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ϵ-isothermic surfaces in pseudo-Euclidean 3-space

Armando M. V. Corro*, Carlos M. C. Riveros† and Marcelo L. Ferro‡

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Abstract

In this paper, we provide a class of surfaces called ϵ-isothermic surface *in the pseudo-Euclidean 3-space and we introduce the* pseudo-Calapso equation. *We prove that for each ϵ-isothermic surface, we can associate two solutions to the pseudo-Calapso equation. In particular, we associate solutions to the Calapso, Zoomeron and Davey-Stewartson III equations. In sequence, we classify the Dupin surfaces in pseudo-Euclidean 3-space having distinct principal curvatures and provide explicit coordinates for such surfaces. As application of the theory, we obtain explicit solutions to the pseudo-Calapso equation and from these solutions, we provide new explicit solutions of the Zoomeron and Davey-Stewartson III equations. Moreover, we also provide explicit solutions to these equations that depend on ϵ_2-holomorphic functions.*

Keywords . Dupin surfaces, Isothermic surfaces, lines of curvature.

1. Introduction. The research of isothermic surfaces is one of the most common more difficult problems of differential geometry and depends on the integration of an equation with fourth-order partial derivatives (see [40]). Particular classes of these surfaces are known and some transformations by means of which it is possible to deduce from isothermic surfaces other isothermic surfaces. All this is known indirectly and independently of the fourth-order differential equation, because it is difficult to integrate.

The theory of isothermic surfaces has a great development for eminent geometers as Christoffel [20], Darboux [21, 22] and Bianchi [1] among others. In the last decades, the theory woke up interest by his connection with the modern theory of integrated systems, see [14, 15], [35], [36] and [38]. Particular classes of isothermic surfaces are the constant mean curvature surfaces, quadrics, surfaces whose lines of curvature has constant geodesic curvature, in particular, the cyclides of Dupin. Transformations of $\mathbb{R}^3$ that preserve isothermic surfaces are isometries, dilations and inversions.

In [4], the authors study surfaces with harmonic inverse mean curvature (HIMC surfaces), they distinguish a subclass of θ-isothermic surfaces, which is a generalization of the isothermic HIMC surfaces, and classify all the θ-isothermic HIMC surfaces, note that when $\theta = 0$, the surfaces are isothermic.

In [15], the author show that theory of soliton surfaces, modified in an appropriate way, can be applied also to isothermic immersions in $\mathbb{R}^3$. In this case the so called Sym's formula gives an explicit expression for the isothermic immersion with prescribed fundamental forms. The complete classification of the isothermic surfaces is an open problem.

In [5], the author establishes an equation with fourth order partial derivatives from which the problem of obtaining isothermic surfaces apparently becomes much simpler. Such equation (called Calapso equation) defined in [5] given by

$$\Delta\left(\frac{\phi_{,12}}{\phi}\right) + (\phi^2)_{,12} = 0,$$

*Instituto de Matemática e Estatística, Universidade Federal de Goiás, Goiânia-GO, Brazil. (avcorro@gmail.com).
†Departamento de Matemática, Universidade de Brasília, Brasília, DF, Brasil. (carlos@mat.unb.br).
‡Instituto de Matemática e Estatística, Universidade Federal de Goiás, Goiânia-GO, Brazil. (marceloferro@ufg.br).

describes isothermic surfaces in $\mathbb{R}^3$, where $\phi_{,12}$ denotes the derivative of ϕ with respect to u_1 and u_2. The Calapso equation is very difficult to solve and is strongly connected to the Painleve ODEs, some authors have found solutions of this equation associated with constant mean curvature surfaces.

It is noted that the transition $u_2 \to iu_2$ takes the Calapso equation to the Zoomeron equation

$$\Delta_{-1}\left(\frac{\phi_{,12}}{\phi}\right) + (\phi^2)_{,12} = 0,$$

where $\Delta_{-1} = \frac{\partial^2}{\partial u_1^2} - \frac{\partial^2}{\partial u_2^2}$. Another equation that also very difficult to solve. The solitons of this equation are referred to as Zoomerons, where they possess the properties of Trappon solitons because they are like particles trapped in a potential well, changing direction indefinitely.

In [23], the authors introduce the Davey-Stewartson (DS) equation. This equation describe the evolution of a three-dimensional wave-packet on water of finite depth in the fluid dynamics.

In [35] (page 196), the authors show that for each $\phi(u_1, u_2)$ solution of the Zoomeron equation, associates a two-parameter family of the solutions $u(u_1, u_2, t)$ given by

$$u = e^{i(\nu u_1 + \mu u_2 + \mu\nu t)}\phi(u_1 + \mu t, u_2 + \nu t), \quad \rho = \frac{|u|_{,12}}{u},$$

to the Davey-Stewartson III equation

$$iu_{,t} = u_{,12} - \rho u, \quad \Delta_{-1}\rho + \left(|u|^2\right)_{,12} = 0.$$

Recently, a study on the fully PT-symmetric nonlocal Davey-Stewartson III equation was reported in [26] and [16], the authors investigated the reverse-time nonlocal of this equation.

In [18], the authors introduce the class of radial inverse mean curvature surfaces (RIMC-surfaces), which are isothermic surfaces. In addition, they show that for each isothermic surface there is another solution to the Calapso equation which depends on the metric and on the skew curvature of the surface. This solution is different from the one presented in [5].

Dupin surfaces were first studied by Dupin in 1822. A surface is said to be Dupin if each principal curvature is constant along its corresponding surface of curvature. A Dupin submanifold M is said to be proper if the number g of distinct principal curvatures is constant on M. The simplest Dupin submanifolds are the isoparametric hypersurfaces, that is, those whose principal curvatures are constant.

Dupin's surfaces in Euclidean space are classified. There are several equivalent definitions of Dupin cyclides, for example, in Euclidean space, they can be defined as any inversion of a torus, cylinder or double cone, i.e, Dupin cyclide is invariant under Möbius transformations. Classically the cyclides of Dupin were characterized by the property that both sheets of the focal set are curves. Another equivalent definition says that such surfaces can also be given as surfaces that are the envelope of two families at 1-parameter spheres (including planes as degenerate spheres). For more on Dupin cyclides see [2, 3].

In this paper, motivated by [5] and [18] we provide a class of surfaces called ϵ-isothermic surfaces in the pseudo-Euclidean 3-space and we introduce the pseudo-Calapso equation

$$\Delta_\epsilon\left(\frac{\phi_{,12}}{\phi}\right) + \epsilon_1\epsilon_3(\phi^2)_{,12} = 0,$$

where $\Delta_\epsilon = \frac{\partial^2}{\partial u_1^2} + \epsilon\frac{\partial^2}{\partial u_2^2}$, $\epsilon_1^2 = \epsilon_2^2 = \epsilon_3^2 = 1$, $\epsilon = \epsilon_1\epsilon_2$.

We show that for each ϵ-isothermic surface of the pseudo-Euclidean 3-space, we can associate to these surfaces two solutions to the pseudo-Calapso equation. Furthermore, for each solution of the pseudo-Calapso equation, we have in particular a solution of the Calapso or Zoomeron equations. Consequently, we obtain solutions of the Davey-Stewartson III equation. In sequence, we consider those proper Dupin surface of the pseudo-Euclidean 3-space having distinct principal curvatures, parametrized by lines of curvature. We prove that every Dupin surface parametrized by lines of curvature is a ϵ-isothermic surface and provide explicit coordinates for such surfaces. As application of the theory, we give explicit solutions of the pseudo-Calapso, that depend on two functions, each one defined in a given variable. In particular, we provide new explicit solutions of the Calapso, Zoomeron and Davey-Stewartson III equations. Moreover, we also provide explicit solutions to these equations that depend on ϵ_2-holomorphic functions.

2. ϵ-isothermic surfaces and the pseudo-Calapso equation. In this section, we briefly review the main definitions, we describe the ϵ-isothermic surfaces in the pseudo-Euclidean 3-space and we obtain the pseudo-Calapso equation.

We consider E^3 as the pseudo-Euclidean 3-space, i.e, $\mathbb{R}^3$ equipped with the metric $\langle , \rangle$, given by

$$\langle (x_1, y_1, z_1), (x_2, y_2, z_2) \rangle = \delta_1 x_1 x_2 + \delta_2 y_1 y_2 + \delta_3 z_1 z_2$$

where $\delta_i^2 = 1, 1 \le i \le 3$.

In particular, making $\delta_1 = 1$, we consider $\mathbb{C}_{\delta_2} = \{z = u_1 + iu_2 / u_1, u_2 \in \mathbb{R}, i^2 = -\delta_2, \delta_2 = \pm 1\}$ with the metric $\langle z, z \rangle_{\delta_2} = u_1^2 + \delta_2 u_2^2$. We note that when $\delta_2 = 1$ we have the complex numbers and when $\delta_2 = -1$ we have the Lorentz numbers.

Definition 2.1. We define the function $f : \mathbb{C}_{\delta_2} \to \mathbb{C}_{\delta_2}$, $f(z) = f(u_1 + iu_2) = u(u_1, u_2) + iv(u_1, u_2)$ as δ_2-*holomorphic* if and only if $u_{,1} = v_{,2}$ and $u_{,2} = -\delta_2 v_{,1}$ (see [27]). Here the subscript $_{,i}$ denotes the derivative with respect to u_i.

