Journal homepage http://revistas.unitru.edu.pe/index.php/SSMM

SELECCIONES MATEMÁTICAS
Universidad Nacional de Trujillo
ISSN: 2411-1783 (Online)
2023; Vol. 10(2): 299-309.

Solutions to the Calapso, Zoomeron, and Davey-Stewartson III equations using Ribaucour transformations

Soluciones para las ecuaciones de Calapso, Zoomeron y Davey-Stewartson III utilizando transformaciones de Ribaucour

Armando M. V. Corro and Marcelo Lopes Ferro

Received, Oct. 27, 2023; Accepted, Nov. 30, 2023; Published, Dec. 27, 2023

CC BY

How to cite this article:
Corro A, Ferro ML. *Solutions to the Calapso, Zoomeron, and Davey-Stewartson III equations using Ribaucour transformations*. Selecciones Matemáticas. 2023;10(2):299–309. http://dx.doi.org/10.17268/sel.mat.2023.02.06

Abstract
In this paper, we present a method for obtaining solutions to the Calapso and Zoomeron equations using Ribaucour transformations. We provide explicit formulas for the solutions of these equations. By applying this method to the generalized cylinder, we provide new solutions of these equations, which are determined by a free function and by two other functions, each one defined in a given variable. As a consequence, we provide new solutions for the Davey-Stewartson III equation.

Keywords . Isothermic surfaces, Ribaucour transformations, Calapso equation, Zoomeron equation, Davey-Stewartson III equation.

Resumen
En este artículo, presentamos un método para obtener soluciones para las ecuaciones de Calapso y Zoomeron utilizando transformaciones de Ribaucour. Proporcionamos fórmulas explícitas para las soluciones de estas ecuaciones. Aplicando este método al cilindro generalizado, proporcionamos nuevas soluciones de estas ecuaciones, que están determinadas por una función libre y por otras dos funciones, cada una definida en una variable dada. Como consecuencia, ofrecemos nuevas soluciones para la ecuación de Davey-Stewartson III.

Palabras clave. Superficies isotérmicas, transformaciones de Ribaucour, ecuación de Calapso, ecuación de Zoomeron, ecuación de Davey-Stewartson III.

1. Introduction. The Ribaucour transformations for hypersurfaces parametrized by lines of curvature were classically studied by Bianchi [1]. They can be applied to obtain surfaces of constant Gaussian curvature and surfaces of constant mean curvature, from a given surface, with constant Gaussian curvature and constant mean curvature, respectively. The first application of this method to minimal and cmc surfaces in $\mathbb{R}^3$ was obtained by Corro, Ferreira, and Tenenblat in [5]-[6]. For more applications of this method, see [4]-[7], [13], [17], [15] and [18].

A regular surface M is isothermic if, locally, near to each non umbilical point of M, it admits isothermic parameters, whose coordinate curves are curvature lines. Particular classes of isothermic surfaces are the constant mean curvature surfaces, quadrics surfaces, and surfaces whose lines of curvature have constant geodesic curvature, in particular, the cyclides of Dupin. Isothermic surfaces are preserved by isometries, dilations and inversions. The classification of isothermic surfaces is an open problem.

*IME, Universidade Federal de Goiás, Caixa Postal 131, 74001-970, Goiânia, GO, Brazil. (corro@ufg.br).
†IME, Universidade Federal de Goiás, Caixa Postal 131, 74001-970, Goiânia, GO, Brazil. **Correspondence author** (marceloferro@ufg.br).

Partial differential equations arise naturally in problems of mathematical physics, physics, engineering and geometry. For instance, they are foundational in the modern scientific understanding of sound, heat, diffusion, electrostatics, electrodynamics, fluid dynamics, elasticity, general relativity, and quantum mechanics. They also arise from many purely mathematical considerations, such as differential geometry and the calculus of variations; among other notable applications, they are the fundamental tool in the proof of the Poincaré conjecture from geometric topology.

In [2], the author establishes an equation with fourth order partial derivatives from which the problem of obtaining isothermic surfaces apparently becomes much simpler. Such equation (called Calapso equation) defined in [2] given by

$$\Delta\left(\frac{\phi_{,12}}{\phi}\right) + (\phi^2)_{,12} = 0,$$

describes isothermic surfaces in $\mathbb{R}^3$, where $\phi_{,12}$ denotes the derivative of ϕ with respect to u_1 and u_2. The Calapso equation is very difficult to solve and is strongly connected to the Painleve ODEs, some authors have found solutions of this equation associated with constant mean curvature surfaces.

It is noted that the transition $u_2 \to iu_2$ takes the Calapso equation to the Zoomeron equation

$$\Delta_{-1}\left(\frac{\phi_{,12}}{\phi}\right) + (\phi^2)_{,12} = 0,$$

where $\Delta_{-1} = \dfrac{\partial^2}{\partial u_1^2} - \dfrac{\partial^2}{\partial u_2^2}$. Another equation that also very difficult to solve. The solitons of this equation are referred to as Zoomerons, where they possess the properties of Trappon solitons because they are like particles trapped in a potential well, changing direction indefinitely.

In [12], the authors introduced the Davey-Stewartson (DS) equation. This equation describe the evolution of a three-dimensional wave-packet on water of finite depth in the fluid dynamics.

In [16] (pag. 196), the authors shows that for each $\phi(u_1, u_2)$ solution of the Zoomeron equation, associates a two-parameter family of the solutions $u(u_1, u_2, t)$ given by

$$u = e^{i(\nu u_1 + \mu u_2 + \mu\nu t)}\phi(u_1 + \mu t, u_2 + \nu t), \quad \rho = \frac{|u|_{,u_1 u_2}}{u},$$

to the Davey-Stewartson III equation

$$iu_t = u_{u_1 u_2} - \rho u, \quad \Delta_\epsilon \rho + \left(|u|^2\right)_{u_1 u_2} = 0.$$

Recently, a study on the fully PT-symmetric nonlocal Davey-Stewartson III equation was reported in [14] and in [3], the authors investigated the reverse-time nonlocal of this equation.

In [8], the authors introduce the class of radial inverse mean curvature surfaces (RIMC - surfaces), which are isothermic surfaces. In addition, they show that for each isothermic surface there is another solution to the Calapso equation which depends on the metric and on the skew curvature of the surface. This solution is different from the one presented in [2].

