $p \geq 1$. In particular, the upper bound for the function $\phi_m(\cdot)$ is also in L^1 and L^2. We observe that, for all $s \in \mathbb{R}$,

$$|\phi_m(s) - g_\lambda(s)| \leq |\phi_m(s)| + |g_\lambda(s)| \leq 2|h(s)| \Longrightarrow |\phi_m(s) - g_\lambda(s)|^2 \leq 4|h(s)|^2, \tag{4.8}$$

where $h \in L^p$, for all $p \geq 1$. Therefore, from the dominated convergence theorem, we conclude that

$$||\phi_m - g_\lambda||_1 \to 0 \ \text{ and } \ ||\phi_m - g_\lambda||_2 \to 0,$$

when $m \to \infty$. Hence, the convergence occurs for both L^1 and L^2, concluding the proof.
□

Further details on the integration concerning the fractional Brownian motion can be found in [24].

Remark 4.2. *Since the fractional Brownian motion $B^H(\cdot)$ has stationary increments, note that*

$$Y^m_{\lambda,d}(t) \sim \sum_{j=-m^2}^{mt} e^{-\lambda\left(t-\frac{j}{m}\right)^2} \left[B^H\left(\frac{j+1}{m}\right) - B^H\left(\frac{j}{m}\right)\right] =$$
$$= \sum_{j=0}^{m^2+mt} e^{-\lambda\left(t-\left(-m+\frac{j}{m}\right)\right)^2} \left[B^H\left(\frac{j+1}{m}\right) - B^H\left(\frac{j}{m}\right)\right] =$$
$$= \sum_{j=0}^{m^2+mt} e^{-\lambda\left(t-\left(-m+\frac{j}{m}\right)\right)^2} \left(\frac{m^2+mt}{m}\right)^H \left[B^H\left(\frac{j+1}{m}\right) - B^H\left(\frac{j}{m}\right)\right], \tag{4.9}$$

when we adapt the discretization set $\{-m^2, -m^2+1, \cdots, 0, \cdots, mt\}$ to $\{0, 1, 2, \cdots, m^2+mt\}$, and use the self-similarity property of fractional Brownian motion noise in the interval $[0, 1]$. For the data generating process, given in (4.1), we use (4.9).

To achieve the goal of this section, we shall present the estimator for the parameter vector $\boldsymbol{\eta}' = (d, \lambda)$, when $d \in \left(0, \frac{1}{2}\right)$ and $\lambda > 0$. To estimate the parameter d of the process defined in (4.1), we consider the parameter $\lambda > 0$ known and the sample autocovariance function for this process. By applying the logarithm function to the expression (3.25) we obtain

$$\hat{d} = \frac{\ln\left(|\widehat{\gamma_{Y_{\lambda,d}}(k)}|\right)}{2\ln(|k|)} - \frac{\ln(\frac{C}{\lambda})}{2\ln(|k|)} + \frac{1}{2}, \tag{4.10}$$

where $\widehat{\gamma_{Y_{\lambda,d}}(k)}$ is the sample autocovariance function of order k and C is the constant defined by

$$C = \frac{\pi}{8\Gamma(2\hat{d})\cos(\pi\hat{d})}. \tag{4.11}$$

Notice that, for large k, the constant C in (4.11) goes to zero and expression (4.10) can be rewritten as

$$\hat{d} = \frac{1}{2} + \frac{\ln\left(|\widehat{\gamma_{Y_{\lambda,d}}(k)}|\right)}{2\ln(|k|)} = \frac{1}{2}\left(1 + \frac{\ln\left(|\widehat{\gamma_{Y_{\lambda,d}}(k)}|\right)}{\ln(|k|)}\right), \tag{4.12}$$

since the second term in expression (4.10) vanishes for large k. Hence, expression (4.12) shall be used as an estimator for d, when $\lambda > 0$ is known.

To estimate the parameter λ, we consider the absolute value of the expression (3.25) to obtain

$$\hat{\lambda} = \frac{C}{|\widehat{\gamma_{Y_{\lambda,d}}(k)}|} k^{2d-1}, \tag{4.13}$$

where the constant C is given in (4.11), d is fixed in $(0, \frac{1}{2})$, k is a positive integer, and $\widehat{\gamma_{Y_{\lambda,d}}(k)}$ is the sample autocovariance function of order k.