Definition 2.2. A surface M in pseudo-Euclidean 3-space is called *ϵ-isothermic surface* if it admits parametrization by lines of curvature and the first fundamental form is conformal to the metric $\epsilon_1 du_1^2 + \epsilon_2 du_2^2$, where for each $1 \le i \le 2$, $\epsilon_i = \delta_k$ for some $1 \le k \le 3$.

Remark 2.1. In possession of the previous definition, we can have up to three classes of ϵ-isothermic surface in pseudo-Euclidean 3-space. This classes are

(i) The ϵ-isothermic surface M with first fundamental form conformal to the metric $\delta_1 du_1^2 + \delta_2 du_2^2$.
(ii) The ϵ-isothermic surface M with first fundamental form conformal to the metric $\delta_1 du_1^2 + \delta_3 du_2^2$.
(iii) The ϵ-isothermic surface M with first fundamental form conformal to the metric $\delta_2 du_1^2 + \delta_3 du_2^2$.

Remark 2.2. If $X : U \subseteq \mathbb{R}^2 \to E^3$ is a parametrization of a ϵ-isothermic surface M, then the first and the second fundamental forms are given by

$$I = e^{2\varphi}\left(\epsilon_1 du_1^2 + \epsilon_2 du_2^2\right), \; II = l du_1^2 + n du_2^2, \tag{2.1}$$

where for each $1 \le i \le 2$, $\epsilon_i = \delta_k$ for some $1 \le k \le 3$ and $\epsilon_i^2 = 1$.
The Codazzi and Gauss equation are given by

$$\begin{aligned} l_{,2} &= (l + \epsilon n)\varphi_{,2}\,, \\ n_{,1} &= \epsilon(l + \epsilon n)\varphi_{,1}\,, \end{aligned} \tag{2.2}$$

$$\Delta_\epsilon \varphi = -\epsilon_2 \epsilon_3 l n e^{-2\varphi}, \tag{2.3}$$

where $\Delta_\epsilon \varphi = \varphi_{,11} + \epsilon \varphi_{,22}$, with $\epsilon = \epsilon_1 \epsilon_2$.
To integrate the system (2.2) we make the following substitution

$$l = \frac{1}{\sqrt{2}}(\omega + \Omega)e^{\varphi} \;,\; n = \frac{\epsilon}{\sqrt{2}}(\omega - \Omega)e^{\varphi}. \tag{2.4}$$

Thus, by (2.4) the system (2.2) can be written as

$$\Omega_{,1} = \omega_{,1} - (\omega + \Omega)\varphi_{,1}\,, \tag{2.5}$$

$$\Omega_{,2} = -\omega_{,2} + (\omega - \Omega)\varphi_{,2}\,, \tag{2.6}$$

$$\Delta_\epsilon \varphi = \frac{\epsilon_1 \epsilon_3}{2}(\Omega^2 - \omega^2), \tag{2.7}$$

where $\epsilon = \epsilon_1 \epsilon_2$.

Definition 2.3. For each function $\omega = \omega(u_1, u_2)$, we define the *pseudo-Calapso equation* as

$$\Delta_\epsilon \left(\frac{\omega_{,12}}{\omega}\right) + \epsilon_1 \epsilon_3 (\omega^2)_{,12} = 0, \tag{2.8}$$

where $\epsilon = \epsilon_1 \epsilon_2$ and $\epsilon_1^2 = \epsilon_2^2 = \epsilon_3^2 = 1$.

Definition 2.4. Let M be a surface with principal curvatures $-\lambda_1$ and $-\lambda_2$. The *skeaw curvature* of M (see [18]) is given by

$$H' = \frac{\lambda_2 - \lambda_1}{2}. \tag{2.9}$$

Theorem 2.1. *Let $X(u_1, u_2)$ be a ϵ-isothermic surface with first fundamental form given by*

$$I = e^{2\varphi}\left(\epsilon_1 du_1^2 + \epsilon_2 du_2^2\right).$$

Then the functions

$$\omega = \epsilon_1\sqrt{2}e^{\varphi}H \quad \textit{and} \quad \Omega = \epsilon_1\sqrt{2}e^{\varphi}H', \tag{2.10}$$

are solutions of the pseudo-Calapso equation (2.8), where $\epsilon = \epsilon_1\epsilon_2$, H is the mean curvature and H' is the skew curvature of X.

Proof: Differentiating the equation (2.5) with respect to u_2 and the equation (2.6) with respect to u_1, adding and subtracting these expression, we obtain

$$\frac{\omega_{,12}}{\omega} = \varphi_{,12} + \varphi_{,1}\varphi_{,2}\,, \qquad \frac{\Omega_{,12}}{\Omega} = -\varphi_{,12} + \varphi_{,1}\varphi_{,2}\,. \tag{2.11}$$

Let

$$A = \Delta_\epsilon\left(\frac{\omega_{,12}}{\omega}\right) + \epsilon_1\epsilon_3\left(\omega^2\right)_{,12}, \tag{2.12}$$

using the first equation of (2.11) and properties of the Laplacian operator, we get

$$A = \epsilon_1\epsilon_3\left(\omega^2\right)_{,12} + \left(\Delta_\epsilon\varphi\right)_{,12} + \varphi_{,2}\left(\Delta_\epsilon\varphi\right)_{,1} + \varphi_{,1}\left(\Delta_\epsilon\varphi\right)_{,2} + 2\varphi_{,12}\left(\Delta_\epsilon\varphi\right),$$

From (2.7), we have

$$A = \frac{\epsilon_1\epsilon_3}{2}\left[\left(\omega^2+\Omega^2\right)_{,12} + \left(\Omega^2-\omega^2\right)_{,1}\varphi_{,2} + \left(\Omega^2-\omega^2\right)_{,2}\varphi_{,1} + \left(\Omega^2-\omega^2\right)\varphi_{,12}\right].$$

Substituting $(\omega^2+\Omega^2)_{,12}$ and using (2.11), we obtain

$$A = \frac{\epsilon_2}{2}\left[2\left(\omega_{,1}\omega_{,2} + \Omega_{,1}\Omega_{,2}\right) + 2\left(\Omega^2+\omega^2\right)\varphi_{,2}\varphi_{,1} + \left(\Omega^2-\omega^2\right)_{,1}\varphi_{,2} + \left(\Omega^2-\omega^2\right)_{,2}\varphi_{,1}\right].$$

Substituting $(\Omega^2-\omega^2)_{,1}$, $(\Omega^2-\omega^2)_{,2}$ and using (2.5), (2.6) and (2.7), we obtain $A = 0$. Using (2.12) and (2.4), we get that $\omega = \epsilon_1\sqrt{2}e^{\varphi}H$ is a solution to the Pseudo-Calapso equation.
On the other hand, using (2.11) we get

$$\Delta_\epsilon\left(\frac{\omega_{,12}}{\omega}\right) - \Delta_\epsilon\left(\frac{\Omega_{,12}}{\Omega}\right) = 2\Delta_\epsilon\left(\varphi_{,12}\right) = 2\left(\Delta_\epsilon\varphi\right)_{,12} = \epsilon_1\epsilon_2\left(\Omega^2-\omega^2\right)_{,12}.$$

This last equation is equivalent to

$$\Delta_\epsilon\left(\frac{\omega_{,12}}{\omega}\right) + \epsilon_1\epsilon_3\left(\omega^2\right)_{,12} = \Delta_\epsilon\left(\frac{\Omega_{,12}}{\Omega}\right) + \epsilon_1\epsilon_3\left(\Omega^2\right)_{,12}.$$

Since ω is a solution to the pseudo-Calapso equation, then we obtain that $\Omega = \epsilon_1\sqrt{2}e^{\varphi}H'$ is also a solution of the pseudo-Calapso equation. The proof is complete. □

Remark 2.3. Consider the pseudo-Calapso equation given by (2.8). Then, for each solution of this equation, we get a solution of the Calapso or Zoomeron equation. In any case, we have a solution for the Davey-Stewartson II or Davey-Stewartson III equation, respectively.
In fact,

(i) If $\epsilon_1\epsilon_3 = 1$ and $\epsilon = \epsilon_1\epsilon_2 = -1$, then the pseudo-Calapso equation becomes the *Zoomeron equation*

$$\Delta_{-1}\left(\frac{\omega_{,12}}{\omega}\right) + \left(\omega^2\right)_{,12} = 0. \tag{2.13}$$

For each $\omega(u_1,u_2)$ solution of the Zoomeron equation, then the function

$$u = e^{i(\nu u_1+\mu u_2+\mu\nu t)}\omega(u_1+\mu t, u_2+\nu t), \quad \rho = \frac{|u|_{,12}}{u}, \tag{2.14}$$

is a solution of the *Davey-Stewartson III equation* (see [35], page 196)

$$iu_{,t} = u_{,12} - \rho u, \quad \Delta_{-1}\rho + \left(|u|^2\right)_{,12} = 0. \tag{2.15}$$

Note that the Zoomeron equation is the stationary case of the Davey-Stewartson III equation (see [35], page 163).
Moreover, if $\epsilon_1\epsilon_3 = \epsilon_1\epsilon_2 = 1$, then the pseudo-Calapso equation becomes in the Calapso equation. In this case, for each $\omega(u_1,u_2)$ solution to the Calapso equation, if the function $\widetilde{\omega}(u_1,u_2) = \omega(u_1, iu_2)$ is a real function, then $\widetilde{\omega}(u_1,u_2)$ is a real solution of the Zoomeron equation (2.13). Therefore, for each solution of this equation, we get also a solution of the Davey-Stewartson III equation (2.15).

