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The region of the unit Euclidean sphere that admits a class of (r, s)-linear Weingarten hypersurfaces

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Dedicated to my mother Victorina Velásquez Romero, in memory.

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Abstract

In the unit Euclidean sphere $\mathbb{S}^{n+1}$*, we deal with a class of hypersurfaces that were characterized in [23] as the critical points of a variational problem, the so-called* (r, s)*-linear Weingarten hypersurfaces* $(0 \leq r \leq s \leq n-1)$*; namely, the hypersurfaces of* $\mathbb{S}^{n+1}$ *that has a linear combination* $a_r H_{r+1} + \cdots + a_s H_{s+1}$ *of their higher order mean curvatures* H_{r+1} *and* H_{s+1} *being a real constant, where* $a_r, \ldots, a_r$ *are nonnegative real numbers (with at least one non zero). By assuming a geometric constraint involving the higher order mean curvatures of these hypersurfaces, we prove a uniqueness result for strongly stable* (r, s)*-linear Weingarten hypersurfaces immersed in a certain region determined by a geodesic sphere of* $\mathbb{S}^{n+1}$*. We also establish a nonexistence result in another region of* $\mathbb{S}^{n+1}$ *for strongly stable Weingarten* (r, s)*-linear hypersurfaces.*

Keywords . Unit Euclidean space, (r, s)-linear Weingarten hypersurfaces, upper (lower) domain enclosed by the geodesic sphere of unit Euclidean space of level τ_0, strong stability, geodesic spheres.

1. Introduction. Associated with the variational problem of minimizing of the area functional

$$\mathcal{A} = \int_{\Sigma^n} d\Sigma$$

of a closed hypersurface $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ for all variations, not necessarily volume-preserving variations, we have the notion of strong stability related to closed hypersurfaces immersed into $\mathbb{S}^{n+1}$ with constant mean curvature H. With regard to this notion, it is well known that (for instance, see [1, Section 2]):

"There are no strongly stable closed hypersurfaces with constant mean curvature in the unit Euclidean sphere $\mathbb{S}^{n+1}$*."*

Another geometric quantity associated with a hypersurface is the (normalized) scalar curvature. With that in mind, when we study the problem of minimizing the 1-area functional

$$\mathcal{A}_1 = \int_{\Sigma^n} H d\Sigma$$

associated to a closed hypersurface $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ for all variations, we get the notion of strong 1-stability for closed hypersurfaces with constant normalized scalar curvature R. In this context, the author in [21, Teorema 1] showed the existence of a region of unit Euclidean sphere $\mathbb{S}^{n+1}$ that admits a specific class of strongly 1-stable closed hyperdufaces with constant normalized scalar curvature:

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"The only strongly 1-stable closed hypersurfaces immersed into the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$, with constant normalized scalar curvature R and mean curvature H satisfying the condition $R - 1 \geq H > 0$, are the geodesic spheres."

For a better understanding of the region described above, the reader is recommended to see Definition 4.1. In the previous statement, when we look at the complementary set in which the hypersurfaces are immersed, we have the following nonexistence result (cf. [21, Teorema 2]):

"There do not exist strongly 1-stable closed hypersurfaces immersed into the lower domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$, with constant normalized scalar curvature R and mean curvature H satisfying the condition $R - 1 \geq H > 0$."

An extension of the variational problems described above is that of minimizing the r-area functional

$$\mathcal{A}_r = \int_{\Sigma^n} F_r \, d\Sigma$$

of a closed hypersurface $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ for all possible variations, where F_r is a suitable function that depends on the higher order mean curvatures H_r of $\Sigma^n \looparrowright \mathbb{S}^{n+1}$, $r \in \{0, 1, \ldots, n\}$. This variational problem generates the notion of strong r-stability for closed hypersurfaces with constant higher order mean curvatures H_r. The concept of higher order mean curvatures of a hypersurface $\Sigma^n \looparrowright \mathbb{S}^{n+1}$, studied initially by R. Reilly [20] in 1973; are such that $H_0 = 1$, H_1 is just the mean curvature H of $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ and H_2 defines a geometric quantity which is related to the normalized scalar curvature R of $\Sigma^n \looparrowright \mathbb{S}^{n+1}$; more specifically, $H_2 = R - 1$. In [22], the author obtained extensions of the above statements for the context of higher order mean curvatures, establishing (cf. [22, Teorema 1]):

"The only strongly r-stable closed hypersurfaces immersed into the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$, with constant $(r+1)$-th mean curvature H_{r+1}, for $r \in \{1, \ldots, n-2\}$, and such that $H_{r+1} \geq H_r > 0$, are the geodesic spheres."

and (cf. [22, Teorema 2])

"There do not exist strongly r-stable closed hypersurfaces immersed into the lower domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$, with constant $(r+1)$-th mean curvature H_{r+1}, for $r \in \{1, \ldots, n-2\}$, and such that $H_{r+1} \geq H_r > 0$."

On the other hand, a natural extension of the hypersurfaces $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ with constant mean curvature H or constant second mean curvature H_2 are those ones whose curvatures H and H_2 obey a linear relation of the type $a_0 H + a_1 H_2 = \text{constant}$, for some real constants a_0 and a_1. These hypersurfaces are called in the literature as *linear Weingarten hypersurfaces* (see, for instance, [5, 6, 7, 12, 16, 17, 18]). A class that extends such hypersurfaces is given by the so-called generalized linear Weingarten hypersurfaces, namely, those hypersurfaces whose higher order mean curvatures H_{r+1} and H_{s+1} (for entire numbers r and s such that $0 \leq r \leq s \leq n-1$) satisfy the linear condition $a_r H_{r+1} + \cdots + a_s H_{s+1} = \text{constant}$, for some real numbers $a_r, \ldots, a_s$. For simplicity, we have named these hypersurfaces as (r, s)-*linear Weingarten*. It is not difficult to observe that geodesic spheres and Clifford torus of $\mathbb{S}^{n+1}$ are examples of (r, s)-linear Weingarten hypersurfaces in $\mathbb{S}^{n+1}$. We also observe that $(0, 1)$-linear Weingarten hypersurfaces are simply linear Weingarten hypersurfaces and (r, r)-linear Weingarten hypersurfaces with $r \in \{0 \ldots, n-1\}$ are just the hypersurfaces having constant $(r+1)$-th mean curvature H_{r+1}. In recent years, several papers have been published showing the interest in understanding the geometry of the (r, s)-linear Weingarten hypersurfaces (see [2, 3, 14, 15, 23]). For instance, we can highlight that the author jointly with H. de Lima and A. de Sousa showed in [23, Section 3] that (r, s)-linear Weingarten closed hypersurfaces compact are critical points of the variational problem of minimizing a suitable linear combination

$$\mathcal{B}_{r,s} = a_r \mathcal{A}_r + \cdots + a_s \mathcal{A}_s$$

of the j-area functionals $\mathcal{A}_j$ of a given compact oriented hypersurface $\Sigma^n \looparrowright \mathbb{S}^{n+1}$, $j \in \{r, \ldots, s\}$, for volume-preserving variations. Furthermore, they established that geodesic spheres of $\mathbb{S}^{n+1}$ are the only stable critical points of $\mathcal{B}_{r,s}$ for volume-preserving variations (cf. [23, Theorem 4.3]).