As a characteristic property of the process in this special case, one can observe in the following result that the version of the process given in (2.20) follows a Gaussian distribution. Theorem 4.2 below gives the distribution of the process $\{Y_{\lambda,d}(t)\}_{t\in\mathbb{R}}$ for the particular case when the noise process is the Brownian motion.

Theorem 4.2. *Let $\{Y_{\lambda,d}(t)\}_{t\in\mathbb{R}}$ be the stochastic process given in (2.20), where the noise process is the Brownian motion, that is, $L(\cdot) := B(\cdot)$. Then*

$$Y_{\lambda,d}(t) \overset{d}{=} X \sim \mathcal{N}\left(0, 2\sigma^2_{\lambda,d}\right),$$

where

$$\sigma^2_{\lambda,d} = \frac{1}{2}\int_0^\infty \frac{x^{2d}}{(\Gamma(d+1))^2}\left({}_2F_2\left(\frac{1}{2}, 1; \frac{d+1}{2}, \frac{d+2}{2}; -\lambda x^2\right)\right)^2 dx.$$

Proof: Considering the control measure when $\alpha = 2$, we obtain the stochastic process given by

$$Y_{\lambda,d}(t) = \int_{-\infty}^t \frac{(t-u)^d}{\Gamma(d+1)}\, {}_2F_2\left(\frac{1}{2}, 1; \frac{d+1}{2}, \frac{d+2}{2}; -\lambda(t-u)^2\right) dB(u).$$

From proposition 3.4.1 in [20]), it follows a $\mathcal{N}(0, 2\sigma^2_{\lambda,d})$ distribution with

$$\sigma^2_{\lambda,d} = \frac{1}{2}\int_{-\infty}^t \frac{(t-u)^{2d}}{(\Gamma(d+1))^2}\left({}_2F_2\left(\frac{1}{2}, 1; \frac{d+1}{2}, \frac{d+2}{2}; -\lambda(t-u)^2\right)\right)^2 du. \tag{4.14}$$

By changing variables $x = t - u$ in (4.14), we obtain that

$$\sigma^2_{\lambda,d} = \frac{1}{2}\int_0^\infty \frac{x^{2d}}{(\Gamma(d+1))^2}\left({}_2F_2\left(\frac{1}{2}, 1; \frac{d+1}{2}, \frac{d+2}{2}; -\lambda x^2\right)\right)^2 dx.$$

□

In the next section, we shall present some Monte Carlo simulation results.

5. Monte Carlo Simulations. Here we present some simulation studies for the process given in expression (4.1). First, Figure 5.1 shows the graphics of three trajectories of the process in (4.1), when $\lambda = 0.01$ and $d \in \{0.15, 0.30, 0.45\}$. To generate samples of size $t = 2,000$ for this process we consider the approximation (4.9). After some simulations, we find out that the approximation is convergent for values of m equal to $m = 200$. Next, we present the estimation results for the parameters of the interested process. Table 5.1 below presents the estimated values for the parameter d (given by $\hat{d}$), defined in expression (4.12), together with its mean ($\hat{d}$), bias ($bias$), and mean squared error (mse) values, for different Monte Carlo simulation scenarios. The sample size n is considered in the set $\{4,000; 5,000; 6,000; 8,000\}$, the number of replications is considered for $re \in \{500, 800\}$, and the sample autocovariance function of order k is considered for $k \in \{1,000; 2,000; \cdots ; 8,000\}$. One observes, from Table 5.1, that when $d = 0.15$, we could not find good estimated values for it, in terms of the smallest bias, in absolute value sense: $\hat{d} = 0.3786$, with $bias = 0.2286$, for $n = 8,000$ and $re = 500$. When $d = 0.30$, the best estimated result was $\hat{d} = 0.3016$, with $bias = 0.0016$, for $n = 6,000$ and $re = 500$. However, when $d = 0.45$, the best estimated value was $\hat{d} = 0.4492$, with $bias = -0.0008$, for $n = 6,000$ and $re = 500$, while for $re = 800$, the best estimated result for this value of d was $\hat{d} = 0.4495$, with $bias = -0.0005$, for both $n = 5,000$ or $n = 8,000$, always for large values of k. For this table, the best-estimated values for the parameter d, for each d, n, and re, in terms of the smallest bias, in the absolute value sense, are always in the last line of it, for any scenario.