(ii) If $\epsilon_1\epsilon_3 = \epsilon_1\epsilon_2 = -1$, then using the transformation $(u_1, u_2) \to (u_2, u_1)$, we obtain that for each solution $\omega(u_1, u_2)$ of the (2.8), the function $\widetilde{\omega}(u_1, u_2) = \omega(u_2, u_1)$ is a solution of the Zoomeron equation (2.13).
Hence, the function

$$u = e^{i(\nu u_1 + \mu u_2 + \mu\nu t)}\omega(u_2 + \mu t, u_1 + \nu t), \quad \rho = \frac{|u|_{,12}}{u}, \tag{2.16}$$

is a solution of the Davey-Stewartson III equation (2.15).
Moreover, if $\epsilon_1\epsilon_3 = -1$ and $\epsilon_1\epsilon_2 = 1$, then using the transformation $(u_1, u_2) \to (iu_1, u_2)$, we have that the pseudo-Calapso equation becomes the Zoomeron equation (2.13). Therefore, as before, for each solution of this equation, we get a solution of the Davey-Stewartson III equation (2.15).

(iii) Note that the Calapso equation is the stationary case of the *Davey-Stewartson II*

$$iu_{,t} = u_{,12} - \rho u, \quad \Delta\rho + \left(|u|^2\right)_{,12} = 0.$$

Thus, given a solution $\omega(u_1, u_2)$ for the Calapso equation the function $u(u_1, u_2) = e^{it}\omega(u_1, u_2)$ is a particular solution for the Davey-Stewartson II equation.

(iv) We observe that if ω is a solution of the pseudo-Calapso equation then $-\omega$ also is a solution of the same equation.

3. Classification of Dupin surfaces in Pseudo-Euclidean 3-space. In this section, we provide a classification of Dupin surfaces parametrized by lines of curvature in pseudo-Euclidean 3-space E^3, with two distinct principal curvatures.

Definition 3.1. An immersion $X : U \subseteq \mathbb{R}^2 \to E^3$ is a *Dupin surface* if each principal curvature is constant along its corresponding line of curvature.

Let $X : U \subseteq \mathbb{R}^2 \to E^3$, be a proper Dupin surface parametrized by lines of curvature, with distinct principal curvatures, $-\lambda_i$, $1 \le i \le 2$, and let $N : U \subseteq \mathbb{R}^2 \to E^3$ be a unit normal vector field of X. Then

$$\begin{aligned} \langle X_{,i}, X_{,j} \rangle &= \delta_{ij} g_{ij}, \ 1 \le i,j \le 2, && (3.1)\\ N_{,i} &= \lambda_i X_{,i}, \ 1 \le i \le 2, && (3.2)\\ \langle N, N \rangle &= \epsilon_3, \\ \lambda_{i,i} &= 0, \end{aligned}$$

where $\langle , \rangle$ denotes the pseudo-Euclidean 3-space on E^3, $\epsilon_j^2 = 1$, $1 \le j \le 3$.
Moreover, for $1 \le i \ne j \le 2$, we have

$$X_{,ij} - \Gamma^i_{ij} X_{,i} - \Gamma^j_{ij} X_{,j} = 0, \tag{3.3}$$

$$\Gamma^i_{ij} = \frac{\lambda_{i,j}}{\lambda_j - \lambda_i}, \tag{3.4}$$

where Γ^i_{ij} are the Christoffel symbols.
The Christoffel symbols in terms of the metric (3.1) are given by

$$\Gamma^i_{ii} = \frac{g_{ii,i}}{2g_{ii}}, \quad \Gamma^j_{ii} = -\frac{g_{ii,j}}{2g_{jj}}, \quad \Gamma^i_{ij} = \frac{g_{ii,j}}{2g_{ii}}, \tag{3.5}$$

where $1 \le i, j \le 2$ are distinct.
It follows from (3.5), that

$$\Gamma^j_{ii} = -\Gamma^i_{ij}\frac{g_{ii}}{g_{jj}} \tag{3.6}$$

From (3.2) and (3.6), we get

$$X_{,ii} = \Gamma^i_{ii} X_{,i} - \Gamma^i_{ij}\frac{g_{ii}}{g_{jj}} X_{,j} - \epsilon_3 \lambda_i g_{ii} N. \tag{3.7}$$

Theorem 3.1. *Let $X : U \subseteq \mathbb{R}^2 \to E^3$, be a Dupin surface parametrized by lines of curvature, with two distinct principal curvatures $-\lambda_1$ and $-\lambda_2$. Then there is a change in each coordinate separately so that the first fundamental form is given by*

$$I = \frac{1}{(\lambda_2 - \lambda_1)^2}\left(\epsilon_1 du_1^2 + \epsilon_2 du_2^2\right), \tag{3.8}$$

i.e. X is a ϵ-isothermic surface.

Proof: Using (3.4) and (3.5), we have

$$\frac{\lambda_{1,2}}{\lambda_2-\lambda_1}=\frac{g_{11,2}}{2g_{11}}, \qquad \frac{\lambda_{2,1}}{\lambda_1-\lambda_2}=\frac{g_{22,1}}{2g_{22}}.$$

These last two equations can be rewritten as

$$\left[\ln\left(\frac{1}{\lambda_2-\lambda_1}\right)^2\right]_{,2}=\left(\ln|g_{11}|\right)_{,2}, \qquad \left[\ln\left(\frac{1}{\lambda_2-\lambda_1}\right)^2\right]_{,1}=\left(\ln|g_{22}|\right)_{,1},$$

so that

$$g_{11}=\epsilon_1\left(\frac{f_1}{\lambda_2-\lambda_1}\right)^2, \qquad g_{22}=\epsilon_2\left(\frac{f_2}{\lambda_2-\lambda_1}\right)^2,$$

where $\epsilon_i^2=1$, f_i $1\leq i\leq 2$ are an arbitrary real functions of the variable u_i.

Therefore, using the change of coordinates $d\widetilde{u}_i=f_i du_i$, we have that X is ϵ-isothermic. □

From now on we will consider surfaces parametrized by lines of curvature with ϵ-isothermic parameters whose first fundamental form is given by (3.8) and we will get all the Dupin surfaces with two distinct principal curvatures $-\lambda_1$ and $-\lambda_2$.

Theorem 3.2. *Let $X:U\subseteq\mathbb{R}^2\to E^3$, be a ϵ-isothermic Dupin surface with two distinct principal curvatures $-\lambda_1$ and $-\lambda_2$. Then*
(i) $\lambda_1\lambda_2=0$,
or
(ii)

1. *If $b_1\neq 0$ and $(\epsilon_3+b_1)\neq 0$, the principal curvatures are given by*

$$\begin{aligned}\lambda_2 &= -\frac{\epsilon_1 c_1}{b_1}+A_1 e^{\sqrt{b_1\epsilon_1}u_1}+A_2 e^{-\sqrt{b_1\epsilon_1}u_1},\\ \lambda_1 &= -\frac{\epsilon_1 c_1}{\epsilon_3+b_1}+A_3 e^{\sqrt{-\epsilon_2(\epsilon_3+b_1)}u_2}+A_4 e^{-\sqrt{-\epsilon_2(\epsilon_3+b_1)}u_2}.\end{aligned} \tag{3.9}$$

where

$$\epsilon_3 c_1^2-4b_1(\epsilon_3+b_1)(b_1A_1A_2-(\epsilon_3+b_1)A_3A_4)=0. \tag{3.10}$$

2. *If $b_1=0$, the principal curvatures are given by*

$$\begin{aligned}\lambda_2 &= \frac{c_1}{2}u_1^2+A_5u_1+A_6,\\ \lambda_1 &= -\epsilon_3\epsilon_1 c_1+A_7\cos(\sqrt{\epsilon_3\epsilon_2}u_2)+A_8\sin(\sqrt{\epsilon_3\epsilon_2}u_2).\end{aligned} \tag{3.11}$$

where

$$\epsilon_1(A_5^2-2c_1A_6)+\epsilon_3(A_7^2+A_8^2-c_1^2)=0. \tag{3.12}$$

3. *If $b_1=-\epsilon_3$, the principal curvatures are given by*

$$\begin{aligned}\lambda_2 &= \epsilon_1\epsilon_3 c_1+B_5\cos(\sqrt{\epsilon_3\epsilon_1}u_1)+B_6\sin(\sqrt{\epsilon_3\epsilon_1}u_1),\\ \lambda_1 &= \frac{-\epsilon_3\epsilon_1\epsilon_2 c_1}{2}u_2^2+B_7u_2+B_8.\end{aligned} \tag{3.13}$$

where

$$\epsilon_2B_7^2+2\epsilon_1c_1B_8+\epsilon_3(B_5^2+B_6^2-c_1^2)=0. \tag{3.14}$$

Conversely, if $\lambda_i:U\subseteq\mathbb{R}^2\to\mathbb{R}$, $1\leq i\leq 2$, are distinct real functions at each point satisfying the condition (i) or (ii), then there is a ϵ-isothermic Dupin surface $X:U\subseteq\mathbb{R}^2\to E^3$, whose principal curvatures are the functions $-\lambda_i$.

Proof: From Theorem 3.1, the first fundamental form of X is given by (3.8) and by Gauss equation (2.3) with $\varphi=-\frac{1}{2}\ln(\lambda_2-\lambda_1)^2$, using (3.4) we have

$$\frac{\epsilon_3\lambda_1\lambda_2}{\left(\lambda_2-\lambda_1\right)^2}+\epsilon_2\Gamma^1_{12,2}+\epsilon_1\Gamma^2_{12,1}=0. \tag{3.15}$$

If λ_1 and λ_2 are constant, then using (3.4), we have $\Gamma_{12}^1 = 0$ and $\Gamma_{12}^2 = 0$. Thus we obtain $\lambda_1\lambda_2 = 0$.