In this work, our objective is to obtain extensions of the results highlighted above in italics for the context of strongly stable (r, s)-linear Weingarten closed hypersurfaces immersed into $\mathbb{S}^{n+1}$. Details about the meaning of (r, s)-linear Weingarten hypersurfaces immersed into $\mathbb{S}^{n+1}$ are given in detail in Section 2, and all the details that lead us to establish strong stability notion for a (r, s)-linear Weingarten hypersurface can be found in Section 3. Indeed, we were able to establish the uniqueness result (see Theorem 4.1):

"Let r and s be entire numbers satisfying $0 \leq r \leq s \leq n-2$ and let $a_r, \ldots, a_s$ be nonnegative real numbers (with at least one non zero). When $r = 0$, assume in addition that $s > r$. The only strongly stable (r, s)-linear Weingarten closed hypersurfaces immersed into the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$, whose higher order mean curvatures $H_{r+1}, \ldots, H_{s+1}$ satisfy the relation $a_r b_r H_{r+1} + \cdots + a_s b_s H_{s+1} = \text{constant}$ and such that

$$\sum_{j=r}^{s}(j+1)a_j b_j H_{j+1} \geq \sum_{j=r}^{s}(j+1)a_j b_j H_j > 0,$$

are the geodesic spheres, where $b_j = (n-j)\binom{n}{j}$ for $j \in \{r, \ldots, s\}$."

and the nonexistence result (see Theorem 4.2):

"Let r and s be entire numbers satisfying the inequalities $0 \leq r \leq s \leq n-2$. When $r = 0$, assume in addition that $s > r$. There do not exist strongly stable (r, s)-linear Weingarten closed hypersurfaces immersed into the lower domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$ whose higher order mean curvatures $H_{r+1}, \ldots, H_{s+1}$ satisfy the relation $a_r b_r H_{r+1} + \cdots + a_s b_s H_{s+1} = \text{constant}$ and such that

$$\sum_{j=r}^{s}(j+1)a_j b_j H_{j+1} \geq \sum_{j=r}^{s}(j+1)a_j b_j H_j > 0,$$

where $a_r, \ldots, a_s$ are some nonnegative real numbers (with at least one non zero) and $b_j = (j+1)\binom{n}{j+1}$ for $j \in \{r, \ldots, s\}$."

The proofs of the main results of this work are presented in Section 4. Finally, in Corollary 4.1 and Corollary 4.2 we establish a version of our main results for the linear Weingarten case.

2. (r, s)-linear Weingarten hypersurfaces in $\mathbb{S}^{n+1}$. Unless stated otherwise, all manifold considered on this work will be connected. Let $\mathbb{S}^{n+1}$ be the $(n+1)$-dimensional Euclidean sphere. We will consider immersions $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ of closed orientable hypersurfaces Σ^n in $\mathbb{S}^{n+1}$, namely, isometric immersions from a n-dimensional orientable Riemannian manifold Σ^n into $\mathbb{S}^{n+1}$. In this setting, we denote by $d\Sigma$ the volume element with respect to the metric induced by x, $C^\infty(\Sigma^n)$ the ring of real functions of class C^∞ defined on Σ^n and by $\mathfrak{X}(\Sigma^n)$ the $C^\infty(\Sigma^n)$-module of vector fields of class C^∞ on Σ^n. Since Σ^n is orientable, one can choose a globally defined unit normal vector field N on Σ^n. The correspondence $N : \Sigma^n \to \mathbb{S}^n$ will be called the *Gauss map* of $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$. Let

$$\begin{array}{rcccl} A & : & \mathfrak{X}(\Sigma^n) & \to & \mathfrak{X}(\Sigma^n) \\ & & Y & \mapsto & A(Y) = -\overline{\nabla}_Y N. \end{array} \tag{2.1}$$

denote the *shape operator* with respect to N, so that, at each $q \in \Sigma^n$, A restricts to a self-adjoint linear map $A_q : T_q(\Sigma^n) \to T_q(\Sigma^n)$. Thus, for fixed $q \in \Sigma^n$, the spectral theorem allows us to choose on $T_q(\Sigma^n)$ an orthonormal basis $\{e_1, \ldots, e_n\}$ of eigenvectors of A_q, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively. For $r \in \{1, \ldots, n\}$, if we let $S_r(p)$ denote the *r-th elementary symmetric function* on the eigenvalues of A_p, we get n smooth functions $S_r : \Sigma^n \to \mathbb{R}$ such that

$$\det(tI - A) = \sum_{k=0}^{n}(-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \ldots, \lambda_n), \tag{2.2}$$

where $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the *r-th elementary symmetric polynomial* on the indeterminates $X_1, \ldots, X_n$.

For $r \in \{0, \ldots, n\}$, one defines the *higher order mean curvature* (or the *r-th mean curvature*) H_r of $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ by

$$\binom{n}{r} H_r = S_r = S_r(\lambda_1, \ldots, \lambda_n). \tag{2.3}$$

In particular, $H_0 = 1$ and for $r = 1$ we have that

$$H_1 = \frac{1}{n}\sum_{i=1}^{n}\lambda_i = H$$

is the *mean curvature* of $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$, which is the main extrinsic curvature of the hypersurface, and for $r = n$,

$$H_n = \lambda_1\lambda_1 \cdots \lambda_n$$

is the *Gauss-Kronecker curvature* of $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$. On the order hand, the second mean curvature

$$H_2 = \frac{2}{n(n-1)} \sum_{i<j} \lambda_i\lambda_j,$$

defines a geometric quantity which is related to the (intrinsic) *normalized scalar curvature* R of $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$. More precisely, it follows from the Gauss equation of $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ that

$$R = 1 + H_2. \tag{2.4}$$

We also define, for $r \in \{0, \ldots, n\}$, the *r-th Newton transformation* $P_r : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ associated to $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ by setting $P_0 = I$ (the identity operator) and, for $r \in \{1, \ldots, n\}$, via the recurrence relation

$$P_r = S_r I - AP_{r-1}. \tag{2.5}$$

A trivial induction shows that

$$P_r = (S_r I - S_{r-1}A + S_{r-2}A^2 - \cdots + rA^r),$$

so that Cayley-Hamilton Theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint whose eigenvalues are $\partial S_{r+1}/\partial\lambda_i$ (where the $\lambda_i's$ are the eigenvalues of A) and commutes with A. Therefore, all bases of $T_q(\Sigma^n)$ diagonalizing A at $q \in \Sigma^n$ also diagonalize all of the P_r at q. Let $\{e_1, \ldots, e_n\}$ be such a basis. Denoting by A_i the restriction of A to $\langle e_i\rangle^\perp \subset T_q(\Sigma^n)$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1} (-1)^k S_k(A_i) t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1\le j_1<\ldots<j_m\le n \\ j_1,\ldots,j_m\neq i}} \lambda_{j_1}\cdots\lambda_{j_m}. \tag{2.6}$$

With the above notations, it is also immediate to check that

$$P_r(e_i) = S_r(A_i)e_i, \tag{2.7}$$

and hence (cf. [8, Lemma 2.1])

$$\begin{cases} \operatorname{tr}(P_r) &= (n-r)S_r = b_r H_r; \\ \operatorname{tr}(AP_r) &= (r+1)S_{r+1} = b_r H_{r+1}; \\ \operatorname{tr}(A^2P_r) &= S_1S_{r+1} - (r+2)S_{r+2} = n\dfrac{b_r}{r+1}HH_{r+1} - b_{r+1}H_{r+2}, \end{cases} \tag{2.8}$$

where $b_r = (r+1)\binom{n}{r+1} = (n-r)\binom{n}{r}$.

Associated to each Newton Transformation P_r, $r \in \{0, \ldots, n\}$, one has the *second order linear differential operator*

$$\begin{aligned} L_r \ :\ C^\infty(\Sigma^n) &\to C^\infty(\Sigma^n) \\ f &\mapsto L_r(f) = \operatorname{tr}(P_r \circ \operatorname{Hess} f). \end{aligned} \tag{2.9}$$

We remark that L_0 is the Laplacian operator Δ and L_1 is the Cheng-Yau's square operator $\square$ defined in [13].

At this point, we are in a position to define our geometric object of study.