In [9], the authors used Ribaucour transformations to obtain new isothermic surfaces in $\mathbb{R}^3$ from the cylinder. Moreover, the authors obtained new solutions to the Calapso and Zoomeron equation. In [10], the same authors obtained new isothermic surfaces in $\mathbb{S}^3$, as well as new solutions for the Calapso, Zoomeron, and Davey-Stewartson III equations.

In [11], the authors defined the Pseudo-Calapso equation, where, in particular, we have the Calapso and Zoomeron equations. Additionally, the authors provided solutions for the Calapso, Zoomeron, and Davey-Stewartson III equations.

In this paper, motivated by [9] and [10] we use the Ribaucour transformations to obtain solutions of the Calapso, Zoomeron and Davey-Stewartson III equations. Such solutions are determined by a free function and by two other functions, each one defined in a given variable. For each choice of the free function, we obtain new solutions for the Calapso, Zoomeron and Davey-Stewartson III equations globally defined or defined in some interval of the real line. This work is organized as follows.In Section 1, we provide a brief description of Ribaucour transformations and highlight three equations that are notoriously difficult to solve. In Section 2, we study Ribaucour transformations for isothermic surfaces and we provide explicit formulas for solutions of the Calapso, Zoomeron and Davey-Stewartson III equations. In Section 3, by applying the theory to the generalized cylinder, we provide new solutions of these three equations that depend on a free function. We also provide explicit examples of such solutions.

2. Preliminary. In this section, we first recall the theory of Ribaucour transformation for surfaces (see [1] and [4] for more details). Subsequently, we introduce the well-known Calapso equation and the results obtained in [2] and [8], which provide solutions to this equation. Following that, we also introduce the well-known Zoomeron and Davey-Stewartson III equations.

Let M be an orientable surface of $\mathbb{R}^3$ without umbilic points, whose Gauss map we denote by N. We say that $\widetilde{M}$ is associated to M by a Ribaucour transformation, if and only if, there exists a differentiable distance function h, defined on M and a diffeomorphism $\psi : M \to \widetilde{M}$

(a) for all $p \in M$, $p + h(p)N(p) = \psi(p) + h(p)\widetilde{N}(\psi(p))$, where $\widetilde{N}$ is the Gauss map of $\widetilde{M}$.
(b) The subset $p + h(p)N(p)$, $p \in M$, is a two-dimensional submanifold.
(c) ψ preserves lines of curvature.

We say that $\widetilde{M}$ is locally associated to M by a *Ribaucour transformation* if, for all $\widetilde{p} \in \widetilde{M}$, there exists a neighborhood of $\widetilde{p}$ in $\widetilde{M}$ which is associated by a Ribaucour transformation to an open subset of M.

We observe that, whenever the surface M is parametrized by orthogonal lines of curvature $X(u_1, u_2)$ we get that the distance function h satisfies (see [4])

$$h_{,12} - \frac{1+h\lambda_1}{1+h\lambda_2}\Gamma^2_{12}h_{,2} - \frac{1+h\lambda_2}{1+h\lambda_1}\Gamma^1_{12}h_{,1} - \left(\frac{\lambda_2}{1+h\lambda_2} + \frac{\lambda_1}{1+h\lambda_1}\right)h_{,1}\, h_{,2} = 0. \tag{2.1}$$

For each nonvanishing function h, which is a solution of this equation, there is a differentiable function Ω (see Proposition 2.2 in [4]) such that, we define

$$\Omega_i = \frac{\Omega_{,i}}{a_i}, \quad a_i = \sqrt{< X_{,i}\ , X_{,i} >}, \quad 1 \le i \le 2, \qquad W = \frac{\Omega}{h},$$

then the nonlinear equation (2.1) becomes a system of linear differential equations as presented in the following theorem, the proof of which can be found in [4] and [5].

Theorem 2.1. *Let M be an orientable surface of $\mathbb{R}^3$ parametrized by $X : U \subseteq \mathbb{R}^2 \to M$, without umbilic points. Assume $e_i = \frac{X_{,i}}{a_i}$, $1 \le i \le 2$ where $a_i = \sqrt{g_{ii}}$ are orthogonal principal directions, $-\lambda_i$ the corresponding principal curvatures, and N is a unit vector field normal to M. A surface $\widetilde{M}$ is locally associated to M by a Ribaucour transformation if and only if there is differentiable functions $W, \Omega, \Omega_i : V \subseteq U \to R$ which satisfy*

$$\begin{aligned} \Omega_{i,j} &= \Omega_j \frac{a_{j,i}}{a_j}, \quad for \;\; i \neq j, \\ \Omega_{,i} &= a_i \Omega_i, \\ W_{,i} &= -a_i \Omega_i \lambda_i. \end{aligned} \tag{2.2}$$

$W(W + \lambda_i\Omega) \neq 0$ and $\widetilde{X} : V \subseteq U \to \widetilde{M}$, is a parametrization of $\widetilde{M}$ given by

$$\widetilde{X} = X - \frac{2\Omega}{S}\left(\sum_{i=1}^{2} \Omega_i e_i - WN\right), \tag{2.3}$$

where

$$S = \sum_{i=1}^{2} (\Omega_i)^2 + W^2. \tag{2.4}$$

Moreover, the normal map of $\widetilde{X}$ is given by

$$\widetilde{N} = N + \frac{2W}{S}\left(\sum_{i=1}^{2} \Omega_i e_i - WN\right), \tag{2.5}$$

and the principal curvatures and coefficients of the first fundamental form of $\widetilde{X}$, are given by

$$\widetilde{\lambda}_i = \frac{WT_i + \lambda_i S}{S - \Omega T_i}, \qquad \widetilde{g}_{ii} = \left(\frac{S - \Omega T_i}{S}\right)^2 g_{ii} \tag{2.6}$$

where Ω_i, Ω and W satisfy (2.2), S is given by (2.4), g_{ii}, $1 \le i \le 2$ are coefficients of the first fundamental form of X, and

$$T_1 = \frac{2}{a_1}\left(\Omega_{1,1} + \frac{a_{1,2}}{a_2}\Omega_2 - W\lambda_1 a_1\right), \qquad T_2 = \frac{2}{a_2}\left(\Omega_{2,2} + \frac{a_{2,1}}{a_1}\Omega_1 - W\lambda_2 a_2\right). \tag{2.7}$$

Definition 2.1. A surface of $\mathbb{R}^3$ is called *isothermic* if near each non umbilic point it admits isothermic parameters, whose coordinate curves are curvature lines.