If $\lambda_1 = h_2(u_2)$ and $\lambda_2 = h_1(u_1)$, with $h_1'(u_1) \neq 0$, using (3.4), we obtain

$$\Gamma_{12,1}^2 = \left(\frac{h_1'}{h_2 - h_1}\right)_{,1} = \frac{h_1''}{h_2 - h_1} + \left(\frac{h_1'}{h_2 - h_1}\right)^2,$$
$$\Gamma_{12,2}^1 = \left(\frac{h_2'}{h_1 - h_2}\right)_{,2} = \frac{h_2''}{h_1 - h_2} + \left(\frac{h_2'}{h_1 - h_2}\right)^2.$$

Therefore,(3.15) can be rewritten as,

$$\epsilon_3 h_1 h_2 + \epsilon_2 h_2''(h_1 - h_2) + \epsilon_1 h_1''(h_2 - h_1) + \epsilon_1 \big(h_1'\big)^2 + \epsilon_2 \big(h_2'\big)^2 = 0. \tag{3.16}$$

Differentiating (3.16) with respect to u_1 and using that $h_1' \neq 0$, we have

$$\frac{\epsilon_1 h_1'''}{h_1'} = \frac{1}{h_1 - h_2}\Big[\epsilon_3 h_2 + \epsilon_2 h_2'' + \epsilon_1 h_1''\Big]. \tag{3.17}$$

Differentiating (3.17) with respect to u_1, we get

$$\left(\epsilon_1 \frac{h_1'''}{h_1'}\right)_{,1} = 0.$$

Therefore

$$h_1'' - \epsilon_1 b_1 h_1 = c_1, \tag{3.18}$$

where b_1 and c_1 are constants.
Substituting (3.18) in (3.17), we have

$$h_2'' + \epsilon_2(\epsilon_3 + b_1)h_2 = -\epsilon_1\epsilon_2 c_1. \tag{3.19}$$

1. If $b_1 \neq 0$ and $(\epsilon_3 + b_1) \neq 0$, the solutions of (3.18) and (3.19) are given by (3.9). Using (3.9) in (3.16) we get (3.10).
2. If $b_1 = 0$, the solutions of (3.18) and (3.19) are given by (3.11). Using (3.11) in (3.16) we get (3.12).
3. If $b_1 = -1$, the solutions of (3.18) and (3.19) are given by (3.13). Using (3.13) in (3.16) we get (3.14).

If $\lambda_1 = h_2(u_2)$ and $\lambda_2 = h_1(u_1)$, with $h_2'(u_2) \neq 0$, with similarly calculus, differentiating (3.16) with respect u_2, we obtain

$$h_2'' - \epsilon_2 b_2 h_2 = c_2, \;\; h_1'' + \epsilon_1(\epsilon_3 + b_2)h_1 = -\epsilon_2\epsilon_1 c_2. \tag{3.20}$$

Defining $b_2 = -\epsilon_3 - b_1$ and $c_1 = -\epsilon_2\epsilon_1 c_2$, then these last two equations coincide with (3.18) and (3.19).

Conversely, let $\lambda_i : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, $1 \leq i \leq 2$, be distinct real functions at each point, given by (i) or (ii). Consider the quadratic forms

$$I = \frac{1}{(\lambda_1 - \lambda_2)^2}\Big(\epsilon_1 du_1^2 + \epsilon_2 du_2^2\Big), \;\; II = \frac{1}{(\lambda_1 - \lambda_2)^2}\Big(- \lambda_1\epsilon_1 du_1^2 - \lambda_2\epsilon_2 du_2^2\Big). \tag{3.21}$$

After very long computations we can show that these functions satisfy the Gauss equation (3.15) and the Coddazi equation (3.4). Therefore, by the fundamental theorem of surfaces, there is a surface whose first and second quadratic forms are given by (3.21). As the functions λ_i given by (i) or (ii) satisfy $\lambda_{i,i} = 0$, $1 \leq i \leq 2$, and by (3.21), we have that surface is a ϵ-isothermic Dupin surface with two distinct principal curvatures $-\lambda_1$ and $-\lambda_2$. □

Theorem 3.3. *Let X be a ϵ-isothermic Dupin surface as in Theorem 3.2. If $-\lambda_1$ and $-\lambda_2$ are constant. Then, up to an isometry of E^3, X is given by*

$$\begin{aligned} X &= \frac{1}{c}\Big(\big(\cos\big(\sqrt{\epsilon_1\epsilon_3}\, u_1\big) - 1\big)\, e_{\epsilon_3} + \epsilon_1\epsilon_3\sqrt{\epsilon_1\epsilon_3}\sin\big(\sqrt{\epsilon_1\epsilon_3}\, u_1\big)\, e_{\epsilon_1} + u_2\, e_{\epsilon_2}\Big), \\ &\textit{or} \\ X &= \frac{1}{c}\Big(\big(\cos\big(\sqrt{\epsilon_2\epsilon_3}\, u_2\big) - 1\big)\, e_{\epsilon_3} + \epsilon_2\epsilon_3\sqrt{\epsilon_2\epsilon_3}\sin\big(\sqrt{\epsilon_2\epsilon_3}\, u_2\big)\, e_{\epsilon_1} + u_1\, e_{\epsilon_2}\Big), \end{aligned} \tag{3.22}$$

where $c \neq 0$ is a real constant, $\epsilon_1^2 = \epsilon_2^2 = \epsilon_3^2 = 1$ and $\langle e_{\epsilon_i}, e_{\epsilon_j} \rangle = \epsilon_i \delta_{ij}$, $1 \leq i, j \leq 3$. Moreover, the Gauss map is, respectively, given by

$$N = \epsilon_1\epsilon_3\sqrt{\epsilon_1\epsilon_3}\sin\left(\sqrt{\epsilon_1\epsilon_3}\,u_1\right)e_{\epsilon_1} + \cos\left(\sqrt{\epsilon_1\epsilon_3}\,u_1\right)e_{\epsilon_3}, \tag{3.23}$$

or

$$N = \epsilon_2\epsilon_3\sqrt{\epsilon_2\epsilon_3}\sin\left(\sqrt{\epsilon_1\epsilon_3}\,u_2\right)e_{\epsilon_1} + \cos\left(\sqrt{\epsilon_2\epsilon_3}\,u_2)\right)e_{\epsilon_3}. \tag{3.24}$$

Proof: Let X be a ϵ-isothermic Dupin surface as in Theorem 3.2 with constant principal curvature $-\lambda_1$ and $-\lambda_2$. By item (i) of the Theorem 3.2, suppose that $\lambda_2 = 0$ and $\lambda_1 = c \neq 0$, then by Theorem 3.1, the first fundamental form of X is $I = \dfrac{1}{c^2}\left(\epsilon_1 du_1^2 + \epsilon_2 du_2^2\right)$, $\epsilon_i^2 = 1$, $1 \leq i \leq 2$ and X satisfy

$$\begin{aligned} X_{,12} &= (0,0,0),\\ X_{,11} &= \frac{-\epsilon_1\epsilon_3 N}{c},\\ X_{,22} &= (0,0,0),\\ N_{,1} &= cX_{,1}\,,\\ N_{,2} &= (0,0,0). \end{aligned} \tag{3.25}$$

Using the last two equations of (3.25), we have

$$N = cX + H_2 \text{ and } N = H_1, \text{ where } H_i = H_i(u_i),\ i = 1, 2, \tag{3.26}$$

are vector valued functions. Therefore

$$X = G_1 - G_2, \tag{3.27}$$

where $G_i = \dfrac{H_i}{c}, i = 1, 2$ are vector valued functions in E^3.

Differentiating (3.27), we have $X_{,11} = G_1''$ and $X_{,22} = -G_2''$.
Thus, follows from the second and thirty equation of (3.25) that $G_1'' = -\epsilon_3\epsilon_1 G_1$ and $G_2'' = 0$.

Finally, given initial conditions $X_{,i}(0,0) = \dfrac{e_{\epsilon_i}}{c}$, $i = 1, 2$, $X(0,0) = (0,0,0)$, $N(0,0) = e_{\epsilon_3}$, we get $G_1'(0) = \dfrac{e_{\epsilon_1}}{c}$, $G_2'(0) = -\dfrac{e_{\epsilon_2}}{c}$ and $G_i(0) = \dfrac{e_{\epsilon_3}}{c}$, where e_{ϵ_k} is a vector of E^3, such that, $\langle e_{\epsilon_k}, e_{\epsilon_k} \rangle = \epsilon_k$.
Hence, G_1 and G_2 are given by

$$G_2 = \frac{1}{c}\left(e_{\epsilon_3} - u_2\, e_{\epsilon_2}\right),\ G_1 = \frac{1}{c}\left(\epsilon_1\epsilon_3\sqrt{\epsilon_1\epsilon_3}\,\sin\left(\sqrt{\epsilon_1\epsilon_3}\,u_1\right)e_{\epsilon_1} + \cos\left(\sqrt{\epsilon_1\epsilon_3}\,u_1\right)e_{\epsilon_3}\right).$$

Therefore, using (3.27), we get (3.22) and from (3.26), the Gauss map is given by (3.23).