Definition 2.1. *Let r and s be any entire numbers satisfying the inequalities $0 \le r \le s \le n-1$. We say that $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ is a (r, s)-linear* Weingarten hypersurface *if there exist nonnegative real numbers $a_r, \ldots, a_s$ (at least one of them nonzero) such that the following linear relation occurs on Σ^n:*

$$a_r H_{r+1} + \cdots + a_s H_{s+1} = \text{constant}, \tag{2.10}$$

where H_j *is the* j*-th mean curvature of* $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$, $j \in \{r, \ldots, s\}$.

Remark 2.1. *Taking into account the relation between* H_2 *and the normalized scalar curvature* R *given in* (2.4)*, we observe from* (2.10) *that the* $(0, 1)$*-linear Weingarten hypersurfaces* $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ *are called simply* linear Weingarten hypersurfaces*, and there is a vast recent literature treating the problem of characterizing these hypersurfaces (see, for instance, [5, 6, 7, 12, 16, 17, 18]). It is because of this observation that the hypersurfaces described in Definition 2.1 are also called, in the current literature, the generalized linear Weingarten hypersurfaces (see [2, 3, 14, 15, 23]). Furthermore, when* $r = s \in \{0, \ldots, n-1\}$ *in our definition, then the hypersurface* $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ *has constant* $(r+1)$*-th mean curvature* H_{r+1}*. In particular, when* $r = s = 0$, $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ *has constant mean curvature* H *and, in turn, if* $r = s = 1$ *then* $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ *is a hypersurface with constant second mean curvature* H_2*, or with constant normalized scalar curvature* R *in view of* (2.4).

Example 2.1. *Here we provide some examples of hypersurfaces in* $\mathbb{S}^{n+1}$ *that meet our Definition 2.1.*

(a) *Let* $\mathbb{E}^n$ *be a* geodesic sphere *of* $\mathbb{S}^{n+1}$ *and let* $\iota : \mathbb{E}^n \looparrowright \mathbb{S}^{n+1}$ *be its inclusion application. In other words,* $\iota(\mathbb{E}^n)$ *is isometric to an* n*-dimensional (totally umbilical) Euclidian sphere whose principal curvatures are all equal to a certain nonzero constant* λ*. From* (2.3) *we get immediately that the higher order mean curvatures of* $\iota : \mathbb{E}^n \looparrowright \mathbb{S}^n$ *are given by* $H_j = \lambda^j$, $j \in \{0, \ldots, n\}$*, and, hence, all of them satisfy* (2.10) *for any real numbers* $a_r, \ldots, a_s$ *(at least one of them nonzero). Therefore, for any entire numbers* r *and* s *satisfying the inequalities* $0 \le r \le s \le n-1$*, all the geodesic spheres in* $\mathbb{S}^{n+1}$ *are* (r, s)*-Linear Weingarten hypersurfaces.*

(b) *Let* $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} = \mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2) \looparrowright \mathbb{S}^{n+1}$ *be a* n-dimensional Clifford torus *immersed into* $\mathbb{S}^{n+1}$*, with* $n_1, n_2 \in \mathbb{N}$ *satisfying* $n = n_1 + n_2$ *and* $\rho_1, \rho_2 \in (0, +\infty)$ *such that* $\rho_1^2 + \rho_2^2 = 1$*. We have that the shape operator* $A : \mathfrak{X}\left(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}\right) \to \mathfrak{X}\left(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}\right)$ *of* $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ *with respect to the Gauss map*

$$\begin{array}{rccl} N : & \mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} & \to & \mathbb{S}^n \\ & (p, q) & \mapsto & N(p,q)) = \left(-\dfrac{\rho_2}{\rho_1}\, p\, , \dfrac{\rho_1}{\rho_2}\, q\right) \end{array}$$

is given by

$$A = \begin{bmatrix} \dfrac{\rho_2}{\rho_1} I_{n_1} & 0 \\ 0 & -\dfrac{\rho_1}{\rho_2} I_{n_2} \end{bmatrix},$$

where $I_{n_1} : \mathfrak{X}(\mathbb{S}^{n_1}(\rho_1)) \to \mathfrak{X}(\mathbb{S}^{n_1}(\rho_1))$ *and* $I_{n_2} : \mathfrak{X}(\mathbb{S}^{n_2}(\rho_2)) \to \mathfrak{X}(\mathbb{S}^{n_2}(\rho_2))$ *denote the identity operators. Thus, the* principal curvatures $\kappa_1, \ldots, \kappa_n$ *of* $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ *are such that*

$$\lambda_1 = \cdots = \lambda_{n_1} = \frac{\rho_1}{\rho_2}\,, \qquad \lambda_{n_1+1} = \cdots = \lambda_n = -\frac{\rho_1}{\rho_2}\,.$$

Hence, for $j \in \{0, \ldots, n\}$*, we have that* j*-th elementary symmetric function* S_j *and the* j*-th mean curvature* H_j *of* $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ *are given by*

$$S_j = \sum_{0 \le k \le j} (-1)^{j-k} \binom{n_1}{k} \binom{n_2}{j-k} \left(\frac{\rho_2}{\rho_1}\right)^k \left(\frac{\rho_1}{\rho_2}\right)^{j-k},$$

and

$$H_j = \frac{1}{\binom{n}{j}} \left\{ \sum_{0 \le k \le j} (-1)^{j-k} \binom{n_1}{k} \binom{n_2}{j-k} \left(\frac{\rho_2}{\rho_1}\right)^k \left(\frac{\rho_1}{\rho_2}\right)^{j-k} \right\},$$

respectively. Since all higher order mean curvatures of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ *are constant, the condition* (2.10) *is satisfied for any real numbers* $a_r, \ldots, a_s$ *(at least one of them nonzero). Therefore, for any entire numbers* r *and* s *satisfying* $0 \le r \le s \le n-1$*, the Clifford hypersurfaces in* $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ *are* (r, s)*-linear Weingarten hypersurfaces.*

(c) *If* $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ *is an* isoparametric *hypersurface, namely, when its principal curvatures* $\lambda_1, \ldots, \lambda_n$ *are constant, we obtain without difficulties from* (2.3) *and* (2.10) *that* $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ *is a* (r, s)*-linear Weingarten hypersurface for all entire numbers* r *and* s *satisfying the inequalities* $0 \le r \le s \le n-1$.

3. (r,s)-linear Weingarten hypersurfaces as minimum points of a functional. Let $x:\Sigma^n \looparrowright \mathbb{S}^{n+1}$ be a *closed* (that is, compact without boundary) hypersurface immersed into $\mathbb{S}^{n+1}$ and let N be its Gauss map.

Following the ideas of [9], we define a *variation* of $x:\Sigma^n \to \mathbb{S}^{n+1}$ as being the smooth mapping

$$\begin{array}{rcl} X: (-\epsilon,\epsilon)\times\Sigma^n & \to & \mathbb{S}^{n+1} \\ (t,p) & \mapsto & X(t,p), \end{array}$$

where $\epsilon>0$, satisfying:

(i) for all $t\in(-\epsilon,\epsilon)$, the map

$$\begin{array}{rcl} X_t: \Sigma^n & \to & \mathbb{S}^{n+1} \\ p & \mapsto & X_t(p) = X(t,p) \end{array} \tag{3.1}$$

is a Riemannian immersion;

(ii) $X_0 = x$.

In all that follows, $d\Sigma_t$ denotes the volume element of Σ^n with respect to the metric induced by X_t.

In this configuration, the *variational field* associated to $X:(-\epsilon,\epsilon)\times\Sigma^n\to\mathbb{S}^{n+1}$ is the smooth vector field

$$K = \left.\frac{\partial X}{\partial t}\right|_{t=0},$$

and we say that the variation X is *normal* if K is parallel to N. Moreover, following [8], we define the *r-th area functional*

$$\begin{array}{rcl} \mathcal{A}_r: (-\epsilon,\epsilon) & \to & \mathbb{R} \\ t & \mapsto & \mathcal{A}_r(t) = \displaystyle\int_{\Sigma^n} F_r\left(S_1(t),S_2(t),\dots,S_r(t)\right)d\Sigma_t, \end{array}$$

where $S_r(t)=S_r(t,\cdot)$ is the r-th elementary symmetric function of Σ^n via the immersion (3.1) and F_r is recursively defined by setting $F_0=1$, $F_1=S_1(t)$ and, for $2\le r\le n-1$,

$$F_r = S_r(t) + \frac{(n-r+1)}{r-1}F_{r-2}.$$

We remark that when $r=0$, the functional $\mathcal{A}_0$ is the classical area functional.