In [2], the author define the Calapso equation by

$$\left(\frac{\phi_{,u_1u_2}}{w}\right)_{,u_1u_1} + \left(\frac{\phi_{,u_1u_2}}{w}\right)_{,u_2u_2} + (\omega^2)_{,u_1u_2} = 0. \tag{2.8}$$

Such equation describes isothermic surfaces in $\mathbb{R}^3$, where $\phi_{,12}$ denotes the derivative of ϕ with respect to u_1 and u_2. The Calapso equation is very difficult to solve and is strongly connected to the Painleve ODEs, some authors have found solutions of this equation associated with constant mean curvature surfaces.

The following result provides solutions of the Calapso equation. (see [2] and [8] for a proof and more details).

Theorem 2.2. *Let $X(u_1, u_2)$ be an isothermic surface with the first fundamental form given by*

$$I = e^{2\varphi}(du_1^2 + du_2^2).$$

Then, the functions $\omega = \sqrt{2}e^{\varphi}H$ and $\Omega = \sqrt{2}e^{\varphi}H'$ are solutions of the Calapso equation, where H is the mean curvature of X and H' is the skew curvature of M.

Remark 2.1. The transition $u_2 \to iu_2$ transforms the Calapso equation (2.8) into the Zoomeron equation,

$$\left(\frac{\omega_{,u_1u_2}}{w}\right)_{,u_1u_1} - \left(\frac{\omega_{,u_1u_2}}{w}\right)_{,u_2u_2} + (\omega^2)_{,u_1u_2} = 0. \tag{2.9}$$

Another equation that poses significant challenges when it comes to solving. The solitons of this equation are referred to as Zoomerons, where they possess the properties of Trappon solitons because they are like particles trapped in a potential well, changing direction indefinitely. For further details about Zoomeron equation, refer to the book [16].

In [16], the authors showed that for each $\phi(u_1, u_2)$ solution of the Zoomeron equation, associates a two-parameter family of the solutions $u(u_1, u_2, t)$ given by

$$u = e^{i(\nu u_1 + \mu u_2 + \mu\nu t)}\phi(u_1 + \mu t, u_2 + \nu t), \quad \rho = \frac{\left|u\right|_{,u_1u_2}}{u}, \tag{2.10}$$

to the Davey-Stewartson III equation

$$iu_{,t} = u_{,u_1u_2} - \rho u, \quad \rho_{,u_1u_1} - \rho_{,u_2u_2} + \left(\left|u\right|^2\right)_{,u_1u_2} = 0. \tag{2.11}$$

This equation was introduced in [12]. It describes the evolution of a three-dimensional wave-packet on water of finite depth in fluid dynamics.

For each solution of the Calapso equation, upon making the transition $u_2 \to iu_2$, we obtain a solution for the Zoomeron equation (2.9). We also obtain solutions for the Davey-Stewartson III equation (2.11) given by (2.10).

Definition 2.2. Consider the flat curve parameterized by the arc length $\alpha : [0, +\infty) \to \mathbb{R}^3$, i.e. $|\alpha'(s)| = 1$. We define the *generalized cylinder* in $\mathbb{R}^3$, $X : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$X(t, s) = \alpha(s) + te_3, \;\; where \;\; e_3 = (0, 0, 1). \tag{2.12}$$

The first fundamental form and principal curvatures of the X are given by $I = ds^2 + dt^2$ and $\lambda_1 = 0$, $\lambda_2 = -k$, where k is the curvature of the curve α given by $k^2 =< \alpha''(s), \alpha''(s) >$. Besides that, the normal map of X is given by $N = \dfrac{1}{k}\alpha''(s)$.

3. Ribaucour Transformation for Isothermic Surface and Solutions for the Calapso Equation. In this section, we first establish a sufficient condition for a Ribaucour transformation to transform one isothermic surface into another. Furthermore, we provide three new solutions for the Calapso equation, each corresponding to an isothermic surface

Theorem 3.1. *Let M be a surface of $\mathbb{R}^3$ parametrized by $X : U \subseteq \mathbb{R}^2 \to M$, without umbilic points and let $\widetilde{M}$ be associated to M by a Ribaucour transformation, such that the normal lines intersect at a distance function h. Assume that $h = \frac{\Omega}{W}$ is not constant along the lines of curvature and consider S and T_i, $1 \le i \le 2$, given by (2.4) and (2.7). In this case, we have* *(i) If* $T_1 + T_2 = \dfrac{2S}{\Omega}$ *or* $T_1 - T_2 = 0$*, then* $\widetilde{M}$ *is a isothermic surface, if and only if* M *is isothermic surface.*

(ii) For each isothermic surface M*, whose first fundamental form is given by* $I = e^{2\varphi}\big(du_1^2 + du_2^2\big)$*, if* $T_1 + T_2 = \dfrac{2S}{\Omega}$*, then the functions*

$$\phi_1 = \frac{\epsilon\sqrt{2}e^{\varphi}\big(2W(S-\Omega T_1) + S\Omega(\lambda_2 - \lambda_1)\big)}{2\Omega S}, \tag{3.1}$$

$$\Phi_1 = \frac{-\epsilon\sqrt{2}e^{\varphi}\big(2W + \Omega(\lambda_2 + \lambda_1)\big)}{2\Omega}, \tag{3.2}$$

are solutions of the Calapso equation. Moreover, if $T_1 - T_2 = 0$*, then the function*

$$\phi_2 = \frac{\epsilon\sqrt{2}e^{\varphi}\big(-2WT_1 - S(\lambda_1 + \lambda_2)\big)}{2S}, \tag{3.3}$$

is also a solution to the Calapso equation, where $\epsilon^2 = 1$ *and* $-\lambda_i$ $1 \le i \le 2$ *are the principal curvatures of the* M*.*

Proof:

(i) Suppose that $\widetilde{M}$ is a isothermic surface, then the coefficients of the first fundamental form of $\widetilde{X}$ satisfy, $\widetilde{g}_{11} = \widetilde{g}_{22}$. So, using (2.6), we have

$$\left(\frac{S - \Omega T_1}{S}\right)^2 g_{11} = \left(\frac{S - \Omega T_2}{S}\right)^2 g_{22}, \tag{3.4}$$

where g_{ii}, $1 \le i \le 2$ are the coefficients of the first fundamental form of X.
If $T_1 + T_2 = \frac{2S}{\Omega}$, then isolating T_1 and substituting in (3.4), we get $g_{11} = g_{22}$.
On the other hand, if $T_1 - T_2 = 0$, then we have from (3.4) that $g_{11} = g_{22}$. Therefore, M is a isothermic surface.
Conversely, suppose that M is a isothermic surface, then using that $T_1 + T_2 = \dfrac{2S}{\Omega}$ or $T_1 - T_2 = 0$, immediately from (2.6), we obtain that $\widetilde{M}$ is a isothermic surface.