On the other hand, if we assume that $\lambda_1 = 0$ and $\lambda_2 = c \neq 0$, then similarly to the previous case, we obtain the expressions (3.22) and (3.24). The proof is complete. □

Theorem 3.4. *Let X be a ϵ-isothermic Dupin surface as in Theorem 3.2. If the principal curvatures $-\lambda_1$ or $-\lambda_2$ are not constant. Then, up to an isometry of E^3, X is given by*

$$X = \frac{G_2 - G_1}{\lambda_2 - \lambda_1}, \tag{3.28}$$

where the vector valued functions $G_i(x_i)$, $1 \leq i \leq 2$, are given by

1. *If $b_1 \neq 0$ and $(\epsilon_3 + b_1) \neq 0$, then*

$$\begin{aligned} G_1 &= \frac{\epsilon_1}{b_1}\left(\cosh\left(\sqrt{b_1\epsilon_1}\,u_1\right) - 1\right)V_1 + \frac{\epsilon_1\sqrt{b_1\epsilon_1}\sinh\left(\sqrt{b_1\epsilon_1}\,u_1\right)}{b_1}e_{\epsilon_1} + \cosh\left(\sqrt{b_1\epsilon_1}\,u_1\right)e_{\epsilon_3},\\ G_2 &= \frac{\epsilon_1}{\epsilon_3 + b_1}\left(\cosh\left(\sqrt{-\epsilon_2(\epsilon_3 + b_1)}\,u_2\right) - 1\right)V_1 + \cosh\left(\sqrt{-\epsilon_2(\epsilon_3 + b_1)}\,u_2\right)e_{\epsilon_3}\\ &\quad - \frac{\epsilon_2\sqrt{-\epsilon_2(\epsilon_3 + b_1)}\sinh\left(\sqrt{-\epsilon_2(\epsilon_3 + b_1)}\,u_2\right)}{\epsilon_3 + b_1}e_{\epsilon_2} \end{aligned} \tag{3.29}$$

where

$$\begin{aligned} V_1 &= \frac{\epsilon_1 b_1(\epsilon_3+b_1)}{D_1}\Big(\epsilon_1(A_1-A_2)\sqrt{b_1\epsilon_1}\,e_{\epsilon_1}+\epsilon_2(A_3-A_4)\sqrt{-\epsilon_2(\epsilon_3+b_1)}\,e_{\epsilon_2}\\ &\quad+\big((\epsilon_3+b_1)(A_3+A_4)-b_1(A_1+A_2)\big)\,e_{\epsilon_3}\Big), \qquad (3.30)\\ D_1 &= -\epsilon_3\epsilon_1 c_1+b_1(\epsilon_3+b_1)(A_1+A_2-A_3-A_4)\neq 0, \end{aligned}$$

and the constants satisfy (3.10).

2. *If* $b_1=0$, *then*

$$G_1 = \frac{u_1^2}{2}\,V_2+u_1\,e_{\epsilon_1}+e_{\epsilon_3}, \qquad (3.31)$$

$$G_2 = \epsilon_3\epsilon_1\big(-1+\cos\big(\sqrt{\epsilon_3\epsilon_2}\,u_2\big)\big)\,V_2+\cos\big(\sqrt{\epsilon_2}\,u_2\big)\,e_{\epsilon_3}+\epsilon_2\epsilon_3\sqrt{\epsilon_2\epsilon_3}\sin\big(\sqrt{\epsilon_3\epsilon_2}\,u_2\big)\,e_{\epsilon_2},$$

where

$$\begin{aligned} V_2 &= \frac{\epsilon_1}{D_2}\Big(-\epsilon_1 A_5\,e_{\epsilon_1}-\epsilon_2 A_8\sqrt{\epsilon_2\epsilon_3}\,e_{\epsilon_2}+\big(\epsilon_1 c_1-\epsilon_3 A_7\big)\,e_{\epsilon_3}\Big), \qquad (3.32)\\ D_2 &= -\epsilon_1\epsilon_3 c_1+A_7-A_6\neq 0, \end{aligned}$$

and the constants satisfy (3.12).

3. *If* $b_1=-\epsilon_3$, *then*

$$G_1 = \epsilon_3\epsilon_1\big(1-\cos\big(\sqrt{\epsilon_3\epsilon_1}\,u_1\big)\big)\,V_3+\cos\big(\sqrt{\epsilon_3\epsilon_1}\,u_1\big)\,e_{\epsilon_3}+\epsilon_1\epsilon_3\sqrt{\epsilon_3\epsilon_1}\sin\big(\sqrt{\epsilon_3\epsilon_1}\,u_1\big)\,e_{\epsilon_1},$$

$$G_2 = \frac{-\epsilon_1\epsilon_2 u_2^2}{2}\,V_3+u_2\,e_{\epsilon_2}+e_{\epsilon_3}, \qquad (3.33)$$

where

$$\begin{aligned} V_3 &= \frac{\epsilon_1}{D_3}\Big(-\epsilon_1 B_6\sqrt{\epsilon_3\epsilon_1}\,e_{\epsilon_1}-\epsilon_2 B_7\,e_{\epsilon_2}-\big(\epsilon_1 c_1-\epsilon_3 B_5\big)\,e_{\epsilon_3}\Big), \qquad (3.34)\\ D_3 &= B_8-B_5-\epsilon_1\epsilon_3 c_1\neq 0, \end{aligned}$$

and the constants satisfy (3.14).

Here $\epsilon_1^2=\epsilon_2^2=\epsilon_3^2=1$ *and* e_{ϵ_i} *are vectors of* E^3, *such that* $\langle e_{\epsilon_i}, e_{\epsilon_j}\rangle=\epsilon_i\delta_{ij}$, $1\le i,j\le 3$.

Conversely, given functions λ_1 *and* λ_2 *by (3.9), (3.11) and (3.13) and the vector valued functions* G_i *given by (3.29), (3.31) and (3.33). Then (3.28) is a* ϵ*-isothermic Dupin surface whose principal curvatures are the functions* $-\lambda_i$, $1\le i\le 2$.

Proof: Let X be a ϵ-isothermic Dupin surface as in Theorem 3.2. If the principal curvatures $-\lambda_1$ or $-\lambda_2$ are not constant, then $-\lambda_i$, $1\le i\le 2$, are given by (3.9), (3.11) and (3.13). For simplicity, define $\lambda_1=h_2(u_2)$, $\lambda_2=h_1(u_1)$ and suppose that $h_1'\neq 0$.

From Theorem 3.1, the first fundamental form of X is given by $I=\dfrac{1}{(h_1-h_2)^2}\Big(\epsilon_1 du_1^2+\epsilon_2 du_2^2\Big)$, $\epsilon_i^2=1$, $1\le i\le 2$ and X satisfy

$$\begin{aligned} &X_{,12}-\Gamma_{12}^2X_{,2}-\Gamma_{12}^1X_{,1}=0,\\ &X_{,11}-\Gamma_{12}^2X_{,1}+\epsilon_1\epsilon_2\Gamma_{12}^1X_{,2}+\frac{\epsilon_3\epsilon_1 h_2 N}{(h_2-h_1)^2}=0,\\ &X_{,22}-\Gamma_{12}^1X_{,2}+\epsilon_1\epsilon_2\Gamma_{12}^2X_{,1}+\frac{\epsilon_3\epsilon_2 h_1 N}{(h_2-h_1)^2}=0, \qquad (3.35)\\ &N_{,1}=h_2X_{,1}\,,\\ &N_{,2}=h_1X_{,2}\,. \end{aligned}$$

Using (3.4), we obtain

$$\begin{aligned} &\Gamma_{12}^2=\frac{h_1'}{h_2-h_1},\ \Gamma_{12}^1=\frac{h_2'}{h_1-h_2}, \qquad (3.36)\\ &\Gamma_{12,1}^2-\big(\Gamma_{12}^2\big)^2=\frac{h_1''}{h_2-h_1},\ \Gamma_{12,2}^1-\big(\Gamma_{12}^1\big)^2=\frac{h_2''}{h_1-h_2}. \end{aligned}$$

The last two equations of (3.35), we get

$$N = h_2X + G_2 \text{ and } N = h_1X + G_1.$$

Hence

$$X = \frac{G_2 - G_1}{h_1 - h_2} \text{ and } N = \frac{h_1G_2 - h_2G_1}{h_1 - h_2}. \tag{3.37}$$

Substituting (3.37) in the first equation of (3.35), we obtain an identity.