In order to relate (r,s)-linear Weingarten hypersurfaces of $\mathbb{S}^{n+1}$ with the critical points of a variational problem, according to [23, Section 3], we consider the functional

$$\begin{array}{rcl} \mathcal{B}_{r,s}: (-\epsilon,\epsilon) & \to & \mathbb{R} \\ t & \mapsto & \mathcal{B}_{r,s}(t) = a_r\mathcal{A}_r(t)+\dots+a_s\mathcal{A}_s(t), \end{array} \tag{3.2}$$

where r and s are entire numbers satisfying the inequalities $0\le r\le s\le n-1$, $a_r,\dots,a_s$ are nonnegative real numbers (with at least one non zero) and $\mathcal{A}_j$ is the j-th area functional, $j\in\{r,\dots,s\}$. It is also necessary to consider the set

$$\mathcal{G}(\Sigma^n) = \left\{ f\in C^\infty(\Sigma^n) : \int_{\Sigma^n} f\,d\Sigma = 0 \right\} \tag{3.3}$$

of all smooth functions defined on Σ^n that admit an integral mean equal to zero. So, according to [9, Lemma 2.2] and [23, Proposition 3.6], every smooth function $f\in\mathcal{G}(\Sigma^n)$ induces a normal variation $X:(-\epsilon,\epsilon)\times\Sigma^n\to\mathbb{S}^{n+1}$ of $x:\Sigma^n\looparrowright\mathbb{S}^{n+1}$ with variational normal field $\frac{\partial X}{\partial t}|_{t=0} = fN$, and with first variation

$$\delta_f\,\mathcal{B}_{r,s} = \left.\frac{d}{dt}\mathcal{B}_{r,s}(t)\right|_{t=0}$$

of the functional $\mathcal{B}_{r,s}$ given by

$$\delta_f\,\mathcal{B}_{r,s} = -\int_{\Sigma^n}\left\{\sum_{j=r}^{s} a_j b_j H_{j+1}\right\} f\,d\Sigma, \tag{3.4}$$

where H_j is the j-th mean curvature of $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ with respect to N and $b_j = (j+1)\binom{n}{j+1}$, for any $j \in \{r, \ldots, s\}$.

As a consequence of (3.4), any (r,s)-linear Weingarten closed hypersurface $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ with higher order mean curvatures $H_{r+1}, \ldots, H_{s+1}$ satisfying the condition

$$a_r b_r H_{r+1} + \cdots + a_s b_s H_{s+1} = \text{constant}$$

is a critical point of $\mathcal{B}_{r,s}$ restricted to functions $f \in \mathcal{G}(\Sigma^n)$. Geometrically, this condition means that the variations under consideration preserve a certain volume function (for more details, see [23, Section 3]). At the moment, we observed that geodesic spheres, Clifford hypersurfaces and closed isoparametric hypersurfaces of $\mathbb{S}^{n+1}$ (all of which are described in Example 2.1) are critical points for the functional $\mathcal{B}_{r,s}$.

For these critical points, [23, Proposition 3.9] asserts that the stability of the corresponding variational problem of minimizing the functional $\mathcal{B}_{r,s}$ for all variations that preserve the volume is given by the second variation

$$\delta_f^2 \mathcal{B}_{r,s} = \frac{d^2}{dt^2}\mathcal{B}_{r,s}(t)\Big|_{t=0} = -\int_{\Sigma^n}\Big\{\mathcal{L}_{r,s}(f) + \sum_{j=r}^{s}(j+1)a_j\{\mathrm{tr}(P_j) + \mathrm{tr}(A^2P_j)\}f\Big\}fd\Sigma \tag{3.5}$$

of $\mathcal{B}_{r,s}$, where $\mathcal{L}_{r,s}$ is the second order linear differential operator on Σ^n given by

$$\begin{array}{rccl} \mathcal{L}_{r,s}: & C^\infty(\Sigma^n) & \to & C^\infty(\Sigma^n) \\ & f & \mapsto & \mathcal{L}_{r,s}(f) = \displaystyle\sum_{j=r}^{s}(j+1)a_jL_j(f), \end{array} \tag{3.6}$$

called the *Jacobi operator* associated with $\mathcal{B}_{r,s}$. Here, A is the shape operator of $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$, P_j is the j-th Newton transformation of $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$, given in (2.5), and L_j is the differential operator on Σ^n defined in (2.9). This will motivate us to establish the following notion of stability.

Definition 3.1. *Let r and s be entire numbers satisfying the inequalities $0 \le r \le s \le n-2$, and let $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ be a (r,s)-linear Weingarten closed hypersurface whose higher order mean curvatures $H_{r+1}, \ldots, H_{s+1}$ satisfying the linear relation*

$$a_r b_r H_{r+1} + \cdots + a_s b_s H_{s+1} = \text{constant},$$

for some nonnegative real numbers $a_r, \ldots, a_s$ (with at least one non zero), where $b_j = (n-j)\binom{n}{j}$ for $j \in \{r, \ldots, s\}$. In addition, we assume that $s > r$ when $r = 0$. We say that $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ is strongly stable *if $\delta_f^2 \mathcal{B}_{r,s} \ge 0$, for all $f \in C^\infty(\Sigma^n)$, where $\mathcal{B}_{r,s}$ the functional defined in (3.2).*

Remark 3.1. *In this previous definition, the restriction $s > r$ when $r = 0$ is due to the fact that there do not exist strongly stable constant mean curvature closed hypersurfaces immersed into $\mathbb{S}^{n+1}$ (cf. [1, Section 2]).*

For entire numbers r and s satisfying the inequalities $0 \le r \le s \le n-2$, with the restriction $s > r$ if $r = 0$, fron [23, Proposition 4.1] it is possible to obtain that the geodesic spheres of $\mathbb{S}^{n+1}$ are *stable* (r,s)-linear Weingarten hypersurface, that is, they are closed (r,s)-linear Weingarten hypersurfaces of $\mathbb{S}^{n+1}$ that satisfy condition $\delta_f^2 \mathcal{B}_{r,s} \ge 0$ for all $f \in \mathcal{G}(\Sigma^n)$, where $\mathcal{G}(\Sigma^n)$ is the set given in (3.3). We note that the proof of this result can be used to affirm that the geodesic spheres of $\mathbb{S}^{n+1}$ are also strongly stable. Here, for completeness of content, we present a proof.

