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# Surfaces with quadratic support function

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## Abstract

*In this paper, we study oriented surfaces $S$ in $\mathbb{R}^3$, called surfaces with quadratic support function (in short QSF-surfaces). We obtain a Weierstrass type representation for the QSF-surfaces which depends on two holomorphic functions. Moreover, classify the QSF-surfaces of rotation. Also, we give some explicit examples of this class of surfaces.*

**Keywords** . Weingarten surfaces, Ribaucour surfaces, support function.

**1. Introduction.** Let $S \subset \mathbb{R}^3$ be a surface oriented by its normal Gauss map $N$. The functions $\Psi, \Lambda : S \to \mathbb{R}^3$ given by $\Psi(p) = \langle p, N(p)\rangle$, $\Lambda(p) = \langle p, p\rangle$, $p \in S$, where $\langle, \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^3$, are called *support function* and *quadratic distance function*, respectively.

Appell in [1], studied a class of oriented surfaces in $\mathbb{R}^3$ associated with area preserving transformations in the sphere. In [2], the authors showed that these surfaces are such that the mean curvature $H$, the Gaussian curvature $K$ and the support function $\Psi$ satisfy $H + \Psi K = 0$. Tzitzéica in [3] estudied oriented hyperbolic surfaces such that there exist a nonzero constant $c \in \mathbb{R}$ for which the following relation is satisfied $K + c^2\Psi^4 = 0$.

In [4], the authors study a special class of oriented surfaces $S \subset \mathbb{R}^3$ that satisfy a relation of the form $2\Psi H + \Lambda K = 0$, this surfaces are called *EDSW-surfaces*. They show that these surfaces are invariant by dilations and inversions. Moreover, they obtain a Weierstrass type representation depending on two holomorphic functions. Given $p \in S$, a sphere with center $p + \frac{H}{K}N(p)$ and radius $\frac{H}{K}$ is called the middle sphere at $p$, the EDSW-surfaces have the geometric property that every middle

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sphere passes through a fixed point. In [5], the authors present a Weierstrass type representation for EDSGW-surfaces with prescribed Gauss map which depends on three holomorphic functions. Also, they classify isothermic EDSGW-surfaces with respect to the third fundamental form parametrized by planar lines of curvature.

In [6], the authors introduce the class of surfaces in Euclidean space motivated by a problem posed by Élie Cartan. This class of surfaces are called *Ribaucour surfaces* and are defined as surfaces where all the medial spheres intercept a fixed sphere along a large circle, these surfaces satisfy a relation of the form

$$2\Psi H + (1+\Lambda)K = 0.$$

The authors obtain holomorphic data for these surfaces and discuss the relation with minimal surfaces.

In [7], the authors study a class of surfaces $S$ in the hyperbolic space that satisfies

$$2ce^{2\mu}(H-1) + (1+ce^{2\mu})K = 0,$$

where $\mu$ is a harmonic function with respect to the quadratic form $\sigma = -KI + 2(H-1)II$, $c$ is a real constant and $I, II$ is the first and second fundamental form of $S$, respectively.

In [8], the authors study the Ribaucour surfaces of harmonic type (in short HR-surfaces), these surfaces satisfy

$$2\Psi H + (ce^{2\mu} + \Lambda)K = 0,$$

where $c$ is a nonzero real constant, $\mu$ a harmonic function with respect to the third fundamental form. These surfaces generalize the Ribaucour surfaces studied in [6].

Corro and Mendez in [9], study the Ribaucour-type surfaces (in short RT-surfaces) what satisfy

$$2\Psi H + (\Lambda + \Psi^2)K = 0.$$

In this paper, we study the QSF-surfaces, these surfaces satisfy

$$2\Psi H + (\Lambda - \Psi^2)K = 0.$$

We obtain a Weierstrass type representation for the QSF-surfaces which depends on two holomorphic functions. Moreover, classify the QSF-surfaces of rotation. Also, we give some explicit examples of this class of surfaces.
We observed that this class of surfaces are different from the surfaces studied in [9].