Subtracting the second and third equation of (3.35), we have

$$\epsilon_2X,_{11} - \epsilon_1X,_{22} = \frac{\epsilon_1\epsilon_2\epsilon_3N}{h_1 - h_2} + 2\epsilon_2\Gamma^2_{12}X,_1 - 2\epsilon_1\Gamma^1_{12}X,_2\,. \tag{3.38}$$

On the other hand, differentiating X given by (3.37) and using (3.36), we obtain

$$\begin{aligned}
X,_1 &= \Gamma^2_{12}X + \frac{G'_1}{h_2 - h_1}, \qquad X,_2 = \Gamma^1_{12}X + \frac{G'_2}{h_1 - h_2},\\
X,_{11} &= \left(\Gamma^2_{12,1} - (\Gamma^2_{12})^2\right)X + 2\Gamma^2_{12}X,_1 + \frac{G''_1}{h_2 - h_1},\\
X,_{22} &= \left(\Gamma^1_{12,2} - (\Gamma^1_{12})^2\right)X + 2\Gamma^1_{12}X,_2 + \frac{G''_2}{h_1 - h_2}.
\end{aligned} \tag{3.39}$$

So, using (3.38) and (3.36), we get

$$-\epsilon_3N = \left(\epsilon_1h''_1 + \epsilon_2h''_2\right)X + \epsilon_1G''_1 + \epsilon_2G''_2.$$

Substituting X and N given by (3.37) and using the Theorem 3.2, we obtain

$$\epsilon_2\left(G''_2 + \epsilon_2(\epsilon_3 + b_1)G_2\right) = -\epsilon_1\left(G''_1 - \epsilon_1b_1G_1\right).$$

Therefore, X is given by (3.28) and the vector valued functions $G_i(x_i)$, $1 \le i \le 2$ satisfy

$$G''_1 - \epsilon_1b_1G_i = V, \tag{3.40}$$

$$G''_2 + \epsilon_2(\epsilon_3 + b_1)G_2 = -\epsilon_1\epsilon_2V, \tag{3.41}$$

Now, substituting (3.37) and (3.39), in the second equation of (3.35), using (3.36), the Gauss equation, Theorem 3.2 and (3.28), we obtain

$$Vh_2 + \epsilon_1(\epsilon_3 + b_1)h_2G_2 + c_1G_2 + \epsilon_1\epsilon_2h'_2G'_2 - Vh_1 - \epsilon_1b_1h_1G_1 - c_1G_1 + h'_1G'_1 = 0. \tag{3.42}$$

Note that $Vh_2 + \epsilon_1(\epsilon_3 + b_1)h_2G_2 + c_1G_2 + \epsilon_1\epsilon_2h'_2G'_2 - Vh_1 - \epsilon_1b_1h_1G_1 - c_1G_1 + h'_1G'_1$, is a constant vector. In fact, it is sufficient differentiate these expression with respect to u_1 and u_2, using (3.18), (3.19), (3.40) and (3.41).

Finally, consider e_{ϵ_j}, $1 \le j \le 3$ vectors of E^3, such that $\langle e_{\epsilon_i}, e_{\epsilon_j}\rangle = \epsilon_i\delta_{ij}$, $1 \le i, j \le 3$.

Hence, given initial conditions $X,_1(0,0) = \frac{-e_{\epsilon_1}}{h_1(0)-h_2(0)}$, $X,_2(0,0) = \frac{e_{\epsilon_2}}{h_1(0)-h_2(0)}$, $N(0,0) = e_{\epsilon_3}$, $X(0,0) = (0,0,0)$ and using (3.37) and (3.39), we have $G'_i(0) = e_{\epsilon_i}$, $G_i(0) = e_{\epsilon_3}$, $1 \le i \le 2$.

Therefore, using (3.42), we get

$$V = \frac{\epsilon_1}{h_2(0) - h_1(0)}\left(- \epsilon_1h'_1(0)\,e_{\epsilon_1} - \epsilon_2h'_2(0)\,e_{\epsilon_2} - \left((\epsilon_3 + b_1)h_2(0) - b_1h_1(0)\right)e_{\epsilon_3}\right). \tag{3.43}$$

1. If $b_1 \neq 0$ and $(\epsilon_3 + b_1) \neq 0$, using (3.9) in (3.43), the solutions of (3.40) and (3.41) are given by (3.29) where V_1 is given by (3.30).
2. If $b_1 = 0$, using (3.11) in (3.43), the solutions of (3.40) and (3.41) are given by (3.31) where V_2 is given by (3.32).
3. If $b_1 = -\epsilon_3$, (3.13) in (3.43), the solutions of (3.40) and (3.41) are given by (3.33) where V_3 is given by (3.34)

Conversely, consider

$$X = \frac{G_2 - G_1}{\lambda_2 - \lambda_1}, \tag{3.44}$$

where the vector valued functions $G_i(x_i)$, $1 \le i \le 2$, are given by (3.29), (3.31) and (3.33), with λ_1 and λ_2 given by (3.9), (3.11) and (3.13).
After very long computations we can show that the first and second fundamental form of X are given by

$$I = \frac{1}{(\lambda_1 - \lambda_2)^2}\Big(\epsilon_1 du_1^2 + \epsilon_2 du_2^2\Big), \ II = \frac{1}{(\lambda_1 - \lambda_2)^2}\Big(- \lambda_1\epsilon_1 du_1^2 - \lambda_2\epsilon_2 du_2^2\Big). \tag{3.45}$$

Therefore (3.44) is a ϵ-isothermic Dupin surface whose principal curvatures are the functions $-\lambda_i$, $1 \le i \le 2$. □

Examples of $\epsilon-$isothermic Dupin surfaces

Example 3.1. Considering $A_1 = A_2 = \frac{1}{2}, A_3 = -\frac{1}{2}, A_4 = -\frac{1}{4}, b_1 = 1, c_1 = 0, \epsilon_1 = 1, \epsilon_2 = -1, \epsilon_3 = 1$ in Theorem 3.4, we have the $\epsilon-$isothermic Dupin surface $X = \dfrac{G_2 - G_1}{\lambda_2 - \lambda_1}$ (see Figure 3.1), where

$$\lambda_2 = \cosh u_1, \ \lambda_1 = -\frac{1}{4}\left(\sinh\left(\sqrt{2}u_2\right) + 3\cosh\left(\sqrt{2}u_2\right)\right),$$

$$\begin{aligned} G_1 &= \left(\sinh u_1, -\frac{\sqrt{2}}{7}\left(\cosh u_1 - 1\right), \frac{1}{7}(10 - 3\cosh u_1)\right), \\ G_2 &= \left(0, -\frac{7\sinh\left(\sqrt{2}u_2\right) + \cosh\left(\sqrt{2}u_2\right) - 1}{7\sqrt{2}}, \frac{1}{7}\left(2\cosh\left(\sqrt{2}u_2\right) + 5\right)\right). \end{aligned}$$

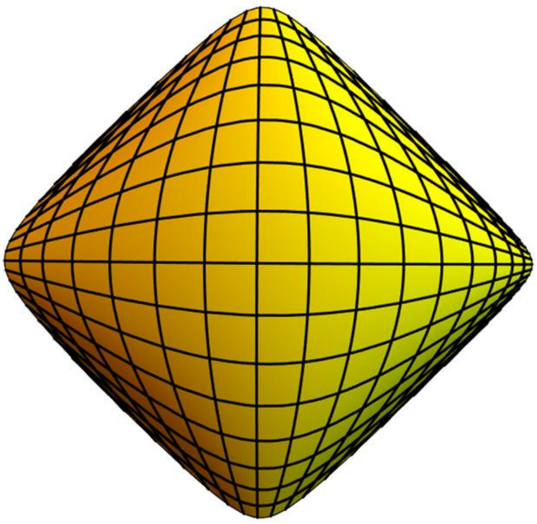

Figure 3.1: ϵ-isothermic Dupin surface

Example 3.2. Considering $A_1 = A_2 = \frac{1}{4}, A_3 = A_4 = \frac{1}{2}, b_1 = 4, c_1 = 2\sqrt{6}, \epsilon_1 = -1, \epsilon_2 = -1, \epsilon_3 = -1$ in Theorem 3.4, we have the $\epsilon-$isothermic Dupin surface $X = \dfrac{G_2 - G_1}{\lambda_2 - \lambda_1}$ (see Figure 3.2), where

$$\lambda_2 = \frac{1}{2}\left(\cos(2u_1) + \sqrt{6}\right), \ \lambda_1 = \cosh\left(\sqrt{3}u_2\right) + 2\sqrt{\frac{2}{3}},$$

$$\begin{aligned} G_1 &= \left(-\sin(u_1)\cos(u_1), 0, \left(\sqrt{6} - 3\right)\sin^2(u_1) - \cos(2u_1)\right), \\ G_2 &= \left(0, -\frac{\sinh\left(\sqrt{3}u_2\right)}{\sqrt{3}}, -\frac{\left(\sqrt{6}+1\right)\cosh\left(\sqrt{3}u_2\right) + 2}{\sqrt{6}+3}\right). \end{aligned}$$

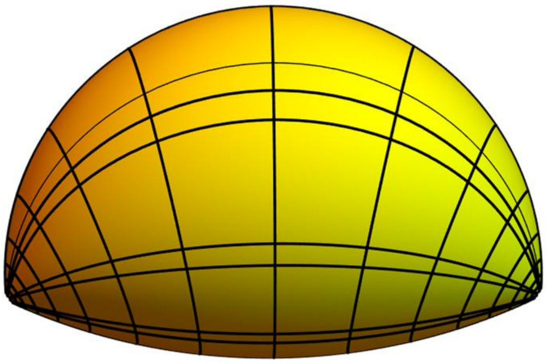

Figure 3.2: ϵ-isothermic Dupin surface

Example 3.3. Considering $A_5 = 1, A_6 = 3, A_7 = \sqrt{15}, A_8 = 0, c_1 = 2, \epsilon_1 = 1, \epsilon_2 = 1, \epsilon_3 = 1$ in Theorem 3.4, we have the $\epsilon-$isothermic Dupin surface $X = \dfrac{G_2 - G_1}{\lambda_2 - \lambda_1}$ (see Figure 3.3), where

$$\lambda_2 = u_1^2 + u_1 + 3, \ \ \lambda_1 = \sqrt{15}\cos(u_2) - 2,$$

$$\begin{aligned} G_1 &= \left(\frac{u_1^2}{10 - 2\sqrt{15}} + u_1, 0, 1 - \frac{\left(\sqrt{15} - 2\right) u_1^2}{2\left(\sqrt{15} - 5\right)} \right), \\ G_2 &= \left(\frac{1 - \cos(u_2)}{\sqrt{15} - 5}, \sin(u_2), -\frac{3\cos(u_2) - \sqrt{15} + 2}{\sqrt{15} - 5} \right). \end{aligned}$$

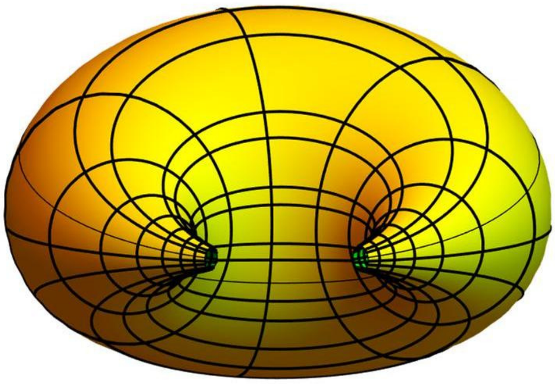

Figure 3.3: ϵ-isothermic Dupin surface

4. Solutions of the pseudo-Calapso and Zoomeron equations . In this section, using the Theorems 2.1 and 3.2, we give explicit solutions to the pseudo-Calapso equation. In particular, we provide new explicit solutions of the Zoomeron, Calapso and Davey-Stewartson III equations.

