Journal homepage http://revistas.unitru.edu.pe/index.php/SSMM

SELECCIONES MATEMÁTICAS
Universidad Nacional de Trujillo
ISSN: 2411-1783 (Online)
2023; Vol. 10(1): 60-68.

Special issue: Peruvian Conference on Scientific Computing 2022, Cusco - Peru

Exploring parameter spaces in complex dynamics

Pedro Iván Suárez Navarro

Received, Jan. 15, 2023 *Accepted, Apr. 10, 2023*

How to cite this article:
Suárez P. *Exploring parameter spaces in complex dynamics*. Selecciones Matemáticas. 2023;10(1):60–68. http://dx.doi.org/10.17268/sel.mat.2023.01.06

Abstract
We show the structure of the parameter space for a family of rational maps containing Blaschke products. Through numerical simulations using the orbit of a single critical point, we reveal the existence of infinitely many Mandelbrot-like sets along the unit circle, as well as eight-like structures in other regions of parameter space. We pose some open questions related to the parameter space of these functions.
Keywords . Complex dynamics, Blaschke products, Mandelbrot set, Julia set.

1. Introduction. The simplest and most studied complex dynamical system is the quadratic family. The idea of perturbing the squaring function by a complex parameter revealed enormous richness both in the dynamical plane and in the parameter plane. The interest in studying the latter begins with the first numerical simulations made by Benoit Mandelbrot with the computational power of the 80's. However, there is a discrepancy in the authorship of the now called Mandelbrot set, because Matelski and Brooks [**?**] had already generated a similar picture in the context of Kleinian Groups, see [1] where discuss the controversy. Beyond the impressive fractal beauty of this object, the main mathematical properties were only revealed with the seminal work of Douady and Hubbard [2].

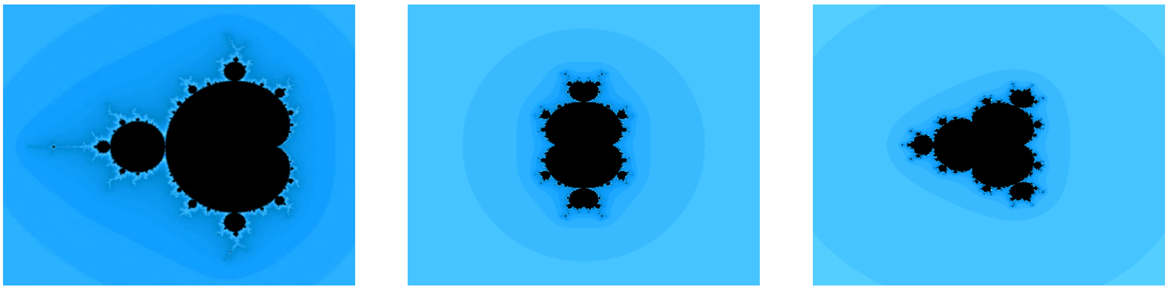
Parameter spaces for other types of complex dynamical systems, beyond the quadratic polynomial family, have also been explored and are less understood, this requires studying the systems one by one, and it is not always possible to determine global properties, see for instance, Newton maps [3], singular perturbations of polynomials maps [4], [5] or transcendental maps [6].

In this work we explore a dynamical system on the Riemann sphere that depends on one or more complex parameters, especially on a family of functions containing Blaschke products. These functions appear in the analytical context due to the work of Wilhelm Blaschke [7] on the convergence of infinite products of automorphisms of the unit disk. From the dynamic point of view, they caught the attention of Michael Herman as the first examples of functions presenting what are now called Herman Rings (see [8]), and since then a lot of work has been done on these functions, revealing its importance from the dynamical point of view as a toy model for the study of polynomial dynamics [9], among other dynamics.

We show through numerical simulations the different types of structures present in the parameter plane of variant of a Blaschke product of degree $d + 1$ with a single free critical point. We reveal different types of structure at different scale levels, especially on the presence of multibrot-like sets sets of degree 3 in the parameter plane. The key point for the simulations depends on the dynamic behavior of the critical point, to be more exact of the critical value.

In section 2, we describe the Blaschke product family and its main properties. In section 3 we draw the non-escape locus and different structures in the parameter plane of a Blaschke family. Finally, in section 4 we present some question related to the Blaschke product. All the images we created were made with the Dynamics Explorer program [10].

*Departamento Académico de Matemática. Facultad de Ciencias Básicas, Universidad Nacional Agraria La Molina, Lima, Perú (psuarez@lamolina.edu.pe).



(a) Mandelbrot set. (b) Multibrot set for $n = 3$. (c) Multibrot set for $n = 4$.