**2. Preliminaries.** In this section we present the definitions and results that will be used in the work. In this paper the inner produt $\langle , \rangle : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ is defined by

$$\langle f, g \rangle = f_1 g_1 + f_2 g_2, \text{ where } f = f_1 + i f_2, \; g = g_1 + i g_2,$$

are holomorphic functions.
In the computation we use the following properties: if $f, g : \mathbb{C} \to \mathbb{C}$ are holomorphic functions of $z = u_1 + iu_2$, then

$$\langle f, g \rangle_{,1} = \langle f', g \rangle + \langle f, g' \rangle, \quad \langle f, g \rangle_{,2} = \langle if', g \rangle + \langle f, ig' \rangle,$$

$$\langle f, g \rangle = \langle 1, \bar{f} g \rangle, \quad \langle 1, f \rangle^2 = \frac{1}{2}\left[ |f|^2 + \langle 1, f^2 \rangle \right] \tag{2.1}$$

Here $\langle f, g \rangle_{,i}$ denotes the derivative of $\langle f, g \rangle$ with respect to $u_i$, $i = 1, 2$.

The next result was obtained in [4].

**Theorem 2.1.** *Let $S$ be a surface with Gaussian curvature $K \neq 0$ and $N$ its normal Gauss map locally parametrized by*

$$N(u) = \frac{1}{1+|g|^2}(2g, 1-|g|^2) \tag{2.2}$$

*where* $\Pi = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : u_3 = 0\}$, $g : U \to \Pi$ *is a holomorphic function with* $|g'| \neq 0$, $U$ *is a connected open subset of* $\mathbb{R}^2$ *and* $u = (u_1, u_2) \in U$.

*Then there is a differentiable function* $h : U \to \mathbb{R}$, *such that* $S$ *can be locally parametrized by*

$$X(u) = \left(\frac{g'}{|g'|^2}\nabla h - \frac{2R}{T}g, -\frac{2R}{T}\right), \tag{2.3}$$

*where*

$$T = 1+|g|^2 \;\; \text{and} \;\; R = \left\langle \nabla h, \frac{g}{g'}\right\rangle - h. \tag{2.4}$$

*Moreover, the coefficients of the first and the second fundamental form of* $X$ *are given by*

$$a_{11} = \frac{|g'|^2}{T^2}[A_1^2 + (TV_{12})^2], \;\; a_{12} = -\frac{|g'|^2}{T}V_{12}[A_1 + A_2], \;\; a_{22} = \frac{|g'|^2}{T^2}[A_2^2 + (TV_{12})^2], \tag{2.5}$$

$$b_{11} = \frac{2|g'|^2}{T^2}A_1, \;\; b_{12} = -\frac{2|g'|^2}{T}V_{12}, \;\; b_{22} = \frac{2|g'|^2}{T^2}A_2, \;\; A_i = 2R - TV_{ii}, \; i = 1, 2, \tag{2.6}$$

*where*

$$\begin{aligned} V_{11} &= \frac{1}{|g'|^2}\left[h_{,11} - \left\langle \frac{g''}{g'}, \nabla h\right\rangle\right], \\ V_{12} &= \frac{1}{|g'|^2}\left[h_{,12} - \left\langle i\frac{g''}{g'}, \nabla h\right\rangle\right], \\ V_{22} &= \frac{1}{|g'|^2}\left[h_{,22} + \left\langle \frac{g''}{g'}, \nabla h\right\rangle\right]. \end{aligned} \tag{2.7}$$

*The regularity condition of* $X$ *is given by*

$$P = (A_1A_2 - T^2V_{12}^2) \neq 0. \tag{2.8}$$

*The third fundamental form is determined by*

$$L_{ii} = \langle N_{,i}, N_{,i}\rangle = \frac{4}{T^2}|g'|^2, \; i = 1, 2, \;\; \langle N_{,1}, N_{,2}\rangle = 0. \tag{2.9}$$

$$H = -\frac{1}{P}\left(T\frac{\Delta h}{|g'|^2} - 4R\right), \qquad K = \frac{4}{P} \tag{2.10}$$

*Conversely, let be a holomorphic function* $g : U \to \Pi$, $U$, *where* $U$ *is a connected open subset of* $\mathbb{R}^2$ *and a differentiable function* $h : U \to \mathbb{R}$. *Then (2.3) define an immersion in* $\mathbb{R}^3$ *with Gaussian curvature non-zero, Gauss map given by (2.2) and (2.5)-(2.10) are satisfied.*

**3. Surfaces with quadratic support function (QSF-surfaces).** In this section we introduce the QSF-surfaces, classify the QSF-surfaces of rotation and we give some explicit examples of this class of surfaces.