Corollary 4.1. *Let $X : U \subseteq \mathbb{R}^2 \to E^3$, be a ϵ-isothermic Dupin surface with two distinct principal curvatures $-\lambda_1$ and $-\lambda_2$. The functions $\omega = \dfrac{\epsilon_1\sqrt{2}(\lambda_2 + \lambda_1)}{2(\lambda_2 - \lambda_1)}$, $\Omega = \dfrac{\epsilon_1\sqrt{2}}{2}$ are solutions of the pseudo-Calapso equation where λ_1 and λ_2 are given in Theorem 3.2.*

Proof: The result it follows from Theorems 2.1 and 3.2. □

Remark 4.1. Consider $\epsilon = \epsilon_1\epsilon_2 = -1$ and $\epsilon_1\epsilon_3 = 1$. In this case the pseudo-Calapso equation becomes the Zoomeron equation (2.13).

By Theorem 3.2 and Corollary 4.1, for each λ_1 and λ_2 given by (3.9), (3.11) and (3.13), the function

$$\omega = \frac{\epsilon_1\sqrt{2}(\lambda_2 + \lambda_1)}{2(\lambda_2 - \lambda_1)} \tag{4.1}$$

is a solution of (2.13).
Moreover, as in Remark 2.3, the function $u(u_1, u_2, t)$ given by

$$u = e^{i(\nu u_1 + \mu u_2 + \mu\nu t)}\omega(u_1 + \mu t, u_2 + \nu t), \quad \rho = \frac{|u|_{,12}}{u},$$

is a solution of the Davey-Stewartson III equation (2.15), where $\omega(u_1, u_2)$ is given by (4.1).

The next using the Remark 4.1, we present some graphs of solutions to the Zoomeron equation.

Example 4.1. Considering $A_1 = A_2 = \frac{1}{2}, A_3 = -\frac{1}{2}, A_4 = -\frac{1}{4}, b_1 = 1, c_1 = 0, \epsilon_1 = 1, \epsilon_2 = -1, \epsilon_3 = 1$ in Theorem 3.2 and using the Corollary 4.1, we have the solution of the Zoomeron equation $\omega = \dfrac{\sqrt{2}(\lambda_2 + \lambda_1)}{2(\lambda_2 - \lambda_1)}$ (see Figure 4.1), where

$$\lambda_2 = \cosh u_1, \ \lambda_1 = -\frac{1}{4}\left(\sinh\left(\sqrt{2}u_2\right) + 3\cosh\left(\sqrt{2}u_2\right)\right).$$

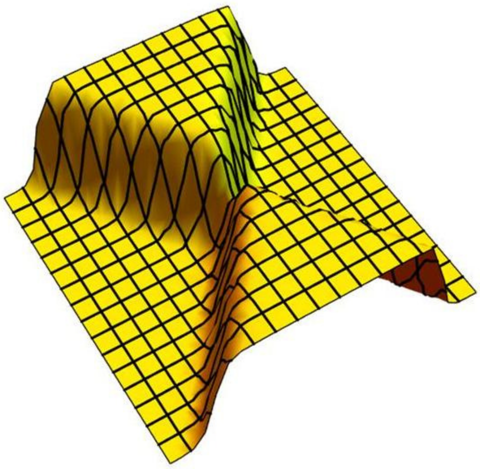

Figure 4.1: Solution of the Zoomeron equation

Example 4.2. Considering $A_1 = A_2 = 1, A_3 = A_4 = -1, b_1 = -\frac{1}{2}, c_1 = 1, \epsilon_1 = 1, \epsilon_2 = -1, \epsilon_3 = 1$ in Theorem 3.2 and using the Corollary 4.1, we have the solution of the Zoomeron equation $\omega = \dfrac{\sqrt{2}(\lambda_2 + \lambda_1)}{2(\lambda_2 - \lambda_1)}$ (see Figure 4.2), where

$$\lambda_2 = 2\left(\cos\left(\frac{u_1}{\sqrt{2}}\right) + 1\right), \ \lambda_1 = -2\left(\cosh\left(\frac{u_2}{\sqrt{2}}\right) + 1\right).$$

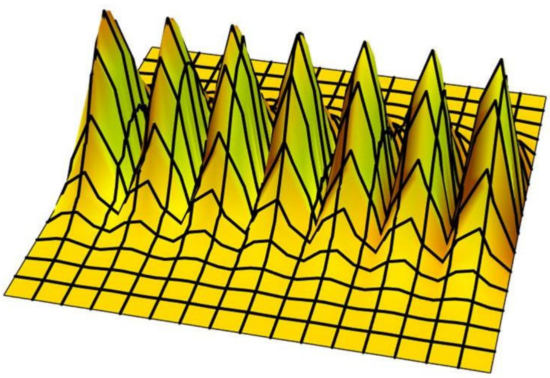

Figure 4.2: Solution of the Zoomeron equation

Remark 4.2. Consider $\epsilon_1 = -1$ and $\epsilon_2 = \epsilon_3 = 1$. In this case the pseudo-Euclidean 3-space E^3 is the Lorentz-Minkowski 3-space $\mathbb{L}^3$.
The pseudo-Calapso equation in $\mathbb{L}^3$ is given by

$$\Delta_\epsilon\left(\frac{\phi_{,12}}{\phi}\right) - (\phi^2)_{,12} = 0, \tag{4.2}$$

where $\Delta_\epsilon = \dfrac{\partial^2}{\partial u_1^2} - \dfrac{\partial^2}{\partial u_2^2}$.

By Theorem 3.2 and Corollary 4.1, for each λ_1 and λ_2 given by (3.9), (3.11) and (3.13), the function

$$\omega(u_1, u_2) = -\frac{\sqrt{2}(\lambda_2 + \lambda_1)}{2(\lambda_2 - \lambda_1)}, \tag{4.3}$$

is a solution of (4.2). Note that, using the transformation $(u_1, u_2) \to (u_2, u_1)$, we obtain that (4.2) becomes the Zoomeron equation. Hence, for each solution $\omega(u_1, u_2)$ of (4.2), we get a solution $\widetilde{\omega}(u_1, u_2) = \omega(u_2, u_1)$ of the Zoomeron equation. Therefore, as in the previous remark, we obtain a solution of the Davey-Stewartson III equation (2.15), given by

$$u = e^{i(\nu u_1 + \mu u_2 + \mu\nu t)}\omega(u_2 + \mu t, u_1 + \nu t), \quad \rho = \frac{|u|_{,12}}{u},$$

where $\omega(u_1, u_2)$ is given by (4.3).

Example 4.3. Considering $A_5 = 1, A_6 = 3, A_7 = \sqrt{15}, A_8 = 0, c_1 = 2, \epsilon_1 = -1, \epsilon_2 = \epsilon_3 = 1$ in Theorem 3.2 and using the Corollary 4.1, we have the solution of the pseudo-Calapso equation $\omega = -\dfrac{\sqrt{2}(\lambda_2 + \lambda_1)}{2(\lambda_2 - \lambda_1)}$ (see Figure 4.3), where

$$\lambda_2 = u_1^2 + u_1 + 3, \;\; \lambda_1 = \sqrt{15}\cos(u_2) + 2.$$

Figure 4.3: Solution of the pseudo-Calapso equation

Example 4.4. Considering $B_5 = 2, B_6 = 0, B_7 = 2, B_8 = -1, c_1 = -2, \epsilon_1 = -1, \epsilon_2 = 1, \epsilon_3 = 1$ in Theorem 3.2 and using the Corollary 4.1, we have the solution of the pseudo-Calapso equation $\omega = -\dfrac{\sqrt{2}(\lambda_2 + \lambda_1)}{2(\lambda_2 - \lambda_1)}$ (see Figure 4.4), where

$$\lambda_2 = 2(\cosh(u_1) + 1), \;\; \lambda_1 = -(u_2 - 1)^2.$$

The following result generalizes the Proposition 5.6 in [18].

Proposition 4.1. *If f is a ϵ_2-holomorphic function, $\epsilon_2 = \pm 1$, then the function ω given by*

$$\omega = \frac{2\sqrt{2|\langle f', f'\rangle_{\epsilon_2}|}}{1 + \epsilon_3\langle f, f\rangle_{\epsilon_2}} \tag{4.4}$$

is a solution to the pseudo-Calapso equation.