Proposition 3.1. *Let r and s be entire numbers satisfying the inequalities $0 \le r \le s \le n-2$. We assume that $s > r$ when $r = 0$. Then, the geodesic spheres of $\mathbb{S}^{n+1}$ are strongly stable (r,s)-linear Weingarten closed hypersurfaces.*

Proof: Let $\mathbb{E}^n$ be a geodesic sphere in $\mathbb{S}^{n+1}$ and let $\iota : \mathbb{E}^n \looparrowright \mathbb{S}^{n+1}$ be its inclusion application into $\mathbb{RP}^n$. As item (a) of Example 2.1, we can conclude that $\iota : \mathbb{E}^n \looparrowright \mathbb{S}^{n+1}$ is a (r,s)-linear Weingarten closed two-side hypersurface with higher order mean curvatures $H_{r+1}, \ldots, H_{s+1}$ satisfying the linear condition $a_r b_r H_{r+1} + \cdots + a_s b_s H_{s+1} = $ constant, for some nonnegative real numbers $a_r, \ldots, a_s$ (with at least one non zero), where $b_j = (n-j)\binom{n}{j}$ for $j \in \{r, \ldots, s\}$, because we can always choose the Gauss map $N : \mathbb{E}^n \to \mathbb{S}^n$ of $\iota : \mathbb{E}^n \looparrowright \mathbb{S}^{n+1}$ in such a way that the principal curvatures of $\iota : \mathbb{E}^n \looparrowright \mathbb{S}^{n+1}$ are all equal to a certain positive constant λ, which in turn implies from (2.3) that its j-th mean curvature is given by

$$H_j = \lambda^j. \tag{3.7}$$

Moreover, from (2.2) and (2.6) we also have

$$S_j = \binom{n}{j}\lambda^j, \qquad S_j(A_i) = \binom{n-1}{j}\lambda^j. \tag{3.8}$$

So if $e_1, ..., e_n$ are principal directions of $\mathbb{E}^n$, from (2.7), (2.9) and (3.8) we get

$$L_j(f) = \sum_{i=1}^{n} \langle P_j(\text{Hess}\, f(e_i)), e_i\rangle = \binom{n-1}{j}\lambda^j \sum_{i=1}^{n}\langle \text{Hess}\, f(e_i), e_i\rangle = \binom{n-1}{j}\lambda^j \Delta(f),$$

for all $f \in C^\infty(\mathbb{E}^n)$ and any $j \in \{0, 1, ..., n\}$. Next, from (3.6),

$$\mathcal{L}_{r,s}(f) = \sum_{j=r}^{s}(j+1)a_j L_j(f) = \sum_{j=r}^{s}(j+1)a_j \binom{n-1}{j}\lambda^j \Delta(f), \tag{3.9}$$

for all $f \in C^\infty(\mathbb{E}^n)$. Consequently, if dv denotes the volume element of $\iota : \mathbb{E}^n \looparrowright \mathbb{RP}^n$, from (2.8), (3.5), (3.7) and (3.9) we obtain

$$\begin{aligned}
\delta_f^2 \mathcal{B}_{r,s} &= -\sum_{j=r}^{s}(j+1)a_j \int_{\mathbb{E}^n}\left\{\binom{n-1}{j}\lambda^j \Delta f + b_j H_j f + \right. \\
&\qquad\qquad \left. + \left(n\frac{b_j}{j+1}HH_{j+1} - b_{j+1}H_{j+2}\right) f\right\} f dv \\
&= -\sum_{j=r}^{s}(j+1)a_j \int_{\mathbb{E}^n}\left\{\binom{n-1}{j}\lambda^j f\Delta f + (n-j)\binom{n}{j}\lambda^j f^2 + \right. \\
&\qquad\qquad \left. + \left[n\binom{n}{j+1}\lambda^{j+2} - (n-j-1)\binom{n}{j+1}\lambda^{j+2}\right] f^2\right\} dv \\
&= \sum_{j=r}^{s}(j+1)\binom{n-1}{j}a_j\lambda^j \int_{\mathbb{E}^n}\left\{-f\Delta f - n(1+\lambda^2)f^2\right\} dv,
\end{aligned} \tag{3.10}$$

for all $f \in C^\infty(\mathbb{E}^n)$.

Now, let η_1 be the first eigenvalue of the Laplacian Δ of $\iota : \mathbb{E}^n \looparrowright \mathbb{RP}^n$, which admits the following min-max characterization (cf. [11])

$$\eta_1 = \min\left\{ \frac{-\int_{\mathbb{E}^n} f\Delta f\, dv}{\int_{\mathbb{E}^n} f^2\, dv} \;:\; f \in C^\infty(\mathbb{E}^n)\;, f \neq 0 \right\}. \tag{3.11}$$

Since a_j are nonnegative real numbers a and λ is a positive real number, from (3.10) and (3.11) we get

$$\delta_f^2 \mathcal{B}_{r,s} \geq \sum_{j=r}^{s}(j+1)\binom{n-1}{j}a_j\lambda^j \int_{\mathbb{E}^n}\left\{\eta_1 - n(1+\lambda^2)f^2\right\} dv,$$

for all $f \in C^\infty(\mathbb{E}^n)$. But, since $\iota(\mathbb{E}^n)$ is isometric to an n-dimensional Euclidian sphere with constant sectional curvature equal to $\lambda^2 + 1$, we have that $\eta_1 = n(\lambda^2 + 1)$. Hence, for every $f \in C^\infty(\mathbb{E}^n)$ we get

$$\delta_f^2 \mathcal{B}_{r,s} \geq \sum_{j=r}^{s}(j+1)\binom{n-1}{j}a_j\lambda^j \int_{\mathbb{E}^n}\left\{\eta_1 - n(1+\lambda^2)f^2\right\} dv = 0.$$

Therefore, according to Definition 3.1, $\iota : \mathbb{E}^n \looparrowright \mathbb{RP}^n$ must be strongly stable. □

4. Main results. In order to obtain our main concerning strongly stable (r, s)-linear Weingarten hypersurfaces immersed in $\mathbb{S}^{n+1}$, we need to describe a warped product that models a certain region of $\mathbb{S}^{n+1}$.

Let $\mathbf{P}$ be the *north pole* of $\mathbb{S}^{n+1}$ and $\mathbb{S}^n$ be the *equator* orthogonal to $\mathbf{P}$. From [19, Example 2], the open region

$$\Omega^{n+1} := \mathbb{S}^{n+1} \setminus \{\mathbf{P}, -\mathbf{P}\} \tag{4.1}$$

is isometric to the Riemannian warped product

$$(0, \pi) \times_{\sin\tau} \mathbb{S}^n, \quad \tau \in (0, \pi)\,. \tag{4.2}$$

At the moment, making $\mathbf{P} = (0, \ldots, 0, 1) \in \mathbb{S}^{n+1}$ and identifying the point $q = (q_1, \ldots, q_{n+1}) \in \mathbb{S}^n$ with $q = (q_1, \ldots, q_{n+1}, 0) \in \mathbb{S}^{n+1}$, we have that the correspondence

$$\begin{aligned}
\Psi \;:\; (0, \pi) \times_{\sin\tau} \mathbb{S}^n &\;\to\; \Omega^{n+1} \subset \mathbb{S}^{n+1} \\
(\tau, q) &\;\mapsto\; \Psi(\tau, q) = (\sin\tau)\, q + (\cos\tau)\, \mathbf{P},
\end{aligned}$$

defines an isometry between (4.2) and (4.1). We denote by

$$\Phi : \Omega^{n+1} \subset \mathbb{S}^{n+1} \to (0,\pi) \times_{\sin\tau} \mathbb{S}^n, \tag{4.3}$$

as being the inverse of Ψ.