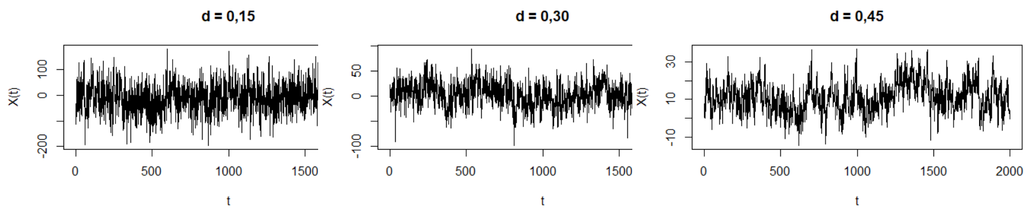

Figure 5.1: Trajectories for the process given in (4.1), when $\lambda = 0.01$, sample size equal to $t = 2,000$, $m = 200$ and $d \in \{a, b, c\}$. First plot, on the left-hand side, for $a = 0.15$, middle plot for $b = 0.30$ and on the right-hand side for $c = 0.45$.

Remark 5.1. *Each process was generated by applying an adequate transformation to a randomly generated fractional Brownian motion. After generating the processes, the d parameter was estimated over each $k \in \{1,000; 2,000; \cdots ; n\}$, granting the possibility to choose the best estimate in terms of its bias and mse values.*

Table 5.2 below presents the results for the estimation of the parameter λ (given by $\hat{\lambda}$), defined in expression (4.13), together with its mean ($\hat{\lambda}$), bias ($bias$) and mean squared error (mse) values, for different Monte Carlo simulation scenarios. The sample size n is considered in the set $\{2,000; 3,000\}$, the number of replications is considered for $re \in \{500, 800\}$ and the sample autocovariance function of order k is considered for $k \in \{1,000; 2,000; \cdots ; n\}$.

Table 5.2 contains values of $\hat{\lambda}$ such that $|bias| < 0.00661$ and $mse < 0.05505$, for all values of the parameter $d \in \{0.15, 0.30, 0.45\}$. From this table one observes the following: when $d = 0.15$, the absolute value of the $bias$ increases for $n = 2,000$ and decreases for $n = 3,000$, for any value of re. The mse decreases whenever n increases. Besides, for this value of d, the lag k for the sample autocovariance function is always large and close to the value of n, for any value of re. We could not find any estimate for λ when $d = 0.15$, $n = 2,000$, and $re = 800$. For the same Table 5.2, when the parameter d is equal to 0.30, we observe the following: for $re = 500$, the best estimate for λ, in terms of the smallest bias and absolute value sense, occurs when $k = 1,260$ and $k = 2,480$, for $n = 2,000$ and $n = 3,000$, respectively. However, for $re = 800$, the best estimate for λ, in terms of the smallest bias and absolute value sense, occurs when $k = 1,040$ and $k = 640$, for $n = 2,000$ and $n = 3,000$, respectively. Finally, when the parameter