Proof: Let $\epsilon_2 = \pm 1$ and f is a ϵ_2-holomorphic function and define the application

$$X(z) = \left(\frac{2f}{1 + \epsilon_3\langle f, f\rangle_{\epsilon_2}}, \frac{\epsilon_3\langle f, f\rangle_{\epsilon_2} - 1}{1 + \epsilon_3\langle f, f\rangle_{\epsilon_2}}\right), \qquad z \in C_{\epsilon_2}$$

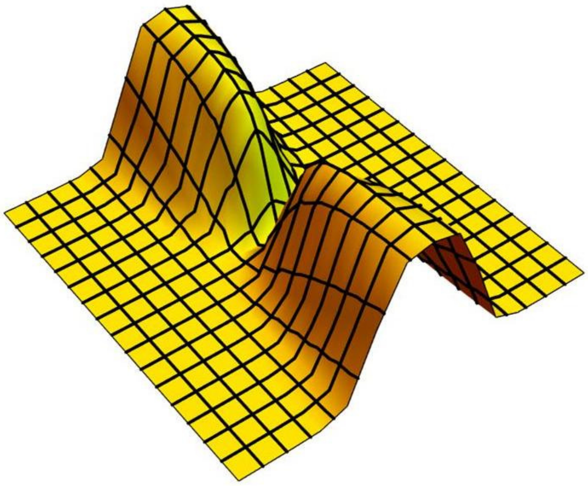

Figure 4.4: Solution of the pseudo-Calapso equation

where $\epsilon_3^2 = 1$.
This application is a parametrization of the sphere in $\mathbb{R}^3$ with metric $du_1^2 + \epsilon_2 du_2^2 + \epsilon_3 du_3^2$. In fact, it is easy to see that $\langle X, X\rangle = \epsilon_3$. Moreover, the first fundamental form of X is given by

$$I = \frac{4|\langle f', f'\rangle_{\epsilon_2}|}{\left(1+\epsilon_3\langle f, f\rangle_{\epsilon_2}\right)^2}\left[du_1^2 + \epsilon_2 du_2^2\right].$$

Using Theorem 2.1, we get the result. □

The next using the Proposition 4.1, we present some graphs of solutions to the Calapso equation.

Example 4.5. Considering the 1-holomorphic functions $f(z) = \cosh z, f(z) = z^3 + 1,\ \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ in Proposition 4.1, we have respectively, the solutions of the Calapso equation

$$\begin{aligned} \omega &= \frac{4\sqrt{\cosh(2u_1) - \cos(2u_2)}}{\cosh(2u_1) + \cos(2u_2) + 2}, \\ \omega &= \frac{6\sqrt{2}(u_1^2 + u_2^2)}{3u_1^2u_2^4 + 3\left(u_1^3 - 2\right)u_1u_2^2 + u_1^6 + 2u_1^3 + u_2^6 + 2}, \end{aligned}$$

whose graphs are given in Figures 4.5 and 4.6.

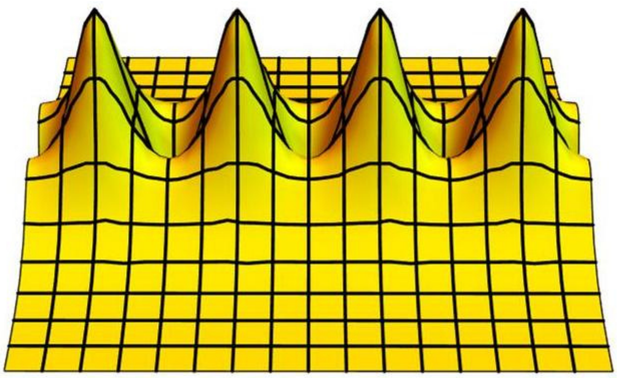

Figure 4.5: Solution of the Calapso equation

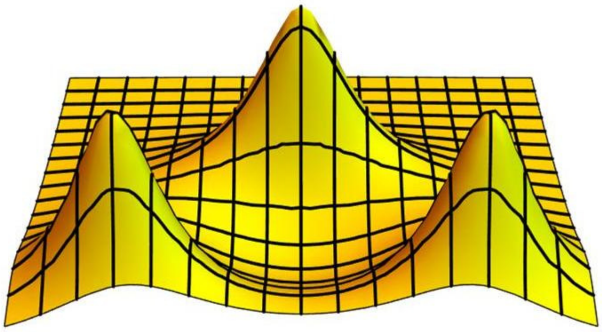

Figure 4.6: Solution of the Calapso equation

Remark 4.3. From Remark 2.3 and Proposition 4.1, we get that if f is a -1-holomorphic function, then

$$\begin{aligned} u &= e^{i(\nu u_1+\mu u_2+\mu\nu t)}\omega(u_1+\mu t, u_2+\nu t), \quad \rho = \frac{|u|_{,12}}{u}, \\ u &= e^{i(\nu u_1+\mu u_2+\mu\nu t)}\phi(u_2+\mu t, u_1+\nu t), \quad \rho = \frac{|u|_{,12}}{u}, \end{aligned}$$

are solutions for the Davey-Stewartson III equation (2.15) where $\omega(u_1,u_2) = \dfrac{2\sqrt{2|\langle f',f'\rangle_{\epsilon_2}|}}{1+\langle f,f\rangle_{\epsilon_2}}$ and $\phi(u_1,u_2) = \dfrac{2\sqrt{2|\langle f',f'\rangle_{\epsilon_2}|}}{1-\langle f,f\rangle_{\epsilon_2}}$. Moreover, $\omega(u_1,u_2)$ and $\widetilde{\phi}(u_1,u_2)=\phi(u_2,u_1)$ are the solutions of the Zoomeron equation.

Example 4.6. Considering the -1-holomorphic functions $f(z) = \sin(u_1)\cos(u_2)+i\cos(u_1)\sin(u_2),\ \epsilon_1 = 1, \epsilon_2 = -1, \epsilon_3 = 1$ in Proposition 4.1 and using the Remark 4.3 we have the solution of the Zoomeron equation

$$\omega = \frac{2\sqrt{2}\sqrt{\cos^2(u_1)+\cos^2(u_2)-1}}{1-\cos^2(u_1)+\cos^2(u_2)},$$

whose graph is given in Figure 4.7.

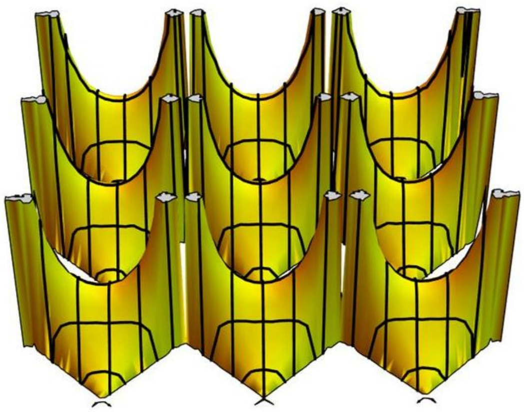

Figure 4.7: Solution of the Zoomeron equation

Remark 4.4. Consider $\epsilon_1=\epsilon_2=\epsilon_3=1$. In this case we get the Euclidean 3-space $\mathbb{R}^3$. By Theorem 3.2 and Corollary 4.1, for each λ_1 and λ_2 given by (3.9), (3.11) and (3.13), the function $\omega = \dfrac{\sqrt{2}(\lambda_2+\lambda_1)}{2(\lambda_2-\lambda_1)}$, is a solution of the Calapso equation

$$\Delta\left(\frac{\phi_{,12}}{\phi}\right) + (\phi^2)_{,12} = 0,$$

where Δ is the usual Laplacian of $\mathbb{R}^2$.
Moreover, by Remark 2.3, if $\omega(u_1,u_2)$ is a solution for the Calapso equation, then the function $\widetilde{\omega}(u_1,u_2) =$

$\omega(u_1, iu_2)$ is a solution of the Zoomeron equation.
Therefore,

$$u = e^{i(\nu x+\mu y+\mu\nu t)}\widetilde{\omega}(u_1 x + \mu t, u_2 + \nu t), \quad \rho = \frac{|u|_{,12}}{u}$$

is a solution of the Davey-Stewartson III equation (2.15).

5. Conclusions. From the results obtained in this work we can make the following conclusions: We show that for each ϵ-isothermic surface of the pseudo-Euclidean 3-space, we can associate to these surfaces two solutions to the pseudo-Calapso equation. Furthermore, for each solution of the pseudo-Calapso equation, we have in particular a solution of the Calapso or Zoomeron equations. Consequently, we obtain solutions of the Davey-Stewartson III equation. Also, we consider those proper Dupin surface of the pseudo-Euclidean 3-space having distinct principal curvatures, parametrized by lines of curvature. We show that every Dupin surface parametrized by lines of curvature is a ϵ-isothermic surface and provide explicit coordinates for such surfaces. Finally, as application of the theory, we give explicit solutions of the pseudo-Calapso, that depend on two functions, each one defined in a given variable. In particular, we provide new explicit solutions of the Calapso, Zoomeron and Davey-Stewartson III equations and we also provide explicit solutions to these equations that depend on ϵ_2-holomorphic functions.

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Armando M. V. Corro* https://orcid.org/0000-0002-6864-3876
Carlos M. C. Riveros† https://orcid.org/0000-0002-1206-7072
Marcelo L. Ferro‡ https://orcid.org/0000-0001-6832-2274

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