(ii) For each isothermic surface M, whose first fundamental form is given by $I = e^{2\varphi}\big(du_1^2 + du_2^2\big)$, let $\widetilde{M}$ be the isothermic surface associated to M by a Ribaucour transformation.
If $T_1 + T_2 = \dfrac{2S}{\Omega}$, then using (2.6) we have that the first fundamental form of $\widetilde{X}$ is given by $\widetilde{I} = \psi^2\big(du_1^2 + du_2^2\big)$, where $\psi = \dfrac{|S - \Omega T_2|e^{\varphi}}{S}$. Additionally, isolating T_1, we have

$$T_1 = \frac{2S - \Omega T_2}{\Omega}. \tag{3.5}$$

Using (2.6), we obtain that the mean and skew curvatures of $\widetilde{X}$ are given by

$$\begin{aligned} \widetilde{H} &= \frac{1}{2\Omega(S - \Omega T_2)}\Big(2W(S - \Omega T_2) + S\Omega(\lambda_2 - \lambda_1)\Big), \\ \widetilde{H}' &= \frac{S}{2\Omega(S - \Omega T_2)}\Big(2W + \Omega(\lambda_2 + \lambda_1)\Big). \end{aligned} \tag{3.6}$$

Therefore, by Theorem 2.2, the functions given by (3.1) and (3.2) are solutions of the Calapso equation.
Proceeding in a similar way, if $T_1 - T_2 = 0$, then using using (2.6), we obtain that the mean and skew curvatures of $\widetilde{X}$ are given by

$$\begin{aligned} \widetilde{H} &= \frac{2WT_2 + S(\lambda_1 + \lambda_2)}{2(S - \Omega T_2)}, \\ \widetilde{H}' &= \frac{S(\lambda_2 - \lambda_1)}{2(S - \Omega T_2)}, \end{aligned} \tag{3.7}$$

Using the Theorem 2.2, we conclude the proof.

□

Next, we make two remarks that will be useful in the subsequent section.

Remark 3.1. *Let $X : U \subseteq \mathbb{R}^2 \to M$ be a parameterization for the isothermic surface M. Thus, the first fundamental form of X is given by $I = e^{2\varphi}(du_1^2 + du_2^2)$. Therefore, the additional relations given by $T_1 + T_2 = \dfrac{2S}{\Omega}$ and $T_1 - T_2 = 0$ are, respectively, equivalent to*

$$\Delta\Omega - e^{2\varphi}(\lambda_1 + \lambda_2)W = \frac{Se^{2\varphi}}{\Omega} \tag{3.8}$$

$$\Omega_{,11} - \Omega_{,22} - 2\varphi_{,1}\Omega_{,1} + 2\varphi_{,2}\Omega_{,2} - e^{2\varphi}(\lambda_1 - \lambda_2)W = 0 \tag{3.9}$$

In fact, under these conditions using (2.2), T_i $1 \le i \le 2$ given by (2.7), can be rewritten as

$$\begin{aligned} T_1 &= \frac{2}{e^{2\varphi}}\Big(\Omega_{,11} - \varphi_{,1}\Omega_{,1} + \varphi_{,2}\Omega_{,2} - W\lambda_1 e^{2\varphi}\Big), \\ T_2 &= \frac{2}{e^{2\varphi}}\Big(\Omega_{,22} - \varphi_{,2}\Omega_{,2} + \varphi_{,1}\Omega_{,1} - W\lambda_2 e^{2\varphi}\Big). \end{aligned} \tag{3.10}$$

Therefore, $T_1 + T_2 = \frac{2}{e^{2\varphi}}\big(\Delta\Omega - e^{2\varphi}W(\lambda_1 + \lambda_2)\big)$ and the additional relation $T_1 + T_2 = \frac{2S}{\Omega}$ is equivalent to (3.8). Finally, by substituting (3.10) in $T_1 - T_2 = 0$, we obtain (3.9).

Remark 3.2. *Let X be as in the previous remark. Then, the parameterization $\widetilde{X}$ of $\widetilde{M}$, locally associated to X by a Ribaucour transformation, given by (2.3), is defined on*

$$V = \{(u_1, u_2) \in U;\ \Omega T_2 - S \neq 0\}.$$

4. Solution of the Calapso, Zoomeron and Davey-Stewartson III Equations. In the section, by applying the Theorem 3.1 tho the generalized cylinder, we provide new explicit solutions of the Calapso, Zoomeron and Davey-Stewartson III Equations (2.8), (2.9) and (2.11).

Theorem 4.1. *For all function $f_2 = f_2(u_2)$ and constants $c \neq 0$, $A_1 > 0$, $b \in \mathbb{R}$, such that $f_2' \neq 0$ and $\dfrac{c^2A_1^2 - (cf_2 - b)^2 - c(f_2')^2}{c} > 0$, the functions*

$$\phi(u_1, u_2) = \frac{\epsilon\sqrt{2}\big[2cf_2'g_2 + g_2'\big(2b + c(f_1 - f_2)\big)\big]}{2f_2'\big(2b + c(f_1 - f_2)\big)}, \tag{4.1}$$

$$\Phi(u_1, u_2) = \frac{\epsilon\sqrt{2}\big(2g_2f_2' - g_2'(f_1 + f_2)\big)}{2f_2'(f_1 + f_2)}, \tag{4.2}$$

are solutions of the Calapso equation (2.8), where $\epsilon^2 = 1$ and the functions f_1 and g_2 are given by

$$f_1 = \begin{cases} A_1\cosh(\sqrt{c}\,u_1) - \frac{b}{c}, & if\ \ c > 0 \\ A_1\sin(\sqrt{-c}\,u_1) - \frac{b}{c}, & if\ \ c < 0 \end{cases} \tag{4.3}$$

$$g_2 = \epsilon_1\sqrt{\frac{c^2A_1^2 - (cf_2 - b)^2 - c(f_2')^2}{c}}, \quad \epsilon_1^2 = 1. \tag{4.4}$$

Moreover,

(i) $\widetilde{\phi}(u_1, u_2) = \phi(u_1, iu_2)$ and $\widetilde{\Phi}(u_1, u_2) = \Phi(u_1, iu_2)$ are solutions of the Zoomeron equation (2.9).