Figure 2.1: Connectedness Locus for $z \mapsto z^n + c$, for $n = 2, 3$ and 4 respectively.

2. Blaschke products. Finite Blaschke products are rational maps of the Riemann sphere that preserving the unit circle. It is not difficult to verify that (see Milnor [11]) every rational maps of degree d that preserving the unit circle can be writting as a finite product

$$B(z) = \lambda \prod_{i=1}^{d} \frac{z - a_i}{1 - \overline{a_i} z},$$

for some constant $\lambda \in \partial\mathbb{D}$ with $a_1, \dots, a_d \in \mathbb{C} - \partial\mathbb{D}$. Every map B commuting with the circle inversion, i.e, $\overline{B(\frac{1}{z})} = \frac{1}{B(z)}$. When the parameters a_i are contained in the disk, the associated Julia set is a subset of the unit circle (see [12]). However, in other cases, the associated Julia sets may have a very complicated structure (see for example Figures 2.2a, 2.2b,2.2c).

In this work we consider a family of rational maps on the Riemann sphere of degree $d + 1$ given by

$$f_\lambda(z) = \lambda z^d \frac{z - \frac{d+1}{d-1}}{1 - \frac{d+1}{d-1} z}, \lambda \in \mathbb{C}^*, d \geq 2. \tag{2.1}$$

The family (2.1) is a degenerate case in the family $z \to z^d \dfrac{z - a}{1 - \overline{a} z}$, where $a \in \mathbb{C}$ (see [13]). We allow the factor λ in the complex plane and not only restricted to the unit circle. In particular (2.1) contains Blaschke products.

It is not difficult to verify that each map f_λ has two super-attracting point at 0 and ∞, a single critical point at $z = 1$ and critical value $\lambda = f_\lambda(1)$. Essentially the behavior of the critical point $z = 1$ determines the different dynamics for the family (2.1). By iteration the orbit of the critical point can be attracted to the attraction basin of one of the two superattractors (but not for both at the same time), it could also be attracted to the basin of some attractor or parabolic point in the complex plane, and even the orbit could be trapped on the unit circle, where the critical point may be to eventually converge to some periodic point or have a dense orbit on the circle (see [14]).

The presence of superattractors in the dynamics is quite a useful feature, as is the case with polynomials, since we can conformally parameterize a neighborhood around 0 or ∞ using the Böttcher coordinates. For example for $d = 3$ (To avoid complications with the degree d), the Böttcher function is the solution of the Böttcher's functional equation

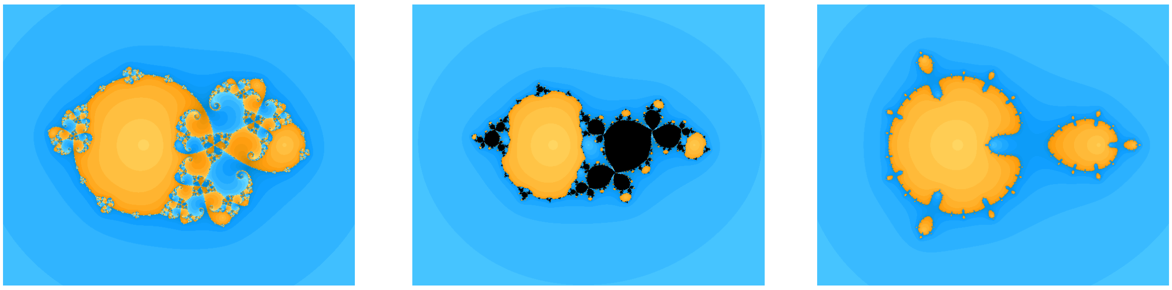
$$\Phi_\lambda^\infty \circ f_\lambda(z) = -\tfrac{\lambda}{2} \Phi_\lambda^\infty(z)^3$$

normalized $\Phi_\lambda^\infty(z) \sim z$ at $z = \infty$. By [15] for $\lambda \in \mathbb{C}^*$, for which the critical point $z = 1$ does not belong to the immediate basin of attraction of infinity denoted by $\mathcal{A}_\lambda(\infty)$, we have $\mathcal{A}_\lambda(\infty)$ is simply connected and Φ_λ^∞ is given by

$$\Phi_\lambda^\infty(z) = \lim_{k \to \infty} \sqrt[3^k]{\frac{f_\lambda^k(z)}{\left(-\frac{\lambda}{2}\right)^{1 + \cdots + 3^{n-1}}}} = \left(-\tfrac{2}{\lambda}\right)^{\frac{1}{2}} \lim_{k \to \infty} \sqrt[3^k]{\left(-\tfrac{\lambda}{2}\right)^{\frac{1}{2}} f_\lambda^k(z)}, \tag{2.2}$$

mapping $\mathscr{A}_\lambda$ conformally onto $|w| > \left(\frac{2}{|\lambda|}\right)^{\frac{1}{2}}$, so that

$$\left(-\tfrac{\lambda}{2}\right)^{\frac{1}{2}} \Phi_\lambda^\infty(z) = \lim_{k \to \infty} \sqrt[3^k]{\left(-\tfrac{\lambda}{2}\right)^{\frac{1}{2}} f_\lambda^k(z)},$$



(a) Julia set of f_λ for λ on the unit circle.