**Definition 3.1.** We say that an oriented surface $S \subset \mathbb{R}^3$ is a *surface with quadratic support function (in short, QSF-surface)* if the mean curvature $H$, the Gaussian curvature $K$, the support function $\Psi$ and quadratic distance $\Lambda$ satisfy

$$2\Psi H + (\Lambda - \Psi^2)K = 0. \tag{3.1}$$

The following Theorem provides a Weierstrass type representation for the QSF-surfaces which depends on two holomorphic functions.

**Theorem 3.1.** *Let $S \subset \mathbb{R}^3$ be a connected orientable Riemann surface. Then $S$ is a QSF-surface if, and only if, there exist holomorphic functions $g, f : S \to \mathbb{C}_\infty$, such that $X(S)$ is locally parametrized by*

$$X(u) = \left( \frac{2c|f|^2}{T^2} \left( T\overline{\left( \frac{f'}{g'f} \right)} - g \right), 0 \right) - \frac{2R}{T}(g, 1), \tag{3.2}$$

*where*

$$R = \frac{2c|f|^2}{T^2} \left( 2T \left\langle 1, \frac{f'g}{fg'} \right\rangle - 3|g|^2 - 1 \right), \; c \neq 0, \; |g'| \neq 0, \; T = 1 + |g|^2. \tag{3.3}$$

*Proof:* From Theorem 2.1, we have that

$$\frac{H}{K} = -\frac{1}{4} \left( \frac{T\Delta h}{|g'|^2} - 4R \right), \tag{3.4}$$

$$\Psi = \frac{2h}{T}, \; \Lambda = \frac{|\nabla h|^2}{|g'|^2} - 4R\frac{h}{T}, \; L_{11} = \frac{4|g'|^2}{T^2}, \; T = 1 + |g|^2. \tag{3.5}$$

Using (3.5) in (3.4) we obtain

$$\frac{H}{K} = -\frac{1}{2\Psi} \left( \frac{h\Delta h - |\nabla h|^2}{|g'|^2} + \Lambda \right) = -\frac{1}{2\Psi} \left( \frac{\Psi^2}{h^2} \left( \frac{h\Delta h - |\nabla h|^2}{L_{11}} \right) + \Lambda \right). \tag{3.6}$$

Thus, $S$ is a QSF-surface if and only if

$$\frac{h\Delta h - |\nabla h|^2}{L_{11}} = -h^2. \tag{3.7}$$

Now, let $h = \dfrac{A}{B}$ where $A$ and $B$ are differentiable functions.
We can show that

$$h\Delta h - |\nabla h|^2 = \frac{1}{B^2} \left( A\Delta A - |\nabla A|^2 \right) - \frac{A^2}{B^4} \left( B\Delta B - |\nabla B|^2 \right). \tag{3.8}$$

Considering $B = T = 1 + |g|^2$ in (3.8) we obtain

$$\frac{h\Delta h - |\nabla h|^2}{L_{11}} = \frac{A\Delta A - |\nabla A|^2}{4|g'|^2} - h^2. \tag{3.9}$$

Thus

$$\frac{h\Delta h - |\nabla h|^2}{L_{11}} = -h^2 \Longleftrightarrow A\Delta A - |\nabla A|^2 = 0. \tag{3.10}$$

The solution of (3.10) is $A = c|f|^2$, where $f$ is a holomorphic function. Thus,

$$h = \frac{c|f|^2}{1+|g|^2}. \tag{3.11}$$

From (3.11) it follows

$$\nabla h = \frac{2c|f|^2}{T}\left(\overline{\left(\frac{f'}{f}\right)} - \frac{g\overline{g'}}{T}\right). \tag{3.12}$$

Using (3.12) in (2.3) we obtain (3.2). The proof is complete. □

The following Theorem classify the QSF-surfaces of rotation.