(ii) $u(u_1, u_2, t) = e^{i(\nu u_1 + \mu u_2 + \mu\nu t)}\widetilde{\phi}(u_1 + \mu t, u_2 + \nu t)$, $\rho = \dfrac{|u|_{,u_1u_2}}{u}$ and
$v(u_1, u_2, t) = e^{i(\nu u_1 + \mu u_2 + \mu\nu t)}\widetilde{\Phi}(u_1 + \mu t, u_2 + \nu t)$, $\rho = \dfrac{|v|_{,u_1u_2}}{v}$
are solution for the Davey-Stewartson III equation (2.11).

Proof: Let $k = k(u_2) > 0$ be an arbitrary smooth function. By the fundamental theorem of flat curves, there is $\alpha : [0, +\infty) \to \mathbb{R}^2$, a curve such that, $|\alpha'(u_2)| = 1$ and $k = |\alpha''(u_2)|$.

Consider the generalized cylinder in $\mathbb{R}^3$, $X : \mathbb{R} \times [0, +\infty) \to \mathbb{R}^3$ given by

$$X(u_1, u_2) = \alpha(u_2) + u_1 e_3, \quad where\ \ e_3 = (0, 0, 1). \tag{4.5}$$

The first fundamental form of the X is given by $ds^2 = du_1^2 + du_2^2$ and the principal curvatures $\lambda_1 = 0$, $\lambda_2 = -k$. Using (2.2), to obtain the Ribaucour transformations, we need to solve the following of equations

$$\Omega_{i,j} = 0, \qquad \Omega_{,i} = \Omega_i, \qquad W_{,i} = -\Omega_i \lambda_i, \ \ 1 \leq i \neq j \leq 2.$$

Since $\Omega_{,12} = 0$, it follows that $\Omega = f_1(u_1) + f_2(u_2)$. Therefore $\Omega_1 = f_1'$ and $\Omega_2 = f_2'$. Moreover, $W = g_2$, where $g_2' = k f_2'$. Thus, from (2.4), $S = (f_1')^2 + (f_2')^2 + (g_2)^2$.

Using (3.8), the associated surface will be isothermic when $\Delta\Omega + kW = \frac{S}{\Omega}$. Therefore, we obtain that the functions f_1, f_2 and g_2 satisfy

$$f_1'' + f_2'' + k g_2 = \frac{(f_1')^2 + (f_2')^2 + (g_2)^2}{f_1 + f_2}. \tag{4.6}$$

Differentiating this last equation with respect to x_1 and x_2, we get

$$f_1''' = f_1' \left(\frac{f_1'' - g_2'' - k g_2}{f_1 + f_2} \right), \qquad f_2''' + \big(k g_2\big)' = -f_2' \left(\frac{f_1'' - g_2'' - k g_2}{f_1 + f_2} \right). \tag{4.7}$$

Therefore

$$\frac{f_1'' - f_2'' - k g_2}{f_1 + f_2} = c,$$

where c is a real constant. Thus, we have that f_1, f_2 and g_2 satisfy

$$f_1'' - c f_1 = b, \qquad g_2' = k f_2' \quad \textit{and} \quad f_2'' + c f_2 + k g_2 = b, \tag{4.8}$$

where real constant $c \neq 0$, because if $c = 0$, then $\widetilde{X}$ is degenerate. Moreover, using (4.6) we get

$$\big(f_1'\big)^2 - c f_1^2 - 2b f_1 + \big(f_2'\big)^2 + c f_2^2 - 2b f_2 + g_2^2 = 0. \tag{4.9}$$

Therefore, using (4.6), we have that $S = (2b + c(f_1 - f_2))(f_1 + f_2)$, $W = g_2$ and $\Omega = f_1 + f_2$. Hence, substituting this function in (3.1)-(3.2) and using (3.10), we have from Theorem 3.1 that (4.1) and (4.2) are solution of the Calapso equation. Moreover, by Theorem 2.1 the isothermic surface $\widetilde{X}$ associated to X by a Ribaucour transformation is given by

$$\widetilde{X} = X - \frac{2}{2b + c\big(f_1 - f_2\big)} \left(f_1' e_3 + f_2' \alpha'(u_2) - \frac{g_2}{k} \alpha''(u_2) \right). \tag{4.10}$$

Now, if $c > 0$, then from (4.8) and (4.9), we get

$$f_1 = a_1 \cosh(\sqrt{c}\, u_1) + b_1 \sinh(\sqrt{c}\, u_1) - \frac{b}{c},$$
$$c g_2^2 + (c f_2 - b)^2 + c\big(f_2'\big)^2 = c^2 (a_1^2 - b_1^2).$$

Hence, $a_1^2 > b_1^2$ and we obtain that f_1 can be rewritten as

$$f_1 = A_1 \cosh(\sqrt{c}\, u_1 + B_1) - \frac{b}{c},$$

where $A_1 = \sqrt{a_1^2 - b_1^2}$. The constant B_1, without loss of generality, my be considered to be zero. One can verify that the surfaces with different values of B_1 are congruent by rigid motions of R^3. In fact, using the notation $\widetilde{X}_{B_1}$ for the surface $\widetilde{X}$ with fixed constant B_1, we have

$$\widetilde{X}_{B_1} = \widetilde{X}_0 \circ h + T_{\frac{-B_1}{\sqrt{c}}},$$

where $h(u_1, u_2) = \left(u_1 + \dfrac{B_1}{\sqrt{c}}, u_2 \right)$ and T_δ is the translation $T_\delta(x, y, z) = (x, y, z + \delta)$.

With an analogous argument, if $c < 0$, from (4.8) and (4.9), we get

$$f_1 = A_1 \sin(\sqrt{-c}\, u_1) - \frac{b}{c},$$
$$c g_2^2 + (c f_2 - b)^2 + c\big(f_2'\big)^2 = c^2 A_1^2.$$

Therefore, f_1 and g_2 are given by (4.3) and (4.4).