(b) Julia set of f_λ for λ in hyperbolic component of period 2.

(c) Julia set of f_λ disconnected for λ outside of non-escape locus .

Figure 2.2: Julia sets for f_λ, $d = 3$.

maps $\mathcal{A}_\lambda(\infty)$ conformally onto $\Delta = \{w : |w| > 1\}$.

Essentially, some of the properties of Julia sets for f_λ can be summarized in the following statement

Proposition 2.1.

1. *f_λ not contain Herman rings.*
2. *The Julia set of f_λ is connected if only if $z = 1$ is not in the immediate basin of* 0 *or* ∞.

Proof: 1) If such a Herman ring exists, by Shishikura [16, Theorem III] then at least the orbits of two different critical points would accumulate on the boundary of the ring, that cannot happen in our case, because our family essentially has only one free critical point.

2) Bottcher's theorem [11, Theorem 9.3] ensures that if the immediate basin of attraction of a superattractor point does not contain another critical point, then the bottcher map defines an isomorphism between the immediate basin and a disk, in particular it is simply connected. In our case, as long as the immediate basin of zero and infinity respectively do not contain the critical point $z = 1$, then both are simply connected. Then, by the Hurtwitz formula [15], all other components of the basin $\mathcal{A}(0)$ (or $\mathcal{A}(\infty)$) are simply connected. In a similar way, but using other isomorphisms (see [11, Teorema 8.2, Theorem 10.5]), it is verified that attractor and parabolic basins and other components of the Fatou set are also simply connected. Finally, the Julia set is connected, since all the components of the Fatou set are simply connected. □

Fig. 2.2 show different Julia sets associated a f_λ. In Fig.2.2a show a Julia sets when the critical point is recurrent in the circle. Fig.2.2c show a Julia set of renormalizable type. Fig.2.2c show the disconnected Julia sets when the critical point belong to the attracting basin of ∞.

We have that the parameter space corresponds to the complex plane, since (2.1) depends on only one complex parameter. We define the non-escape locus in the parameter plane, i,e, the set of parameters λ for which the critical point is not trapped by basins of 0 or ∞. In symbols we have

$$\mathcal{M}_d := \{\lambda \in \mathbb{C}^* : f_\lambda^n(1) \nrightarrow \infty, f_\lambda^n(1) \nrightarrow 0, \text{ when } n \to \infty\}$$

The non-escape locus $\mathcal{M}_d$ has the following properties

Proposition 2.2.

1. *$\mathcal{M}_d$ does not coincide with the connectedness locus.*
2. *$\mathbb{S}^1 \subset \mathcal{M}_d$.*
3. *$\mathcal{M}_d$ is symmetric in relation to real axis.*
4. *$\mathcal{M}_d$ is bounded.*

Proof: 1) By the Proposition 2.1, we have that $\mathcal{M}_d$ is a subset of the connectedness set (the set of parameters λ for which the Julia set is connected). However, this inclusion is strict. Indeed, it is enough to consider the set of parameters λ for which $f_\lambda^n(1) = 2$ (or $f_\lambda^n(1) = 1/2$), for each $n \geq 2$. For any of these parameters λ, the Julia set of f_λ is connected, that is, each λ belongs to the connectedness set, however, by definition they do not belong to $\mathcal{M}_d$. 2) As a consequence of the previous Proposition 2.1, because for $\lambda \in \mathbb{S}^1$, the critical orbit is contain on the circle, hence $\lambda \in \mathcal{M}_d$. 3) Consequence of symmetries of the critical orbit in the dynamical plane, $\overline{f_\lambda^n(1)} = f_{\bar{\lambda}}^n(1)$, for all n. 4) Without loss of generality, for the case $d = 3$, we have that $\mathcal{M}_3 \subset \mathbb{A}(\frac{1}{3}, 3)$ (A ring with external radius and internal radius 3 and $1/3$ respectively.), see [14] for more details. □

Remark 2.1. *The parameters mentioned in the first part of Proposition 2.2 can be located in the smallest blue and oranges bubbles (see Figure 2.4b).*

We divide the λ-parameter space into the following hyperbolic components observed in numerical simulations. Every hyperbolic component in the λ -plane belongs to one of the following four types.