**Theorem 3.2.** *An oriented connected surface $S$ with nonzero Gauss curvature is a QSF-surface of rotation if, and only if, locally can be parameterized by*

$$X(u) = (A(u_1)\cos u_2, A(u_1)\sin u_2, B(u_1)) \tag{3.13}$$

*where*

$$A(u_1) = \frac{2c(k+(2-k)e^{4u_1})e^{(2k-1)u_1+2a}}{(1+e^{2u_1})^3}, \tag{3.14}$$

$$B(u_1) = -\frac{2c(2k+(2k-3)e^{2u_1}-1)e^{2ku_1+2a}}{(1+e^{2u_1})^3}. \tag{3.15}$$

*Proof:* Note that taking $g(w) = w, w \in \mathbb{C}$, $S$ is a QSF-surface of rotation if, and only if, $h$ is a radial function i.e., $h(w) = r(|w|)$ for any differentiable function $r$.
Making the change of parameters

$$w = e^z,\ z = u_1 + iu_2 \in \mathbb{C},$$

we have that $g(z) = e^z$ and $h_{,2} = 0$.
From (3.11) and $h_{,2} = 0$ we obtain $\langle f, if'\rangle = 0$, consequently, $f(z) = e^{kz+z_0}$, $z_0 = a+ib, a, b \in \mathbb{R}, k \neq 0$.
Thus

$$h = \frac{ce^{2ku_1+2a}}{1+e^{2u_1}}. \tag{3.16}$$

From (3.16) and (3.2), we obtain that $X$ is defined by (3.13)-(3.15). The proof is complete. □

**Example 3.1.** Considering $a = 1, c = 2, k = 2$ in Theorem 3.2, we obtain

$$X(u) = (A(u_1)\cos u_2, A(u_1)\sin u_2, B(u_1)),$$

and the profile curve is given by

$$\alpha(u_1) = (A(u_1), 0, B(u_1)),$$

where

$$A(u_1) = \frac{8e^{3u_1+2}}{(e^{2u_1}+1)^3},\quad B(u_1) = -\frac{4e^{4u_1+2}\left(e^{2u_1}+3\right)}{(e^{2u_1}+1)^3}$$

Here $A(u_1) > 0, \forall u_1 \in \mathbb{R}$ and the profile curve is regular, therefore, the QSF-surface of rotation is complete (see Figures 3.1 and 3.2).

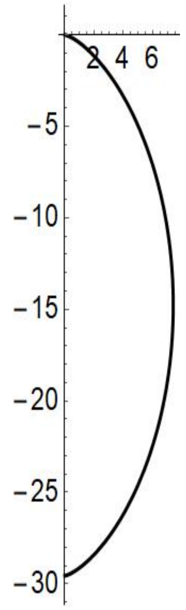


Figure 3.1: Profile curve

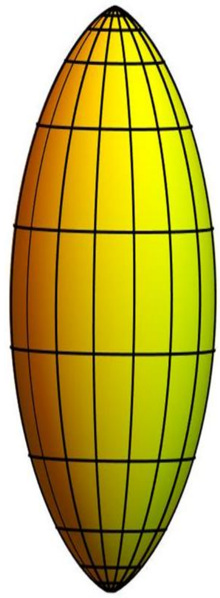

Figure 3.2: QSF-surface of rotation

**Example 3.2.** Considering $a = 2, c = -2, k = \frac{1}{30}$ in Theorem 3.2, we obtain

$$X(u) = (A(u_1)\cos u_2, A(u_1)\sin u_2, B(u_1)),$$

and the profile curve is given by

$$\alpha(u_1) = (A(u_1), 0, B(u_1)),$$

where

$$A(u_1) = -\frac{2e^{4-\frac{14u_1}{15}}\left(59e^{4u_1}+1\right)}{15\left(e^{2u_1}+1\right)^3}, \quad B(u_1) = -\frac{8e^{\frac{u_1}{15}+4}\left(22e^{2u_1}+7\right)}{15\left(e^{2u_1}+1\right)^3}.$$

Here $A(u_1) < 0$, $\forall u_1 \in \mathbb{R}$ and the profile curve is not regular only in one point, therefore, the QSF-surface of rotation has a circle of singularities (see Figures 3.3 and 3.4).