Finally, using the Remark 2.1, we obtain (i) and (ii). □

Theorem 4.2. *For all function $f_2 = f_2(u_2)$ and constants $b,\ c_1 \in \mathbb{R}$, such that $f_2' \neq 0$ and $c_1 + 2bf_2 - (f_2')^2 > 0$, the function*

$$\phi = \frac{2\epsilon\, b g_2\sqrt{2}}{b^2u_1^2 + c_1 + 2bf_2} - \frac{\epsilon\, g_2'\sqrt{2}}{2f_2'}, \tag{4.11}$$

is solution of the Calapso equation (2.8), where $\epsilon^2 = 1$ and the function g_2 is given by

$$g_2 = \epsilon_1\sqrt{c_1 - (f_2')^2 + 2bf_2}, \quad \epsilon_1^2 = 1. \tag{4.12}$$

Moreover,

(i) $\widetilde{\phi}(u_1,u_2) = \phi(u_1, iu_2)$ and $\widetilde{\Phi}(u_1,u_2) = \Phi(u_1, iu_2)$ are solutions of the Zoomeron equation (2.9).

(ii) $u(u_1,u_2,t) = e^{i(\nu u_1+\mu u_2+\mu\nu t)}\widetilde{\phi}(u_1+\mu t, u_2+\nu t)$, $\rho = \dfrac{|u|_{,u_1u_2}}{u}$ and
$v(u_1,u_2,t) = e^{i(\nu u_1+\mu u_2+\mu\nu t)}\widetilde{\Phi}(u_1+\mu t, u_2+\nu t)$, $\rho = \dfrac{|v|_{,u_1u_2}}{v}$
are solution for the Davey-Stewartson III equation (2.11).

Proof: Let $k = k(u_2) > 0$ be an arbitrary smooth function. By the fundamental theorem of flat curves, there is $\alpha : [0,+\infty) \to \mathbb{R}^2$, a curve such that, $|\alpha'(u_2)| = 1$ and $k = |\alpha''(u_2)|$.
Consider the generalized cylinder in $\mathbb{R}^3$, $X : \mathbb{R}\times[0,+\infty) \to \mathbb{R}^3$ given by

$$X(u_1,u_2) = \alpha(u_2) + u_1e_3, \ \ where\ \ e_3 = (0,0,1). \tag{4.13}$$

The first fundamental form of the generalized cylinder $ds^2 = du_1^2 + du_2^2$ and the principal curvatures $\lambda_1 = 0$, $\lambda_2 = -k$. Using (2.2), to obtain the Ribaucour transformations, we need to solve the following of equations

$$\Omega_{i,j} = 0, \qquad \Omega_{,i} = \Omega_i, \qquad W_{,i} = -\Omega_i\lambda_i, \ \ 1 \le i \neq j \le 2.$$

Since $\Omega_{,12} = 0$, it follows that $\Omega = f_1(u_1) + f_2(u_2)$. Therefore $\Omega_1 = f_1'$ and $\Omega_2 = f_2'$. Moreover, $W = g_2$, where $g_2' = kf_2'$. Thus, from (2.4), $S = (f_1')^2 + (f_2')^2 + (g_2)^2$.

Using (3.9), the associated surface will be isothermic when $\Omega_{,11} - \Omega_{,22} - kW = 0$. Therefore, we obtain that the functions f_1, f_2 and g_2 satisfy

$$f_1'' - f_2'' - kg_2 = 0.$$

From this last equation, we get

$$f_1'' = f_2'' + kg_2 = b.$$

Therefore, as in the proof of Theorem 4.1, we obtain that f_1 is given by

$$f_1 = \frac{bu_1^2}{2} + a_1, \tag{4.14}$$

and f_2 and g_2 satisfy

$$f_2'' + kg_2 = b \quad and \quad g_2' = kf_2'. \tag{4.15}$$

Using (4.15), we get $\left(f_2'\right)^2 + g_2^2 - 2bf_2 = c_1$, therefore g_2 is given by (4.12) and $S = b^2u_1^2 + c_1 + 2bf_2$, $W = g_2$ and $\Omega = f_2 + \frac{bu_1^2}{2} + a_1$. Hence, substituting in (3.3), and using (3.10), by Theorem 2.2, we obtain that (4.11) is solution of the Calapso equation. Finally, using the Remark 2.1, we conclude the proof. □

By using the Theorem 4.1 and 4.2, we get the following examples of the solutions of the Calapso, Zoomeron and Davey-Stewartson III Equations.

Example 4.1. *For all $A_1 > \sqrt{2}$ and $c > 0$ we consider $f_2 = \frac{b}{c} + e^{-\sqrt{c}\,u_2}$, where using (4.4), we have that $g_2 = \epsilon_1\sqrt{c(A_1^2 - 2e^{-2\sqrt{c}\,u_2})}$ is defined in $(0,\infty)$, $\epsilon_1^2 = 1$. By Theorem 4.1 we obtain two solutions of the Calapso equation. Such solutions are given by*

$$\phi = \frac{\epsilon c\sqrt{2}\left[A_1^2 - e^{-\sqrt{c}\,u_2}(A_1\cosh(\sqrt{c}\,u_1) + e^{-\sqrt{c}\,u_2}\right]}{\sqrt{c(A_1^2 - 2e^{-2\sqrt{c}\,u_2})}\left(A_1\cosh(\sqrt{c}\,u_1 - e^{-\sqrt{c}\,u_2}\right)}, \tag{4.16}$$

$$\Phi = \frac{\epsilon c\sqrt{2}\left[A_1^2 + e^{-\sqrt{c}\,u_2}(A_1\cosh(\sqrt{c}\,u_1) - e^{-\sqrt{c}\,u_2})\right]}{\sqrt{c(A_1^2 - 2e^{-2\sqrt{c}\,u_2})}\left(A_1\cosh(\sqrt{c}\,u_1) + e^{-\sqrt{c}\,u_2}\right)}. \tag{4.17}$$

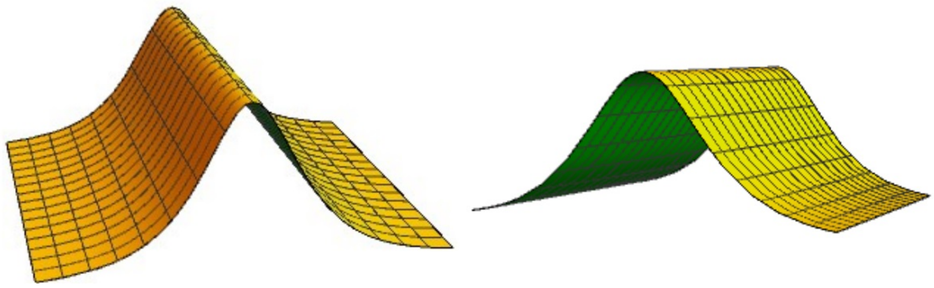

Figure 4.1: Solutions of the Calapso equation given by Example 4.1 with $b = 1$, $A_1 = 3$ and $c = 4$.