- The central orange region in Figure 2.4a, is the hyperbolic component centered at the singular constant map at 0. In this case, the critical point $z = 1$ necessarily lies in the basin of zero .
- The unbounded blue region in Figure 2.4b, is the hyperbolic component centered at the singular constant map at ∞. In this case, the critical point $z = 1$ necessarily lies in the basin of infinity.
- Mandelbrot type. Here there are attracting orbits of periods 1, 2, or more, with one free critical point in the immediate basin of this. For an example in the parameter plane of $\mathcal{M}_2$, see Figure 2.5.
- Capture type. Here the free critical points are not in the immediate basin of zero or infinity, but some forward image of each free critical point belongs in one basin or the other. (These regions are either blue or orange in Figures 2.4a, 2.4b,2.6a,2.6b, 2.7c, according as the orbit of $z = 1$ converges to ∞ or 0.)

The set of all λ for which the Julia set is disconnected consists of the union of the the central orange region and the unbounded blue region (see Figures 2.4a and 2.4b).

Using techniques of quasiconformal surgery [12], it is possible to show that the hyperbolic escape components both of zero and infinity are simply connected, and consequently that $\mathcal{M}_d$ is connected (see [14]).

In the parameter spaces of (2.1) we not observed Arnold tongues and tricorn-like set, because our dynamics is not anti-holomorphic, contrary to what was observed in Canela [13] for the maps $z \mapsto z^3 \frac{z-a}{1-\bar{a}z}, a \in \mathbb{C}$. One type of shrimp-shaped structure present in the parameter space in Fig. 2.3, had already been observed in other contexts (see, for instance, [17]).

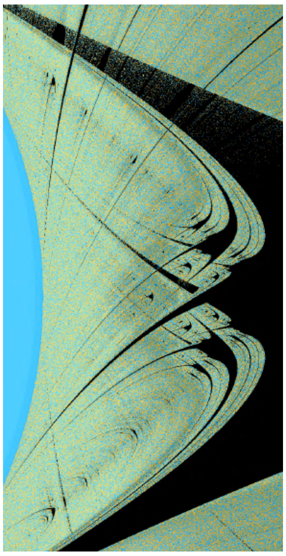


Figure 2.3: Shrimps in the parameter space of $z \mapsto z^3 \frac{z-a}{1-\bar{a}z}, a \in \mathbb{C}$ in [13]

Considering the case $d = 3$ (for other cases it is also possible), a direct computation allows us to make some special parameters explicit.

Proposition 2.3. *The following statements are valid*

1. *The Blaschke product f_{λ_j}, for $j = 1, 2, 3, 4$, has a critical orbit of period two, where*

$$\lambda_1 = \frac{1}{4}\left(1 + \sqrt{13} + \sqrt{2(\sqrt{13} - 1)}\right) \tag{2.3}$$

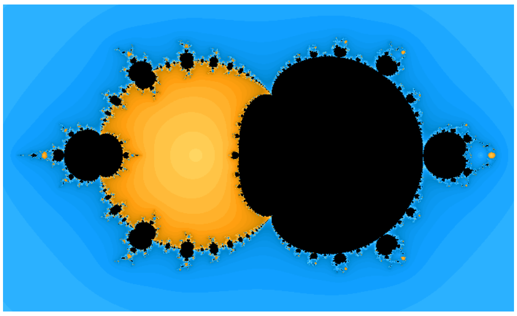
$$\lambda_2 = \frac{1}{4}\left(1 - \sqrt{13} + i\sqrt{2(\sqrt{13} + 1)}\right), \tag{2.4}$$

   and $\lambda_3 = \frac{1}{\lambda_1}$, $\lambda_4 = \overline{\lambda_2}$, and $|\lambda_2| = 1$.

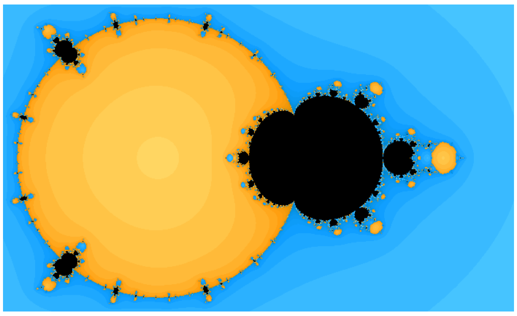
2. *For $\lambda_j = \frac{1}{64}(61 \pm 5i\sqrt{15})$, ($j = 1, 2$), the Blaschke product f_{λ_j} have a fixed point parabolic in the circle of multiplier 1 given by $z_{\lambda_j} = \frac{1}{8}(7 \pm i\sqrt{15})$.*
3. *For $\lambda = -1$ the critical value $f_\lambda(1)$ is repulsive fixed point $p = -1$.*
4. *For $\lambda = 1/2$ ou $\lambda = 2$ the second image of the critical point $c = 1$ by f_λ is the super attractor $z = 0$ or $z = \infty$.*