If $d\tau^2$ and $d\sigma^2$ denote the metrics of $(0,\pi)$ and $\mathbb{S}^n$, respectively, then

$$\langle\, , \rangle = (\pi_I)^* \left(d\tau^2\right) + (\sin\tau)^2 (\pi_{\mathbb{S}^n})^* \left(d\sigma^2\right),$$

is the tensor metric of the Riemannian warped product (4.2), where π_I and $\pi_{\mathbb{S}^n}$ denote the projections onto the $(0,\pi)$ and $\mathbb{S}^n$, respectively. In this context, the vector field

$$(\sin\tau)\,\frac{\partial}{\partial\tau} \in \mathfrak{X}\left((0,\pi) \times_{\sin\tau} \mathbb{S}^n\right)$$

is a *conformal* and *closed* one (in the sense that its dual 1-form is closed), with conformal factor $\cos\tau$. Moreover, from [19, Proposition 1], for each $\tau_0 \in (0,\pi)$, the *slice* $\{\tau_0\} \times \mathbb{S}^n$ of the *foliation*

$$(0,\pi) \ni \tau_0 \longmapsto \{\tau_0\} \times \mathbb{S}^n$$

is a n-dimensional geodesic sphere of $\mathbb{S}^{n+1}$, parallel to the equator $\mathbb{S}^n$, with shape operator (see (2.1)) A_{τ_0} given by

$$\begin{array}{rcl} A_{\tau_0} \; : \; \mathfrak{X}(\{\tau_0\} \times \mathbb{S}^n) & \to & \mathfrak{X}(\{\tau_0\} \times \mathbb{S}^n) \\ Y & \mapsto & A_{\tau_0}(Y) \;=\; -\overline{\nabla}_Y(-\partial_\tau) = \dfrac{(\cos\tau_0)}{(\sin\tau_0)}\, Y \end{array} \tag{4.4}$$

with respect to the orientation given by $-\frac{\partial}{\partial\tau}$. Thus, from (2.2), (2.3) and (4.4), we get for $r \in \{0,\dots,n\}$ that the r-th elementary symmetric function $\mathcal{S}_r$ and the r-th mean curvature $\mathcal{H}_r$ of each slice $\{\tau_0\} \times \mathbb{S}^n \looparrowright \Omega^{n+1} \subset \mathbb{S}^{n+1}$ are

$$\mathcal{S}_r = \binom{n}{r} (\cot\tau_0)^r \quad \text{and} \quad \mathcal{H}_r = (\cot\tau_0)^r\,, \tag{4.5}$$

respectively.

Remark 4.1.

(a) *From (4.5) we get that $\mathcal{S}_r$ and $\mathcal{H}_r$ are constant on each slice $\{\tau_0\} \times \mathbb{S}^n \looparrowright \Omega^{n+1} \subset \mathbb{S}^{n+1}$, $\tau_0 \in (0,\pi)$. All of these slides correspond to the geodesic spheres of $\mathbb{S}^{n+1}$ described in item (a) of Example 2.1, that, according to Proposition 3.1, we already know that they are strongly stable (r,s)-linear Weingarten closed hypersurfaces for any entire numbers r and s satisfying $0 \le r \le s \le n-2$, with $s > r$ when $r = 0$.*

(b) *In the warped product $(0,\pi) \times_{\sin\tau} \mathbb{S}^n$, when $\tau_0 \in (0, \frac{\pi}{4}]$, from (4.5) we can observe that the higher order mean curvatures $\mathcal{H}_j = (\cos\tau_0/\sin\tau_0)^j$, $j \in \{0,\dots,n\}$, of a slice $\{\tau_0\} \times \mathbb{S}^n \looparrowright \Omega^{n+1} \subset \mathbb{S}^{n+1}$ verify the inequalities*

$$\cdots \ge \mathcal{H}_{j+1} \ge \mathcal{H}_j \ge \mathcal{H}_{j-1} \ge \cdots \ge \mathcal{H}_2 \ge \mathcal{H}_1 \ge 1.$$

Then, for any nonnegative real numbers $a_r,\dots,a_s$ (with at least one non zero) we have that the slices $\{\tau_0\} \times \mathbb{S}^n \looparrowright \Omega^{n+1} \subset \mathbb{S}^{n+1}$, with $\tau_0 \in (0,\frac{\pi}{4}]$, are strongly stable (r,s)-linear Weingarten closed hypersurfaces that satisfy the condition

$$\sum_{j=r}^{s} (j+1) a_j b_j \mathcal{H}_{j+1} \ge \sum_{j=r}^{s} (j+1) a_j b_j \mathcal{H}_j > 0,$$

where $b_j = (n-j)\binom{n}{j}$ for $j \in \{r,\dots,s\}$.

According to item (b) of Remark 4.1, we can see that the slice $\{\frac{\pi}{4}\} \times \mathbb{S}^n \looparrowright \Omega^{n+1} \subset \mathbb{S}^{n+1}$ divides the open region $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ into two connected components, which motivates us to establish the following notions of regions of $\mathbb{S}^{n+1}$, notions that will make it possible to write the statements of the main results in a clearer way.

Definition 4.1. *Fixed $\tau_0 \in (0,\pi)$, the region*

$$\Phi^{-1}\left((0,\tau_0) \times_{\sin\tau} \mathbb{S}^n\right) = \{\, q \in \mathbb{S}^{n+1} \; : \; \Phi(q) \in (0,\tau_0) \times_{\sin\tau} \mathbb{S}^n \,\},$$

of $\mathbb{S}^{n+1}$ *that corresponds to*

$$(0,\tau_0)\times_{\sin\tau}\mathbb{S}^n \subset (0,\pi)\times_{\sin\tau}\mathbb{S}^n,$$

will be called of upper domain enclosed by the geodesic sphere of Ω^{n+1} of level τ_0. *Similarly, the region*

$$\Phi^{-1}\left((\tau_0,\pi)\times_{\sin\tau}\mathbb{S}^n\right)=\{\,q\in\mathbb{S}^{n+1}\,:\,\Phi(q)\in(\tau_0,\pi)\times_{\sin\tau}\mathbb{S}^n\,\},$$

of $\mathbb{S}^{n+1}$ *that corresponds to*

$$(\tau_0,\pi)\times_{\sin\tau}\mathbb{S}^n \subset (0,\pi)\times_{\sin\tau}\mathbb{S}^n$$

will be called of lower domain enclosed by the geodesic sphere of Ω^{n+1} of level τ_0. *In turn, the regions*

$$\Phi^{-1}\left((0,\tau_0]\times_{\sin\tau}\mathbb{S}^n\right)=\{\,q\in\mathbb{S}^{n+1}\,:\,\Phi(q)\in(0,\tau_0]\times_{\sin\tau}\mathbb{S}^n\,\}$$

and

$$\Phi^{-1}\left([\tau_0,\pi)\times_{\sin\tau}\mathbb{S}^n\right)=\{\,q\in\mathbb{S}^{n+1}\,:\,\Phi(q)\in[\tau_0,\pi)\times_{\sin\tau}\mathbb{S}^n\,\},$$

of $\mathbb{S}^{n+1}$ *that corresponds to*

$$(0,\tau_0]\times_{\sin\tau}\mathbb{S}^n \subset (0,\pi)\times_{\sin\tau}\mathbb{S}^n$$

and

$$[\tau_0,\pi)\times_{\sin\tau}\mathbb{S}^n \subset (0,\pi)\times_{\sin\tau}\mathbb{S}^n,$$

respectively, will be called of closure of the upper domain *and* closure of the lower domain enclosed by the geodesic sphere of Ω^{n+1} of level τ_0, *where* Φ *is the isometry given in* (4.3).

Following the ideas established in [4], we will consider that hypersurfaces $x:\Sigma^n \looparrowright \Omega^{n+1}\subset\mathbb{S}^{n+1}$ whose Gauss map N satisfies

$$-1\le\left\langle\Phi_*(N(q)),\frac{\partial}{\partial\tau}\right\rangle_{\Phi(x(q))}<0,$$

for all $q\in\Sigma^n$. In this setting, for such a hypersurface $x:\Sigma^n\looparrowright\Omega^{n+1}\subset\mathbb{S}^{n+1}$ we define the *normal angle* θ as being the smooth function

$$\begin{array}{rccl}\theta: & \Sigma^n & \to & \left[0,\dfrac{\pi}{2}\right) \\ & q & \mapsto & \theta(q) \;=\; \arccos\left(-\left\langle\Phi_*(N(q)),\dfrac{\partial}{\partial\tau}\right\rangle_{\Phi(x(q))}\right).\end{array} \tag{4.6}$$

Thus, on Σ^n the normal angle θ verifies

$$0<\cos\theta=-\left\langle\Phi_*(N),\frac{\partial}{\partial\tau}\right\rangle\le 1. \tag{4.7}$$

Moreover, since the orientation of the slice $\{\tau_0\}\times\mathbb{S}^n$ is given by $-\frac{\partial}{\partial\tau}$, the normal angle θ of $\{\tau_0\}\times\mathbb{S}^n$ is such that $\cos\theta=1$.