**Example 3.3.** Considering $a = -2, c = -3, k = -\frac{1}{30}$ in Theorem 3.2, we obtain

$$X(u) = (A(u_1)\cos u_2, A(u_1)\sin u_2, B(u_1)),$$

and the profile curve is given by

$$\alpha(u_1) = (A(u_1), 0, B(u_1)),$$

where

$$A(u_1) = \frac{e^{-\frac{16u_1}{15}-4}\left(1-61e^{4u_1}\right)}{5\left(e^{2u_1}+1\right)^3}, \quad B(u_1) = -\frac{4e^{-\frac{u_1}{15}-4}\left(23e^{2u_1}+8\right)}{5\left(e^{2u_1}+1\right)^3}.$$

Here $A(u_1) = 0$ only in one point and the profile curve is not regular only in one point, therefore, the QSF-surface of rotation has one isolated singularity (see Figures 3.5 and 3.6).

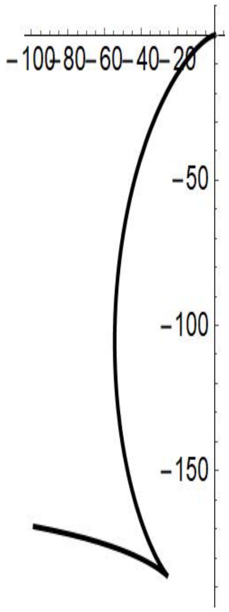


Figure 3.3: Profile curve

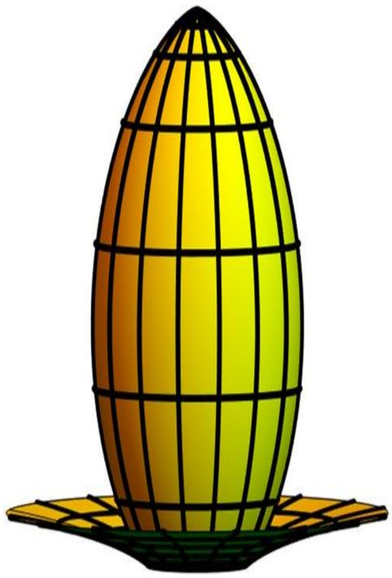

Figure 3.4: QSF-surface of rotation

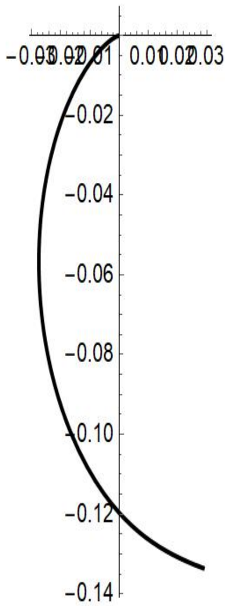


Figure 3.5: Profile curve

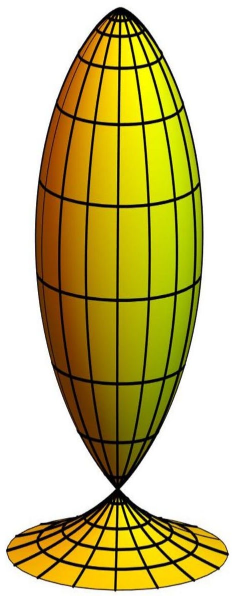

Figure 3.6: QSF-surface of rotation

**Example 3.4.** Considering $a = 1, c = -1, k = 1$ in Theorem 3.2, we obtain

$$X(u) = (A(u_1)\cos u_2, A(u_1)\sin u_2, B(u_1)),$$

and the profile curve is given by

$$\alpha(u_1) = (A(u_1), 0, B(u_1)),$$

where

$$A(u_1) = -\frac{2e^{u_1+2}\left(e^{4u_1}+1\right)}{\left(e^{2u_1}+1\right)^3}, \quad B(u_1) = \frac{2e^{2u_1+2}\left(1-e^{2u_1}\right)}{\left(e^{2u_1}+1\right)^3}.$$

Here $A(u_1) < 0, \forall\, u_1 \in \mathbb{R}$ and the profile curve is not regular, therefore, the QSF-surface of rotation has two circle of singularities (see Figures 3.7 and 3.8).