2.1. Simulations.

Non-escape set. The set $\mathcal{M}_d$ was defined in a similar way to how the Multibrot set (see Fig. 2.1) is defined for the case of unicritical polynomials, i.e, the set of parameters c for which the iterates of the critical point $z = 0$ of $z \mapsto z^d + c$ do not converge to infinity. Fig. 2.4a shows the structure of the non-escape set $\mathcal{M}_2$, this is a special case, because for $|\lambda| = 1$ each f_λ are diffeomorphisms of the circle, and it is possible to classify them by their number of rotation. For the cases $d > 2$, the situation is different, for $|\lambda| = 1$, f_λ are critical covering map of the circle. Regarding the structure of the parameter space, the Fig. 2.4b confirm what we know from Proposition 2.2, that the unit circle is contained in each $\mathcal{M}_d$, $d \geq 2$.



(a) $\mathcal{M}_2$.



(b) $\mathcal{M}_3$.

Figure 2.4: Non-escape locus $\mathcal{M}_d$, $d = 2, 3$.

Mandelbrot-like sets. The presence of Mandelbrot-like sets is related to the fact that the dynamics is renormalizable, that is, the adequate restriction of the iterates of the functions f_λ have a polynomial-like behavior in the Douady-Hubbard sense [18]. We do not prove that these sets are in fact quasiconformal copies of Mandelbrot sets associated to $z \mapsto z^3 + c, c \in \mathbb{C}$, that requires the techniques of the aforementioned article. We limit ourselves to numerical simulations that show the presence of an infinite number of these baby Mandelbrot sets stuck along the unit circle in the parameter plane.

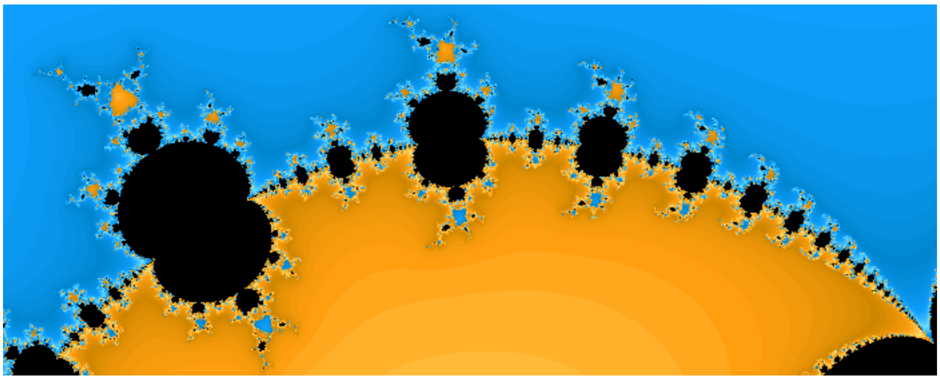
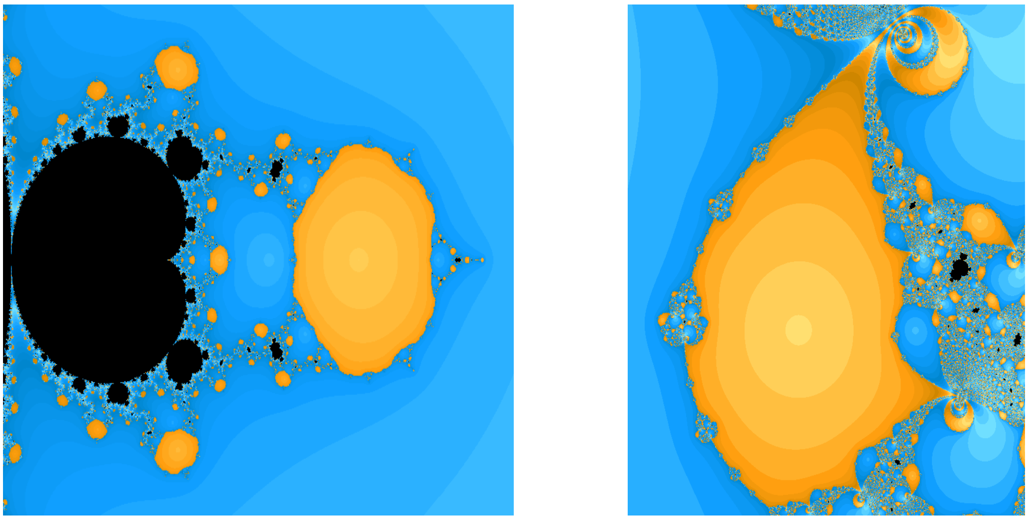


