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New Isothermic surfaces in $\mathbb{S}^3$

Nuevas superficies isotérmicas en $\mathbb{S}^3$

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Abstract

In this paper, we consider a method of constructing isothermic surfaces in $\mathbb{S}^3$ *based on Ribaucour transformations. By applying the theory to the flat torus, we obtain two family of complete isothermic surfaces in* $\mathbb{S}^3$*. One four-parameter family of complete isothermic surfaces that contains n-bubble surfaces inside and outside of the torus. We also get another four-parameter of complete isothermic surfaces which are Dupin surfaces. As aplication we obtain explicit solutions of the Calapso equation.*

Keywords . Isothermic surfaces, Ribaucour transformations, Dupin surfaces, Calapso equation.

Resumen

En este artículo, consideramos un método para construir superficies isotérmicas en $\mathbb{S}^3$ *basadas en transformaciones de Ribaucour. Aplicando la teoría al toro plano, obtenemos dos familias de superficies isotérmicas completas en* $\mathbb{S}^3$*. Una familia de cuatro parámetros de superficies isotérmicas completas que contiene superficies de n burbujas dentro y fuera del toro. También obtenemos otros cuatro parámetros de superficies isotérmicas completas que son superficies de Dupin. Como aplicación obtenemos soluciones explícitas de la ecuación de Calapso.*

Palabras clave. Superficies isotérmicas, Transformaciones de Ribaucour, Superficie de Dupin, ecuación de Calapso.

1. Introduction. The Ribaucour transformations for hypersurfaces parametrized by lines of curvature were classically studied by Bianchi [1]. They can be applied to obtain surfaces of constant Gaussian curvature and surfaces of constant mean curvature, from a given surface, with constant Gaussian curvature and constant mean curvature, respectively. An application of this method to minimal and cmc surfaces in $\mathbb{R}^3$ was obtained by Corro, Ferreira, and Tenenblat in [11]-[12]. For more applications of this method, see [10]-[13], [19], [24], [22] and [25].

A regular surface M is isothermic if, locally, near to each non umbilical point of M, it admits isothermic parameters, whose coordinate curves are curvature lines. Particular classes of isothermic surfaces are the constant mean curvature surfaces, quadrics surfaces, and surfaces whose lines of curvature have constant geodesic curvature. Isothermic surfaces are preserved by isometries of the ambient space $\mathbb{S}^3$, dilations and inversions. The classification of isothermic surfaces is an open problem.

The theory of isothermic surfaces has been widely studied by several eminent geometers as Christoffel [8], Darboux [17] and [16], Bianchi [1]. In their works the authors present transformations between surfaces that preserve the property of being isothermic, which are called transformations of Darboux-Bianchi. Cieśliński in [6] explores these Darboux-Bianchi transformations and in [7] the author study the Darboux

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transformations iterated via Clifford numbers. More recently in [9], the authors shows that two-step Darboux transforms with the same spectral parameter are obtained by a Sym-type construction.

In [2], the authors study surfaces with harmonic inverse mean curvature (HIMC surfaces), they also distinguish a subclass of $\theta - isothermic$ susrfaces, and if $\theta = 0$, the surfaces are isothermic.

In [5], the author shows that the theory of soliton surfaces, modified in an appropriate way, can be applied also to isothermic immersions in $\mathbb{R}^3$. In this case, the so called Sym's formula gives an explicit expression for the isothermic immersion with prescribed fundamental forms.

In [3], the authors shows how pairs of isothermic surfaces are given by curved flats in a pseudo Riemannian symmetric space and vice versa.

In [20], the author use quaternionic calculus to discuss the relation between curved flats in the symmetric space of point pairs and Darboux and Christoffel pairs of isothermic surfaces. In [21], using quaternionic calculus, the authors develop isothermic surface theory in codimension 2.

In [4], the author shows that for each isothermic surface there is a solution of an equation with partial derivatives of fourth order, called the Calapso equation, given by

$$\left(\frac{\omega_{,u_1u_2}}{w}\right)_{,u_1u_1} + \left(\frac{\omega_{,u_1u_2}}{w}\right)_{,u_2u_2} + \left(\omega^2\right)_{,u_1u_2} = 0.$$

This solution depends on the metric and on the mean curvature of the surface. The Calapso equation is very difficult to solve and is strongly connected to the Painleve ODEs, some authors have found solutions of this equation associated with constant mean curvature surfaces.