We need the following result, which gives us an expression of Jacobi operator $\mathcal{L}_{r,s}$ acting in an appropriate support function associated with a (r,s)-linear Weingarten hypersurface in unit Euclidean sphere.

Proposition 4.1. *Let* r *and* s *be entire numbers satisfying the inequalities* $0\le r\le s\le n-2$, *and let* $x:\Sigma^n\looparrowright\Omega^{n+1}\subset\mathbb{S}^{n+1}$ *be a* (r,s)*-linear Weingarten hypersurface whose higher order mean curvatures* $H_{r+1},\dots,H_{s+1}$ *satisfying the relation* $a_rb_rH_{r+1}+\cdots+a_sb_sH_{s+1}=\text{constant}$, *for some nonnegative real numbers* $a_r,\dots,a_s$ *(with at least one non zero), where* $b_j=(n-j)\binom{n}{j}$ *for* $j\in\{r,\dots,s\}$. *If*

$$\begin{array}{rccl}\xi: & \Sigma^n & \to & \mathbb{R} \\ & q & \mapsto & \xi(q) \;=\; -\sin\tau\cos\theta(q),\end{array} \tag{4.8}$$

where θ *is the normal angle of* $x:\Sigma^n\looparrowright\Omega^{n+1}$ *defined in* (4.6), *then the Jacobi operator* $\mathcal{L}_{r,s}$ *defined in* (3.6) *acting on* ξ *is given by*

$$\mathcal{L}_{r,s}(\xi)=-\sum_{j=r}^{s}(j+1)a_j\left\{\operatorname{tr}(A^2P_j)\xi+b_jH_{j+1}\cos\tau\right\},$$

where A and P_j are the shape operator and the j-th Newton transformation of $x : \Sigma^n \looparrowright \mathbb{S}^{n+1}$, respectively. Here, for simplicity, we are adopting the notations $H_{j+1} = H_{j+1} \circ x^{-1} \circ \Phi^{-1}$ for all $j \in \{r, \dots, s\}$, where Φ is the isometry described in (4.3).

Proof: In fact, from [10, Theorem 2] we obtain

$$\begin{aligned} L_j(\xi) &= -\mathrm{tr}(A^2P_j)\xi - b_jH_j\xi - b_jH_j\,\Phi_*(N)(\cos\tau) - b_jH_{j+1}\cos\tau \\ &\quad - \left\langle \frac{\partial}{\partial\tau}, \nabla\left(\frac{b_j}{j+1}H_{j+1}\right)\right\rangle \sin\tau, \end{aligned} \tag{4.9}$$

for all $j \in \{0, \dots, n-2\}$. By noting that

$$\overline{\nabla}\cos\tau = \left\langle \overline{\nabla}\cos\tau, \frac{\partial}{\partial\tau}\right\rangle \frac{\partial}{\partial\tau} = (\cos\tau)'\frac{\partial}{\partial\tau} = -\sinh\tau\frac{\partial}{\partial\tau},$$

from (4.6) we have that

$$\Phi_*(N)(\cos\tau) = \langle\overline{\nabla}\cos\tau, \Phi_*(N)\rangle = -\sin\tau\left\langle\frac{\partial}{\partial\tau}, \Phi_*(N)\right\rangle = \sin\tau\cos\theta = -\xi. \tag{4.10}$$

Next, when we replace (4.10) into (4.9) we obtain

$$L_j(\xi) = -\mathrm{tr}(A^2P_j)\xi - b_jH_{j+1}\cos\tau - \left\langle \frac{\partial}{\partial\tau}, \nabla\left(\frac{b_j}{j+1}H_{j+1}\right)\right\rangle \sin\tau, \tag{4.11}$$

for all $j \in \{0, \dots, n-2\}$. Therefore, from (3.6) and (4.11) we get

$$\mathcal{L}_{r,s}(\xi) = -\sum_{j=r}^{s}(j+1)a_j\left\{\mathrm{tr}(A^2P_j)\xi + b_jH_{j+1}\cos\tau\right\} - \left\langle\frac{\partial}{\partial\tau}, \nabla\underbrace{\left(\sum_{j=r}^{s}a_jb_jH_{j+1}\right)}_{0}\right\rangle\sin\tau.$$

□

4.1. The region of $\mathbb{S}^{n+1}$ that admits a class of (r, s)-linear Weingarten hypersurfaces. Now, we are in a position to establish the following uniqueness result for strongly stable (r, s)-linear Weingarten hypersurfaces immersed in $\mathbb{S}^{n+1}$.

Theorem 4.1. *Let r and s be entire numbers satisfying $0 \le r \le s \le n-2$ and let $a_r, \dots, a_s$ be nonnegative real numbers (with at least one non zero). When $r = 0$, assume in addition that $s > r$. The only strongly stable (r, s)-linear Weingarten closed hypersurfaces immersed into the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$, whose higher order mean curvatures $H_{r+1}, \dots, H_{s+1}$ satisfy the relation*

$$a_rb_rH_{r+1} + \cdots + a_sb_sH_{s+1} = \text{constant} \tag{4.12}$$

and such that

$$\sum_{j=r}^{s}(j+1)a_jb_jH_{j+1} \ge \sum_{j=r}^{s}(j+1)a_jb_jH_j > 0, \tag{4.13}$$

are the geodesic spheres, where $b_j = (n-j)\binom{n}{j}$ for $j \in \{r, \dots, s\}$.

Proof: Taking Remark 4.1 into account, we have that any geodesic sphere $\{\tau_0\} \times \mathbb{S}^n \looparrowright \Omega^{n+1} \subset \mathbb{S}^{n+1}$ with $\tau_0 \in (0, \frac{\pi}{4}]$ is a strongly stable (r, s)-linear Weingarten closed hypersurface that satisfy (4.12) and (4.13).

Let $x : \Sigma^n \looparrowright \Omega^{n+1} \subset \mathbb{S}^{n+1}$ be a strongly stable (r, s)-linear Weingarten closed hypersurface whose higher order mean curvatures satisfy (4.12) and (4.13). Let Φ be the isometry given in (4.3). Since the (r, s)-linear Weingarten closed hypersurface

$$\Phi \circ x : \Sigma^n \looparrowright \left(0, \frac{\pi}{2}\right) \times_{\sin\tau} \mathbb{S}^n$$

is strongly stable, from Definition 3.1 and (3.5) we get

$$0 \le \delta_f^2\mathcal{B}_{r,s}(f) = -\int_{\Phi(x(\Sigma^n))}\left\{\mathcal{L}_{r,s}(f) + \sum_{j=r}^{s}(j+1)a_j\left\{\mathrm{tr}(P_j) + \mathrm{tr}(A^2P_j)\right\}f\right\}fd\Phi(\Sigma),$$

for all $f \in C^\infty(\Sigma^n)$, where $\mathcal{L}_{r,s}$ is the Jacobi operator defined in (3.6), $d\Phi(\Sigma)$ denotes the volume element of Σ^n induced by $\Phi \circ x$ and, for simplicity in notations, we identify H_{j+1} with $H_{j+1} \circ x^{-1} \circ \Phi^{-1}$ for any $j \in \{r, \ldots, s\}$. In particular, considering f to be the function $\xi = -\sin\tau\cos\theta \in C^\infty(\Sigma^n)$ defined in (4.8), from Proposition 4.1 and (2.8) we obtain