Figure 2.5: Mandelbrot-like sets for $d = 2$

McMullen [19] proved that Mandelbrot set is universal, i.e, the copies of generalized Mandelbrot set are dense in the bifurcation locus for generic families of rational maps. Fig. 2.5 show an infinite collection of cubic Multibrot sets along a part of the unit circle for $d = 2$. Similar observations are obtained for any value of d.

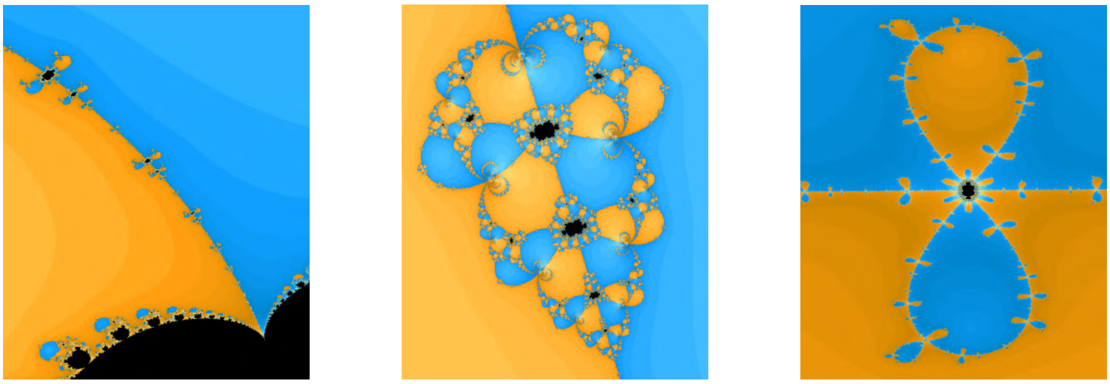
Escape components. Fig. 2.6a show a (orange) escape component of period 2, i.e, the critical point $z = 1$ after two iterations belongs to the immediate basin of attraction of 0. Escape bubbles can have quite complicated geometry, for example see Fig. 2.6b, where an escape region is shown stretching along a spiral. On the other hand, at a smaller scale it is possible to visualize eight-shaped structures around Mandelbrot-like sets, for example in Fig. 2.7c, near the parameter $\lambda = i$, where the escape components of zero and infinity are also shown in orange and blue respectively.



(a) Escape component of period 2.

(b) A escape component along a spiral.

Figure 2.6: Escape components.



(a) Near to parabolic parameter λ_p.

(b) Zoom near to parameter λ_p.

(c) Near to parameter $\lambda = i$.

Figure 2.7: Zooms in the parameter space.

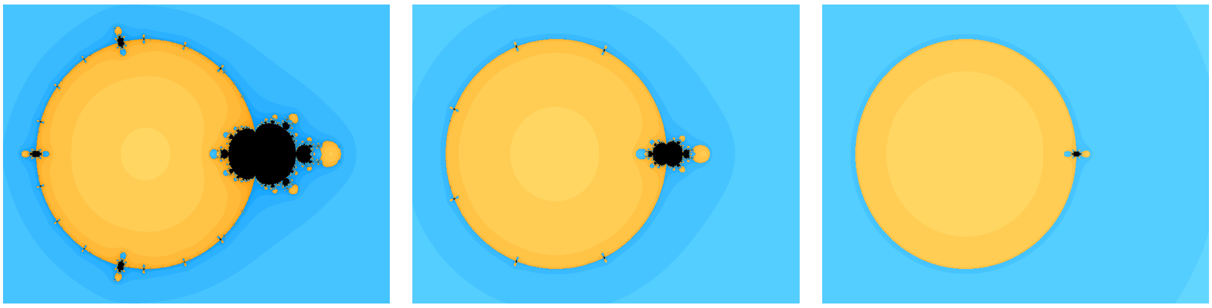
Near to the parabolic parameter. For $d = 3$, it is not difficult to determine the parameters for which the critical point $z = 1$ has a parabolic fixed point on the circle, by Proposition 2.3 this exactly corresponds to the parameters $\lambda_p = \frac{1}{64}(61 \pm 5i\sqrt{15})$. For parameters λ in a neighborhood of parameter λ_p, without leaving the circle, we have the phenomenon of parabolic explosion.