Example 4.2. *Let A_1, b and $c > 0$ be real constants, such that $2c_1^2 A_1^2 \geq (b+c)^2$. Consider the function $f_2 = \dfrac{\sqrt{2c_1^2 A_1^2 - (b+c)^2}}{2c} \sin(\sqrt{2c}\, u_2) + \dfrac{b-c}{2c}$, where using (4.4) we obtain that $g_2 = \epsilon_1 \sqrt{c}(f_2 + 1)$ is defined in $\mathbb{R}$, $\epsilon_1^2 = 1$. By Theorem 4.1 we have two solutions of the Calapso equation. Such solutions are given by*

$$\phi = \frac{\epsilon\sqrt{2c}\big[\sqrt{2c_1^2 A_1^2 - (b+c)^2}\sin(\sqrt{2c}\,u_2) + 2cA_1\cosh(\sqrt{c}\,u_1) + 3b + 3c\big]}{2b + 2c + 4cA_1\cosh(\sqrt{c}\,u_1) - 2\sqrt{2c_1^2 A_1^2 - (b+c)^2}\sin(\sqrt{2c}\,u_2)}, \tag{4.18}$$

$$\Phi = \frac{\epsilon\sqrt{2c}\big[\sqrt{2c_1^2 A_1^2 - (b+c)^2}\sin(\sqrt{2c}\,u_2) - 2cA_1\cosh(\sqrt{c}\,u_1) + 3b + 3c\big]}{-2b - 2c + 4cA_1\cosh(\sqrt{c}\,u_1) + 2\sqrt{2c_1^2 A_1^2 - (b+c)^2}\sin(\sqrt{2c}\,u_2)}. \tag{4.19}$$

Note that the isothermic surface associated with this solution of the Calapso equation is a cylinder.

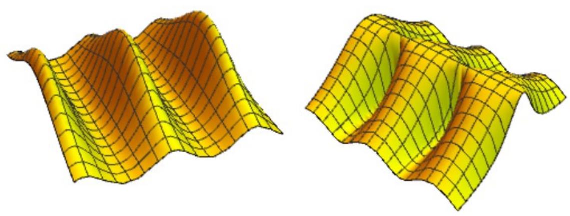

Figure 4.2: Solutions of the Calapso equation given by Example 4.2 with $c = b = 1$ and $A_1 = 2$.

Example 4.3. Let A_1, b be real constants and $c < 0$. Consider $f_2 = \dfrac{1}{2c}\left(-\sqrt{8c^2A_1^2 + 2b^2}\cosh\left(\dfrac{\sqrt{-2c}}{2}\,u_2\right) + b\right)$, where using (4.4) we obtain that $g_2 = \dfrac{-\epsilon_1\sqrt{-2c}}{4c}\left(-2cf_2 + 3b\right)$ is defined in $\mathbb{R}$, $\epsilon_1^2 = 1$. By Theorem 4.1 we have two solutions of the Calapso equation. Such solutions are given by

$$\phi = \frac{\epsilon\sqrt{-c}\big[\sqrt{8c^2A_1^2 + 2b^2}\cosh\big(\frac{\sqrt{-2c}}{2}\,u_2\big) + 3b - 2cA_1\sin(\sqrt{-c}\,u_1)\big]}{2b + 4cA_1\sin(\sqrt{-c}\,x_1) + 2\sqrt{8c^2A_1^2 + 2b^2}\cosh\big(\frac{\sqrt{-2c}}{2}\,u_2\big)}, \tag{4.20}$$

$$\Phi = \frac{\epsilon\sqrt{-c}\big[\sqrt{8c^2A_1^2 + 2b^2}\cosh\big(\frac{\sqrt{-2c}}{2}\,u_2\big) + 3b2cA_1\sin(\sqrt{-c}\,u_1)\big]}{-2b + 4cA_1\sin(\sqrt{-c}\,x_1) - 2\sqrt{8c^2A_1^2 + 2b^2}\cosh\big(\frac{\sqrt{-2c}}{2}\,u_2\big)}. \tag{4.21}$$

Example 4.4. *Consider the solutions of the Calapso equation (4.20) and (4.21). Then, the functions*

$$\widetilde{\phi} = \frac{\epsilon\sqrt{-c}\big[\sqrt{8c^2A_1^2 + 2b^2}\cosh\big(\frac{\sqrt{-2c}}{2}\,u_2\big) + 3b - 2cA_1\sin(\sqrt{-c}\,u_1)\big]}{2b + 4cA_1\sin(\sqrt{-c}\,x_1) + 2\sqrt{8c^2A_1^2 + 2b^2}\cosh\big(\frac{\sqrt{-2c}}{2}\,u_2\big)}, \tag{4.22}$$

$$\widetilde{\Phi} = \frac{\epsilon\sqrt{-c}\big[\sqrt{8c^2A_1^2 + 2b^2}\cosh\big(\frac{\sqrt{-2c}}{2}\,u_2\big) + 3b2cA_1\sin(\sqrt{-c}\,u_1)\big]}{-2b + 4cA_1\sin(\sqrt{-c}\,x_1) - 2\sqrt{8c^2A_1^2 + 2b^2}\cosh\big(\frac{\sqrt{-2c}}{2}\,u_2\big)}, \tag{4.23}$$

are solutions of the Zoomeron equation (2.9). Moreover, the functions

$$u = e^{i(\nu x+\mu y+\mu\nu t)}\widetilde{\phi}(u_1+\mu t, u_2+\nu t), \quad \rho = \frac{|u|_{,xy}}{u}, \tag{4.24}$$

$$u = e^{i(\nu x+\mu y+\mu\nu t)}\widetilde{\Phi}(u_1+\mu t, u_2+\nu t), \quad \rho = \frac{|u|_{,xy}}{u}, \tag{4.25}$$

are solutions of the Davey-Stewartson III equation.

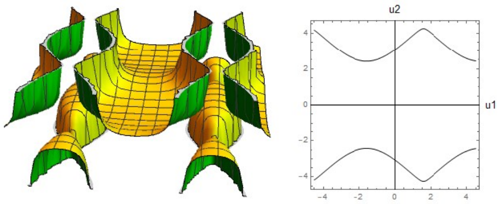

Figure 4.3: Solutions of the Zoomeron equation given by (4.22) in Example 4.4, with $c = -1$, $b = 4$ and $A_1 = \frac{3}{2}$. The curves presented represent the singularities of this solution.