$d = 0.15$				$d = 0.30$				$d = 0.45$			
$\hat{d}$	$bias$	mse	k	$\hat{d}$	$bias$	mse	k	$\hat{d}$	$bias$	mse	k
$n = 4,000; re = 500$											
0.7312	0.5812	0.3435	1,000	0.6580	0.3580	0.1349	1,000	0.5634	0.1134	0.0198	1,000
0.6999	0.5499	0.3073	2,000	0.6275	0.3275	0.1128	2,000	0.5599	0.1099	0.0180	2,000
0.6712	0.5212	0.2757	3,000	0.5992	0.2992	0.0955	3,000	0.5238	0.0738	0.0118	3,000
								0.4566	0.0066	0.0058	3,820
0.4057	0.2557	0.0744	4,000	0.3232	0.0232	0.0090	4,000	0.4472	-0.0028	0.0065	3,840
$n = 5,000; re = 500$											
0.7268	0.5768	0.3391	1,000	0.6490	0.3400	0.1285	1,000	0.5641	0.1141	0.0189	1,000
0.6970	0.5470	0.3046	2,000	0.6320	0.3330	0.1163	2,000	0.5543	0.1043	0.0167	2,000
0.6719	0.5219	0.2779	3,000	0.6180	0.3180	0.1057	3,000	0.5420	0.0920	0.0141	3,000
0.6479	0.4979	0.2523	4,000	0.5870	0.2870	0.0872	4,000	0.5157	0.0657	0.0099	4,000
								0.4506	0.0006	0.0053	4,780
0.4022	0.2522	0.0724	5,000	0.3050	0.0050	0.0078	5,000	0.4437	-0.0063	0.0060	4,800
$n = 5,000; re = 800$											
0.7323	0.5823	0.3457	1,000	0.6493	0.3493	0.1285	1,000	0.5582	0.1082	0.0196	1,000
0.6971	0.5471	0.3052	2,000	0.6303	0.3303	0.1150	2,000	0.5514	0.1014	0.0172	2,000
0.6768	0.5268	0.2823	3,000	0.6141	0.3141	0.1037	3,000	0.5440	0.0940	0.0141	3,000
0.6429	0.4929	0.2482	4,000	0.5831	0.2831	0.0852	4,000	0.5158	0.0658	0.0093	4,000
								0.4534	0.0034	0.0058	4,780
0.3964	0.2464	0.0695	5,000	0.3052	0.0052	0.0096	5,000	0.4495	-0.0005	0.0057	4,800
$n = 6,000; re = 500$											
0.7291	0.5791	0.3421	1,000	0.6467	0.3467	0.1281	1,000	0.5641	0.1141	0.0202	1,000
0.6951	0.5451	0.3023	2,000	0.6294	0.3294	0.1144	2,000	0.5486	0.0986	0.0161	2,000
0.6759	0.5259	0.2811	3,000	0.6129	0.3129	0.1037	3,000	0.5528	0.1028	0.0159	3,000
0.6579	0.5079	0.2627	4,000	0.5981	0.2981	0.0938	4,000	0.5358	0.0858	0.0130	4,000
0.6336	0.4836	0.2382	5,000	0.5693	0.2693	0.0777	5,000	0.5061	0.0561	0.0082	5,000
								0.4493	-0.0006	0.0059	5,720
0.3898	0.2398	0.0650	6,000	0.3016	0.0016	0.0076	6,000	0.4492	-0.0008	0.0046	5,760
$n = 8,000; re = 500$											
0.7194	0.5694	0.3303	1,000	0.6475	0.3475	0.1261	1,000	0.5703	0.1203	0.0212	1,000
0.6918	0.5418	0.2980	2,000	0.6191	0.3191	0.1075	2,000	0.5485	0.0985	0.0151	2,000
0.6719	0.5219	0.2776	3,000	0.6156	0.3156	0.1047	3,000	0.5458	0.0958	0.0148	3,000
0.6639	0.5139	0.2684	4,000	0.6073	0.3073	0.0994	4,000	0.5470	0.0970	0.0143	4,000
0.6507	0.5007	0.2550	5,000	0.5881	0.2881	0.0875	5,000	0.5319	0.0819	0.0122	5,000
0.6364	0.4864	0.2100	6,000	0.5787	0.2787	0.0814	6,000	0.5202	0.0702	0.0098	6,000
0.6124	0.4624	0.2175	7,000	0.5583	0.2583	0.0704	7,000	0.4938	0.0438	0.0069	7,000
								0.4505	0.0005	0.0050	7,700
0.3786	0.2286	0.0593	8,000	0.2915	-0.0085	0.0068	8,000	0.4495	-0.0005	0.0005	7,720

Table 5.1: Estimation values for the parameter d, with its mean ($\hat{d}$), bias and mean squared error (mse) values when sample size $n \in \{4,000; 5,000; 6,000; 8,000\}$, replications $re \in \{500, 800\}$, $\lambda = 0.01$, $d \in \{0.15, 0.30, 0.45\}$ and $m = 200$.

$d = 0.45$, we observe a more repetitive pattern, that is, the value $k = 20$ occurs for all possible values of n and re. However, the $bias$ value for $\hat{\lambda}$, in absolute value sense, was always larger than any other $bias$ values $\hat{\lambda}(bias)$ when compared to $d \in \{0.15, 0.30\}$. That is, the best estimate values for the parameter λ occur when $d \in \{0.15, 0.30\}$, in terms of small $bias$ value.

For Table 5.2, the best-estimated values for the parameter λ, for each d, n, and re, in terms of the smallest bias, in the absolute value sense, are always in the last line of the table, for each scenario.