Fig. 2.7a shows us a region of the parameter space near a parameter λ_p on the boundary of the hyperbolic component of period 1. Fig. 2.7b shows us a zoom in the region of Fig. 2.7a , where we can observe a structure in the parameter spaces formed by Mandelbrot-like sets accumulating near the parabolic parameter.

3. Some Questions.

Geometric Limits. In this work we have considered f_λ a family of functions of degree $d + 1$. As we increase the degree d, the numerical simulations that both the Julia sets and the non-escape set begin to decrease geometrically, showing in the limit (in some suitable topology) a tendency to a set limited to a ring around the unit circle. Recently, [20] studied this type of problem for Julia sets on a family of rational functions. The study on geometric limits for the non-escape locus of the Blaschke products is an open problem.

In Fig. 3.1 shows the convergence of the structure of $\mathcal{M}_d$, when the degree d increases. In Figures 3.1a,3.1b,3.1c, we observe that both the component of period 1 (the largest component), as well as all the escape components and the Mandelbrot-like sets are geometrically smaller and smaller, accumulating around the unit circle.



(a) $\mathcal{M}_4$. (b) $\mathcal{M}_7$. (c) $\mathcal{M}_{20}$.

Figure 3.1: Non-escape locus $\mathcal{M}_d$, when the degree d increases.

Combinatorial structure in the parameter space. As we already mentioned, the structure of the parameter space for non-polynomial functions is far from being fully understood, even in the quadratic polynomial case the major conjecture about local connectedness of the Mandelbrot set remains unresolved. Topological problems such as the local connectedness of both the Julia sets, as well as the corresponding parameters, are generally difficult problems. In the polynomial case, a combinatorial tool is Yoccoz puzzles, which have been successful in solving problems of a topological and metric nature related to Julia sets. Also, there is the version of this tool in the parameter space. However, such partitions are only possible to construct for certain types of rational functions, see for example [21], [22]. In the case of the family (2.1), a special case of such a construction was studied in [14]. Studying the feasibility of building such structures for the parameter space of Blaschke products would be an interesting project to consider.

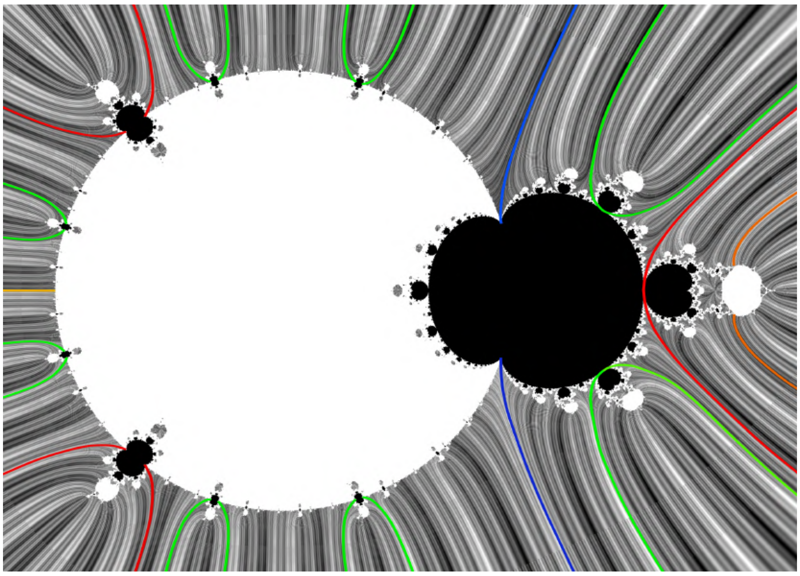


Figure 3.2: Some external rays in the parameter space

Fig. 3.2 shows one of Yoccoz's puzzle-building ingredients, external rays. Here we are showing the approximations of outer rays in parameter space converging for some special parameters on the boundary of $\mathcal{M}_3$. Blue rays converging to parameters on the boundary of the period 1 component. Red rays landing on parameters of the boundary of the period 2 components and green rays landing on the boundary of period 3 components. A priori, we do not know if all the rays external converge to parameters on the unit circle. A starting point to address this conjecture would be studying the local connectivity in the parameters of the unit circle.

Julia-like sets in the parameter space. Recently the author [23] presented a family of rational maps depending on two complex parameters in particular containing Blaschke product, where numerical simulations show the presence of infinite Julia-like sets in the slice of parameter space. This phenomenon is particularly interesting because the natural habitat of the Julia sets is the dynamical plane, and its presence shows the chaotic structure also in the parameter plane. For instance, Buff and Henriksen [24] show that the bifurcation locus of a two-parameter cubic polynomial family contains quasiconformal copies of quadratic Julia sets. Exploring other dynamical systems beyond the polynomial where this phenomenon occurs would be interesting.