Example 4.5. Let b, b_1, and b_2 be real constants and consider the function

$$f_2 = \frac{u_2}{4}\left(2(b_1+b_2)\cos\big(\ln(u_2)\big) + 2(b_2-b_1)\sin\big(\ln(u_2)\big) + bu_2 + \frac{b_1 b_2}{b}\right) + \frac{b_1 b_2}{b}. \tag{4.26}$$

We define $c_1 = (b_2-b_1)^2$, then using (4.12), we obtain that the

$$g_2 = \epsilon_1\left(b_2\sin\big(\ln(u_2)\big) + b_1\cos\big(\ln(u_2)\big) + \frac{bu_2}{2}\right),$$

is defined in $(0,\infty)$, $\epsilon_1^2 = 1$. By Theorem 4.2 we have a solution of the Calapso equation. Such solution is given by

$$\phi = \frac{\epsilon\, 2\sqrt{2}\, b\big(2b_1\sin\big(\ln(u_2)\big) + 2b_2\cos\big(\ln(u_2)\big) + bu_2\big)}{2b^2u_1^2 + 2b_1^2 + 2b_2^2 + bu_2\big(2(b_1+b_2)\cos\big(\ln(u_2)\big) + 2(b_2-b_1)\sin\big(\ln(u_2)\big) + bu_2\big)} - \frac{\epsilon\sqrt{2}}{2u_2}.$$

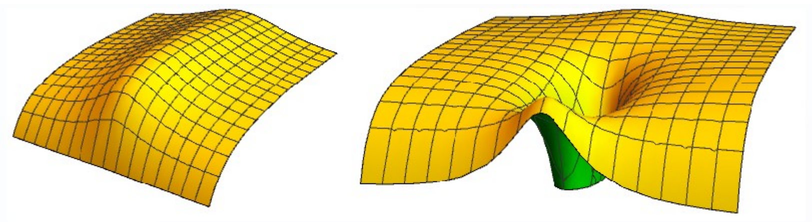

Figure 4.4: Solution of the Calapso equation given by in Example 4.5, where $b_1 = 3$, $b_2 = 5$ and $b = 4$ in the first one and $b = -4$ in the second one.

5. Conclusions. From the results obtained in this work we can make the following conclusions:
For each isothermic surface, using Ribaucour Transformations, we obtain three new solutions for the Calapso equation. For each solution of the Calapso equation, we associate a new solution for the Zoomeron equation and subsequently a new solution for the Davey-Stewartson III equation.

6. Author contributions. The authors of this publication contributed equally in the following aspects: Conceptualization, Corro AMV and Ferro ML; investigation, Corro AMV and Ferro ML; formal analysis, Corro AMV and Ferro ML; metodologia, Corro AMV; validation, Corro AMV and Ferro ML; writing - original draft, Ferro ML; Writing - review and editing, Corro AMV and Ferro ML.

Conflicts of interest. The authors declare no conflict of interest.

ORCID and License

Armando M. V. Corro https://orcid.org/0000-0002-6864-3876
Marcelo Lopes Ferro https://orcid.org/0000-0001-6832-2274

This work is licensed under the Creative Commons - Attribution 4.0 International (CC BY 4.0)

References

[1] Bianchi L. Lezioni di geometria Differenziale. Terza Edicione, Nicola Zanichelli Editore; 1927.
[2] Calapso P. Sulle superficie a linee di curvatura isoterme. Palermo Rend. 1903; 17:275-286.
[3] Changyan S, Heming F, Chengfa W. Soliton solutions to the reverse-time nonlocal Davey-Stewartson III equation. Wave Motion. 2021; 104: 102744. DOI: https://doi.org/10.1016/j.wavemoti.2021.102744.
[4] Corro AMV, Ferreira WP, Tenenblat K. On Ribaucour transformations for hypersurfaces. Mat. Contemp. 1999; 17:137-160.
[5] Corro AMV, Ferreira WP, Tenenblat K. Ribaucour transformations for Constant mean curvature and linear weingarten surfaces. Pacific Journal of Mathematics. 2003; 212(2):265-296.
[6] Corro AMV, Ferreira WP, Tenenblat K. Minimal surfaces obtained by Ribaucour transformations. Geometriae Dedicata, Netherlands. 2003; 96(1):117-150.
[7] Corro AMV, Tenenblat K. Ribaucour transformation revisited, Comum. Geom. 2004; 12(5):1055-1082.
[8] Corro AMV, Fernandes KV, Riveros CMC. Isothermic surfaces and solutions of the Calapso equation. Serdica Math. J. 2018; 44:341-364.
[9] Corro AMV, Ferro ML. New Isothermic Surfaces. arXiv. 2021; Available from: https://doi.org/10.48550/arXiv.2103.04679.
[10] Corro, AMV, Ferro ML. New Isothermic Surfaces in $\mathbb{S}^3$. Selecciones Matemáticas. 2023; 10(1):114-128.
[11] Corro, AMV, Ferro ML, Riveros CMC. ϵ-isothermic surfaces in pseudo-Euclidean 3-space. Selecciones Matemáticas. 2023; 10(1):129-146.
[12] Davey A, Stewartson K. On Three-dimensional Packets of Surface Waves. Proc. R. Soc. Lond. A. 1974; 338;101-110.
[13] Ferreira WP, Tenenblat K. Hypersurfaces with flat r-mean curvature and Ribaucour transformations. International Journal of Applied Mathematics and Statistics. 2007; 11:38-51.
[14] Heming F, Chenzhen R, Weiying H. Soliton soltuions to the nonlocal Davey-Stewartson III equation. Mod. Phys. Lett. B. 2021;35(1) 2150026. DOI: 10.1142/S0217984921500263
[15] Lemes MV, Tenenblat K. On Ribaucour transformations and minimal surfaces. Matemática Contemporânea. 2005; 29:13-40.
[16] Rogers R, Schief WK. Bäcklund and Darboux transformations. Cambridge University Press; 2002.
[17] Tenenblat K. On Ribaucour transformations and applications to linear Weingarten surfaces. Anais da Academia Brasileira de Ciências (Impresso), ABC. 2002; 74:559-575.
[18] Tenenblat K, Wang Q. Ribaucour Transformations for Hypersurfaces in Space Forms. Annals of Global Analysis and Geometry. 2006; 29:157-185.