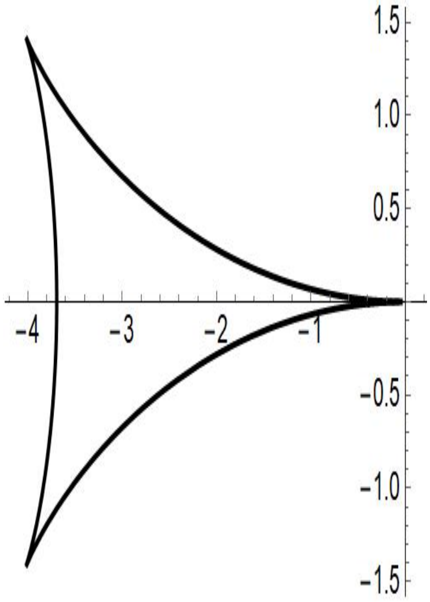


Figure 3.7: Profile curve

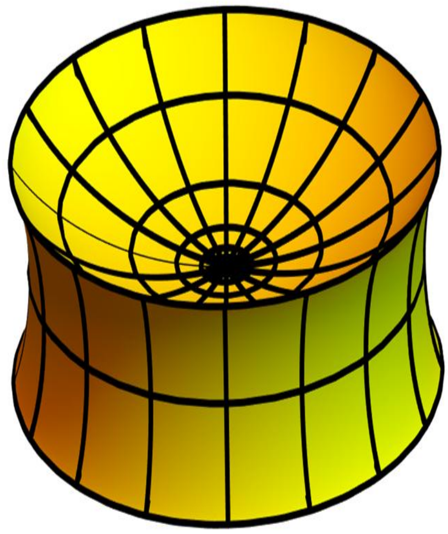

Figure 3.8: QSF-surface of rotation

**Example 3.5.** Considering $a = -\frac{1}{6}, c = -2, k = \frac{1}{2}$ in Theorem 3.2, we obtain

$$X(u) = (A(u_1)\cos u_2, A(u_1)\sin u_2, B(u_1)),$$

and the profile curve is given by

$$\alpha(u_1) = (A(u_1), 0, B(u_1)),$$

where

$$A(u_1) = -\frac{2\left(3e^{4u_1}+1\right)}{\sqrt[3]{e}\left(e^{2u_1}+1\right)^3}, \quad B(u_1) = -\frac{8e^{3u_1-\frac{1}{3}}}{\left(e^{2u_1}+1\right)^3}.$$

Here $A(u_1) < 0$, $\forall\, u_1 \in \mathbb{R}$ and the profile curve is not regular only in one point, therefore, the QSF-surface of rotation has one circle of singularities (see Figures 3.9 and 3.10).

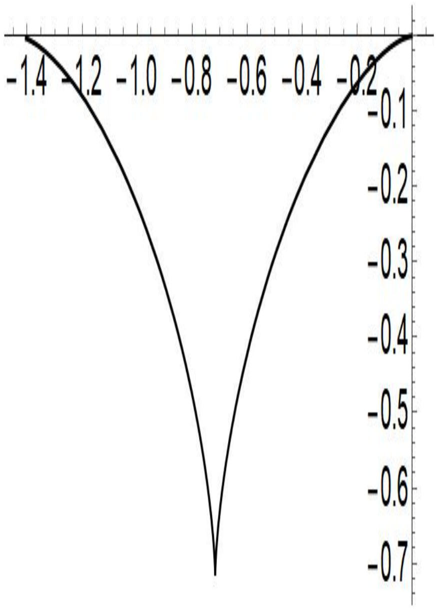


Figure 3.9: Profile curve

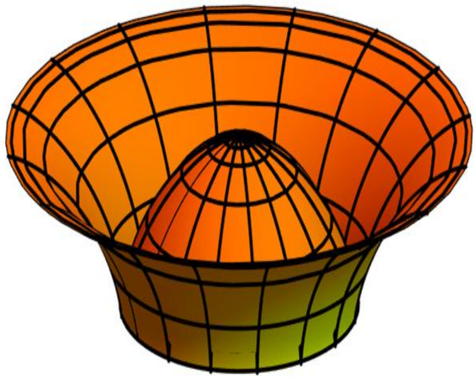

Figure 3.10: QSF-surface of rotation

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