4. Conclusions. In this work, we show the structure of the parameter space for a family of Blaschke products. We first described the decomposition of parameter space in hyperbolic components and showed by numerical simulations how the behavior of the critical value described all dynamics. The importance of performing numerical simulations using the power of current computers is undeniable, and it is a useful tool not only to visualize objects geometrically but also allows us, with adequate care, to make conjectures about certain topological properties of the complicated structures that appear in the plane of parameters in dynamic systems in a complex variable.

We hope that the questions posed in the last section will attract the attention of readers interested in problems of holomorphic dynamics, both from a theoretical and computational point of view.

Acknowledgment. The author wishes to thank the anonymous referee for their helpful comments.

ORCID and License

Pedro Iván Suárez Navarro https://orcid.org/0000-0003-0963-0829

References

[1] Horgan J. Mandelbrot set-to. Scientific American. 1990 Apr 1; 262(4):30-5.
[2] Douady A, Hubbard JH. Exploring the mandelbrot set. the orsay notes. Publ. Math. Orsay. 1984.
[3] Roesch P, Wang X, Yin Y. Moduli space of cubic Newton maps. Advances in Mathematics. 2017 Dec 15; 322:1-59.
[4] Devaney R. Structure of the McMullen domain in the parameter planes for rational maps. Fundamenta Mathematicae. 2005; 3(185):267-85.
[5] Devaney R. The McMullen domain: satellite Mandelbrot sets and Sierpinski holes. Conformal Geometry and Dynamics of the American Mathematical Society. 2007; 11(12):164-90.
[6] Fagella N, Keen L. Stable components in the parameter plane of transcendental functions of finite type. The Journal of Geometric Analysis. 2021; 31(5):4816-55.
[7] Garcia SR, Mashreghi J, Ross WT. Finite Blaschke products and their connections. Cham: Springer; 2018 May 24.
[8] Herman MR. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. Publications Mathématiques de l'IHÉS. 1979; 49:5-233.
[9] Milnor J. Hyperbolic components. Conformal dynamics and hyperbolic geometry. 2012; 573:183-232.
[10] Boyd B, Boyd S. Dynamics explorer, Computer software[Internet]. 2012 Mar. 24 Available on https://sourceforge.net/projects/detool/
[11] Milnor J. Dynamics in One Complex Variable.(AM-160):(AM-160)-. Princeton University Press; 2011 Feb 11.
[12] Branner B, Fagella N, Buff X. Quasiconformal surgery in holomorphic dynamics. Cambridge University Press; 2014 Jan 23.
[13] Canela J, Fagella N, Garijo A. Tongues in degree 4 Blaschke products. Nonlinearity. 2016 Sep 21; 29(11):3464.
[14] Navarro PI. Dinâmica de um produto de Blaschke[Tese de Douorado], Universidade de São Paulo; 2021.
[15] Steinmetz N. Rational iteration: complex analytical dynamical systems.De gruter. InRational Iteration 2011 Jul 22. de Gruyter studies in mathematics; 1993.
[16] Schleicher D, editor. Complex dynamics: families and friends. CRC Press; 2009 Nov 3.
[17] Gallas JA. Structure of the parameter space of the Hénon map. Physical Review Letters. 1993 May 3; 70(18):2714.
[18] Douady A, Hubbard JH. On the dynamics of polynomial-like mappings. InAnnales scientifiques de l'École normale supérieure 1985; Vol. 18(2):287-343).
[19] McMullen C. The Mandelbrot set is universal, in "The Mandelbrot set, theme and variations", LMS Lecture Note Series 274. Cambridge University Press, 2000.
[20] Alves AM, Silva e Silva BP, Salarinoghabi M. Geometric limit of Julia set of a family of rational functions with odd degree. Dynamical Systems. 2021 Oct 2; 36(4):699-713.
[21] Roesch P. On local connectivity for the Julia set of rational maps: Newton's famous example. Annals of mathematics. 2008; 168(1):127-74.
[22] Qiu W, Wang X, Yin Y. Dynamics of McMullen maps. Advances in Mathematics. 2012 Mar 1; 229(4):2525-77.
[23] Suarez PI. Julia-like sets in the parameter space of a family of quartic rational maps. Poster presented at Congress Complex Dynamics in the Tropics, IMPA; 2022 Nov 8- Dic 2; Rio de Janeiro. Brazil. Available from: https://impa.br/en_US/eventos-do-impa/2022-2/complex-dynamics-in-the-tropics/.
[24] Buff X, Henriksen C. Julia sets in parameter spaces. Communications in Mathematical Physics. 2001 Jul; 220(2):333-75.