$$\begin{aligned}
0 &\leq -\int_{\Phi(x(\Sigma^n))} \Big\{\mathcal{L}_{r,s}(\xi) + \sum_{j=r}^{s}(j+1)a_j \mathrm{tr}(A^2P_j)\xi + \sum_{j=r}^{s}(j+1)a_j \underbrace{\mathrm{tr}(P_j)}_{b_jH_j}\, \xi\Big\}\xi d\Phi(\Sigma) \\
&= -\int_{\Phi(x(\Sigma^n))} \Big\{-\sum_{j=r}^{s}(j+1)a_jb_jH_{j+1}\cos\tau + \sum_{j=r}^{s}(j+1)a_jb_jH_j\,\xi\Big\}\xi d\Phi(\Sigma) \\
&= \int_{\Phi(x(\Sigma^n))} \Big\{\sum_{j=r}^{s}(j+1)a_jb_jH_{j+1}\cos\tau - \sum_{j=r}^{s}(j+1)a_jb_jH_j\,\xi\Big\}\xi d\Phi(\Sigma) \\
&= \int_{\Phi(x(\Sigma^n))} \Big\{\sum_{j=r}^{s}(j+1)a_jb_jH_{j+1}(-\cos\tau) \\
&\qquad\qquad + \sum_{j=r}^{s}(j+1)a_jb_jH_j(-\sin\tau)\cos\theta\Big\}\sin\tau\cos\theta d\Phi(\Sigma) \\
&\leq \int_{\Phi(x(\Sigma^n))} \Big\{-\sum_{j=r}^{s}(j+1)a_jb_jH_{j+1} + \sum_{j=r}^{s}(j+1)a_jb_jH_j\cos\theta\Big\}\cos\tau\sin\tau\cos\theta d\Phi(\Sigma),
\end{aligned}$$

where we use that $\cos\tau \geq -\sin\tau$ on $n(0, \pi/2)$. Hence, from (4.13),

$$\begin{aligned}
0 &\leq \int_{\Phi(x(\Sigma^n))} \Big\{-\sum_{j=r}^{s}(j+1)a_jb_jH_{j+1} + \sum_{j=r}^{s}(j+1)a_jb_jH_j\cos\theta\Big\}\cos\tau\sin\tau\cos\theta d\Phi(\Sigma) \\
&\leq \int_{\Phi(x(\Sigma^n))} \Big(-1+\cos\theta\Big)\Big(\sum_{j=r}^{s}(j+1)a_jb_jH_j\Big)\cos\tau\sin\tau\cos\theta d\Phi(\Sigma).
\end{aligned}$$

Since the normal angle θ is such that $0 < \cos\theta \leq 1$ (see (4.7)), $\sum_{j=r}^{s}(j+1)a_jb_jH_j > 0$ (see (4.13)) and the functions $\cos\tau$ and $\sin\tau$ are strictly positive on $(0, \pi/2)$, we get

$$0 \leq \int_{\Phi(x(\Sigma^n))} \Big(-1+\cos\theta\Big)\Big(\sum_{j=r}^{s}(j+1)a_jb_jH_j\Big)\cos\tau\sin\tau\cos\theta d\Phi(\Sigma) \leq 0.$$

Thus, $\cos\theta = 1$ on Σ^n and, consequently, there exists $\tau_0 \in (0, \pi/2)$ such that $\Phi(x(\Sigma^n)) = \{\tau_0\} \times \mathbb{S}^n$. But, since the inequalitie given in (4.13) are valid on Σ^n, we must restrict the values of τ_0 to the interval $(0, \pi/4]$ (see Remark 4.1). Therefore, we can conclude that $x(\Sigma^n)$ is isometric to a geodesic sphere contained in the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$. □

Regarding the complement of the set where hypersurfaces are considered in Theorem 4.1, we can establish the following nonexistence result.

Theorem 4.2. *Let r and s be entire numbers satisfying the inequalities $0 \leq r \leq s \leq n-2$. When $r = 0$, assume in addition that $s > r$. There do not exist strongly stable (r, s)-linear Weingarten closed hypersurfaces immersed into the lower domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$ whose higher order mean curvatures $H_{r+1}, \ldots, H_{s+1}$ satisfy the relation*

$$a_rb_rH_{r+1} + \cdots + a_sb_sH_{s+1} = \text{constant}$$

and such that

$$\sum_{j=r}^{s}(j+1)a_jb_jH_{j+1} \geq \sum_{j=r}^{s}(j+1)a_jb_jH_j > 0,$$

where $a_r, \ldots, a_s$ are some nonnegative real numbers (with at least one non zero) and $b_j = (j+1)\binom{n}{j+1}$ for $j \in \{r, \ldots, s\}$.

Proof: By contradiction, let us suppose the existence of such a hypersurface $x : \Sigma^n \looparrowright \Omega^{n+1} \subset \mathbb{S}^{n+1}$ immersed into into the lower domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$. Taking into account the arguments used at the end of the proof of Theorem 4.1, we get that $x(\Sigma^n)$ is isometric to a geodesic sphere contained in the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$, which is absurd. □

4.2. Some particular cases. In this last subsection we analyze the main results, given in Section 4, for the case of some specific curvatures (extrinsic or intrinsic) associated with a hypersurface immersed into $\mathbb{S}^{n+1}$.

For starters, knowing that the $(0, 1)$-linear Weingarten hypersurfaces are recorded in the literature as the classical linear Weingarten hypersurfaces (see Remark 2.1), from Theorem 4.2 we have the following uniqueness result.

Corollary 4.1. *The only strongly stable linear Weingarten closed hypersurfaces immersed into the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$, whose mean curvature H and normalized scalar curvature R satisfy the relation*

$$na_0 H + n(n-1)a_1(R-1) = \text{constant}$$

and such that $R - 1 \geq H \geq 1$, are the geodesic spheres, for some nonnegative real number a_0 and some positive real number a_1.

Proof: Initially, from (2.4) we get $H_2 = R - 1$. Now, observing that from conditions $H_2 \geq H \geq 1$, $a_0 \geq 0$ and $a_1 > 0$ we obtain the inequality $na_0(H-1) + 2n(n-1)a_1(H_2 - H) \geq 0$, or even

$$na_0 H + 2n(n-1)a_1 H_2 \geq na_0 + 2n(n-1)a_1 H > 0,$$

the result immediately follows from Theorem 4.1 making $r = 0$ and $s = 1$. □

Thinking similarly as in the proof of Corollary 4.1, from Theorem 4.2 we have the following nonexistent result.

Corollary 4.2. *There do not exist strongly stable linear Weingarten closed hypersurfaces immersed into the lower domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$, whose mean curvature H and normalized scalar curvature R satisfy the relation*

$$na_0 H + n(n-1)a_1(R-1) = \text{constant}$$

and such that $R - 1 \geq H \geq 1$, for some nonnegative real number a_0 and some positive real number a_1.

On the other hand, according to one of the established statements of Remark 2.1, when we consider $r = s \in \{1, \ldots, n-2\}$ we have that a (r, r)-linear Weingarten hypersurface immersed into $\mathbb{S}^{n+1}$ becomes a hypersurface with constant $(r+1)$-th mean curvature H_{r+1}. In this case, from Theorem 4.2 and Theorem 4.1 we have recovered [22, Theorem 1] and [22, Theorem 2], respectively.

Corollary 4.3. *The only strongly stable closed hypersurfaces immersed into the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$, with constant $(r+1)$-th mean curvature H_{r+1}, for $r \in \{1, \ldots, n-2\}$, and such that $H_{r+1} \geq H_r > 0$, are the geodesic spheres. In particular, the only strongly stable closed hypersurfaces immersed into the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$, with constant normalized scalar curvature R and mean curvature H satisfying the condition $R - 1 \geq H > 0$, are the geodesic spheres.*

Corollary 4.4. *There do not exist strongly stable closed hypersurfaces immersed into the lower domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$, with constant $(r+1)$-th mean curvature H_{r+1}, for $r \in \{1, \ldots, n-2\}$, and such that $H_{r+1} \geq H_r > 0$. In particular, there do not exist strongly stable closed hypersurfaces immersed into the lower domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$, with constant normalized scalar curvature R and mean curvature H satisfying the condition $R - 1 \geq H > 0$.*

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