Remark 5.2. *Each process was generated by applying an adequate transformation*

$d = 0.15$				$d = 0.30$				$d = 0.45$			
$\hat{\lambda}$	$bias$	mse	k	$\hat{\lambda}$	$bias$	mse	k	$\hat{\lambda}$	$bias$	mse	k
$n = 2,000; re = 500$											
0.01060	0.00060	0.05505	620	0.00982	-0.00018	0.00064	160	0.09465	0.08465	0.01401	20
0.01661	0.00661	0.02384	2,000	0.01006	0.00006	0.00456	1,260				
$n = 3,000; re = 500$											
0.00721	-0.00279	0.01960	2,987	0.00992	-0.00008	0.00081	260	0.08065	0.07065	0.00609	20
0.00991	-0.00009	0.01735	2,999	0.01000	0.00000	0.00155	2,480				
$n = 2,000; re = 800$											
				0.01173	0.00173	0.01344	40	0.09400	0.08400	0.00092	20
				0.01005	0.00006	0.00192	1,040				
$n = 3,000; re = 800$											
0.00514	-0.00486	0.01875	2,720	0.01022	0.00022	0.00214	380	0.08185	0.07184	0.00063	20
0.01083	0.00083	0.00722	3,000	0.01008	0.00008	0.00255	640				

Table 5.2: Estimated results for the parameter $\lambda = 0.01$ with its mean $(\hat{\lambda})$, bias and mean squared error (mse) values when sample size $n \in \{2,000; 3,000\}$, replications $re \in \{500, 800\}$, $d \in \{0.15, 0.30, 0.45\}$, and $m = 200$.

to a randomly generated fractional Brownian motion. After generating the processes, the λ parameter was estimated over each $k \in \{20, 40, \cdots, n\}$, granting us the possibility to choose the best estimate in terms of its $bias$ and mse values.

6. Application. Various financial studies have been carried out in the areas of probability and statistics seeking a relationship between financial market time series and stochastic processes with a long-range dependence property. One of the first motivations emerged from the work by [25] seeking the behavior of market variables and their relationship with dependence on time. Research carried out in the 1990s gave impulse to this theory, as seen in [26] and [27]. These works showed the existence of the long-range dependence property giving rise to new alternative models for the financial time series.

In this section, we consider a real data set for the study proposed in Section 5, based on the process given by equation (4.1). As an application for the process given in (3.1), we consider the closing values of shares traded on the Brazilian financial market. Specifically, we analyze the BOVESPA Index, São Paulo's Stock Exchange Index data set. The chosen data presents the index's closing values, minute by minute, from 17 hours and 51 minutes on March 13, 2008, to 19 hours and 38 minutes on April 18, 2008, totaling $8,000$ observations. For analyzing this data set, we consider the continuous stochastic process defined by the difference of two consecutive closing values of the BOVESPA Index, observed minute by minute. The data series was centered at zero mean, by subtracting each data set value from its sample mean given by 0.6040. The original sample variance was 2203.012. Figure 6.1 below presents this data set.

By considering the estimator proposed in (4.12), we obtain the value of k that optimizes the estimator $\hat{d}$ with the smallest bias, resulting in $\hat{d} = 0.1162$. From the theoretical point of view, we know that when $d \in \left(0, \frac{1}{2}\right)$, the stochastic process has LRD, and the estimated value obtained agrees with this finding. Table 6.1 presents the estimation results for the parameter d, given by the expression (4.12), with its respective bias value, for different values of k in the set $\{5,000; 6,000; 7,000; 7,200; 7,400; ...; 8,000\}$. We observe that $k = 8,000$ is the value that minimizes the bias for the $\hat{d}$ estimator proposed in (4.12). This estimation result is in accordance with Table 5.1, noting that for values of d closer to zero, the estimate improves the higher the value of k. This also agrees with the estimator expression given in (4.12). For comparison reasons, we also considered Whittle's estimator (see [28]) which is widely used in the study of LRD properties in stochastic processes. We use the notation $\hat{d}_{Whittle}$ to represent this estimator and, for this data set, the value

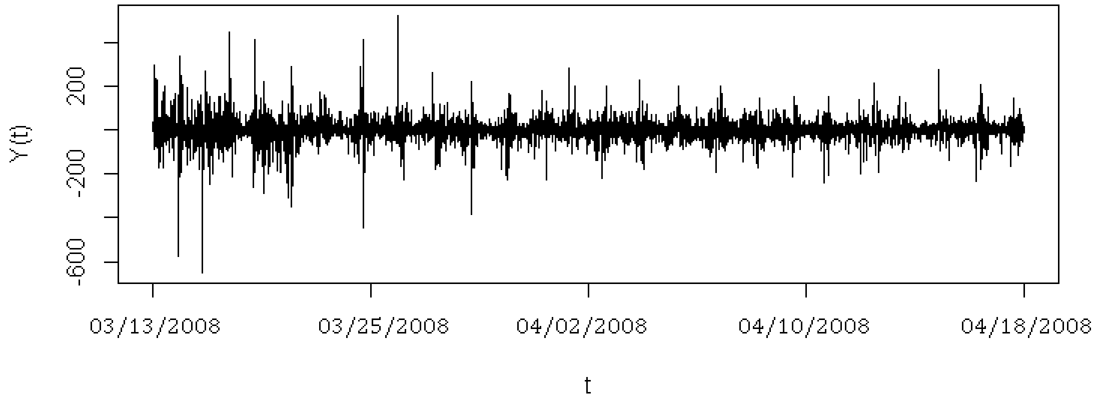

Figure 6.1: The BOVESPA Index closing values, minute by minute, from March 13, 2008, to April 18, 2008.

obtained is $\hat{d}_{Whittle} = 0.1151$.

k	$\hat{d}$	$bias$	k	$\hat{d}$	$bias$
5,000	0.5159	0.4008	7,400	0.5645	0.4494
6,000	0.5928	0.4777	7,600	0.6607	0.5456
7,000	0.6280	0.5129	7,800	0.5552	0.4401
7,200	0.6278	0.5127	**8,000**	**0.1162**	**-0.0011**

Table 6.1: Estimation results for the parameter d, given by expression (4.12), with its mean ($\hat{d}$) and bias ($bias$) values, for different values of $k \in \{5,000; 6,000; 7,000; 7,200; 7,400; ...; 8,000\}$.

From Table 6.1, the value $\hat{d} = 0.1162$ is used to obtain the estimator $\hat{\lambda}$ via the expression (4.13), since $\hat{\lambda}$ depends on $\hat{d}$. We found the value $\hat{\lambda} = 0.1074$.

7. Conclusions. Given the current importance of studies carried out on stochastic processes with the long-range dependence property and its relation to applications in the economic and finance area, we have presented in this work a new process model, given in two versions, which has this property.

Firstly, we have presented a kernel function in (2.3), with real parameter $\lambda > 0$, associated with a stationary moving average process given in (2.4). We used the Riemann-Liouville fractional integral to the right with $d \in \left(0, \frac{1}{2}\right)$ to generate a fractionally integrated kernel, presented in (2.8). With this fractionally integrated kernel, we presented the first version of the main process on this work, denoted by $Y_{\lambda,d}(\cdot)$. It is a variation of the initial process, where the primary kernel was modified to a fractionally integrated kernel. This process belongs to the class of FIMA processes and was defined in Definition 2.5, through the expression (2.20), which has the associated parameters d and λ. By studying this first version of the main process, we showed in Lemma 2.1 its autocovariance and its spectral density functions. However, due to the high complexity arising from the fractionally integrated kernel, it has become an unlikely way to prove the LRD property of the process through its autocovariance function.

Secondly, from [2], it was possible to calculate and rewrite the first version of the process as a second model, given in (3.1), whose representation for the process was given in (2.20) in terms of the fractional Lévy process integrator. In this new version, equivalent to the first one, we showed results for the first and second-order moments and their autocovariance function. Through the autocovariance function, given in (3.6), Theorem 3.3 presents its rate decay and we obtained that this process has LRD. Thus, we reached the main goal of this work by presenting a new stochastic process with the LRD property.

For the study with Monte Carlo simulations and the real data application, we considered a particular case of our process, where the fractional Brownian motion was chosen as the noise of the process given in (3.1), that is, the second version of the process. Through the approximation given in (4.2), we find estimators for the parameters d and λ, presented, respectively, in (4.12) and (4.13). In Section 5, we performed a few Monte Carlo simulations, generating processes with time $t = 2,000$, considering the parameters $d \in \{0.15, 0.30, 0.45\}$ and $\lambda = 0.01$. The trajectories are presented in Figure 5.1. Afterward, we performed estimation tests for the $\hat{d}$ and $\hat{\lambda}$ estimators whose results are in Tables 5.1 and 5.2.

We concluded this work by performing an application of this particular process to a real data set. We considered $8,000$ observations from the BOVESPA Index from March 13, 2008, to April 18, 2008. The main parameters were estimated and the values $\hat{d} = 0.1162$ and $\hat{\lambda} = 0.1074$ have been found. The value of the parameter estimator $\hat{d}$ was compared with Whittle's estimator, showing their similarities.

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Conflicts of interest. "The authors declare no conflict of interest".

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