It is noted that the transition $u_2 \to iu_2$ takes the Calapso equation to the Zoomeron equation

$$\left(\frac{\omega_{,u_1u_2}}{w}\right)_{,u_1u_1} - \left(\frac{\omega_{,u_1u_2}}{w}\right)_{,u_2u_2} + \left(\omega^2\right)_{,u_1u_2} = 0.$$

Another equation that also is very difficult to solve. The solitons of this equation are referred to as Zoomerons, where they possess the properties of Trappon solitons because they are like particles trapped in a potential well, changing direction indefinitely.

In [18], the authors introduced the Davey-Stewartson (DS) equation. This equation describe the evolution of a three-dimensional wave-packet on water of finite depth in the fluid dynamics.

In [23], the authors shows that for each $\phi(u_1, u_2)$ solution of the Zoomeron equation, associates a two-parameter family of the solutions $u(u_1, u_2, t)$ given by

$$u = e^{i(\nu u_1 + \mu u_2 + \mu\nu t)}\phi(u_1 + \mu t, u_2 + \nu t), \quad \rho = \frac{\left|u\right|_{,u_1u_2}}{u},$$

to the Davey-Stewartson III equation

$$iu_{,t} = u_{,u_1u_2} - \rho u, \quad \rho_{,u_1u_1} - \rho_{,u_2u_2} + \left(\left|u\right|^2\right)_{,u_1u_2} = 0.$$

In [14], the authors introduce the class of radial inverse mean curvature surfaces (RIMC - surfaces), which are isothermic surfaces. In addition, they show that for each isothermic surface there is another solution to the Calapso equation which depends on the metric and on the skew curvature of the surface. This solution is different from the one presented in [4].

In [15], the authors used Ribaucour transformations to obtain new isothermic surfaces in $\mathbb{R}^3$ associated to cylinder. Moreover, the authors obtained new solutions to the Calapso and Zoomeron equation.

In this paper, motivated by [15], we use the Ribaucour transformations to obtain a family of isothermic surfaces from a given such surface. As an application of the theory, we obtain two families of complete isothermic surfaces associated to the torus $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$. One of them families depends on four parameters. One of the parameters is b: when $b = 0$, the surfaces have constant mean curvature $\frac{r_2^2 - r_1^2}{2r_1r_2}$ and depending on the sign of b and using stereographic projection, we have surfaces in $\mathbb{R}^3$ with n-bubbles inside or outside the torus of the $\mathbb{R}^3$. A second parameter is c: if $\sqrt{|r_1^2 + cr_1^2r_2^2|}$ is not a rational number, then the stereographic projection of the surfaces obtained are complete immersions of $\mathbb{R}^2$ into $\mathbb{R}^3$, and they are not periodic in any of the two variables. However, if $\sqrt{|r_1^2 + cr_1^2r_2^2|} = \frac{n}{m}$ is an irreducible rational number, then the stereographic projection of the surfaces obtained are torus immersed in $\mathbb{R}^3$ with n-bubbles, two ends of geometric index m, and n isolated points of maximum and of minimum for the Gaussian curvature. The last parameters appear from the integration of the Ribaucour transformation.

Moreover, motivated by [4] and [14], we associate explicit solutions to the Calapso equation for each isothermic surface obtained by the Ribaucour transformations. Such solutions depend on two functions, each one defined in a given variable. Applying isometries, dilations and inversions, we obtain new isothermic surfaces and new solutions to the Calapso equation.

2. Preliminary. This section contains the definitions and the basic theory of Ribaucour transformations for surfaces in a space form. (For more details see [25])

(i) A congruence of geodesic spheres in $\mathbb{S}^3$ is a family of n-parameter geodesic spheres in $\mathbb{S}^3$ such that the set of centers of the geodesic spheres is a surface of $\mathbb{S}^3$ and the radii of the geodesic spheres are given by a differentiable function on the surface.

(ii) An involute of a congruence of geodesic spheres is an two-dimensional submanifold M of $\mathbb{S}^3$, such that each point of M is tangent to a geodesic sphere of the congruence of geodesic spheres.

(iii) Let M and $\widetilde{M}$ be the surfaces in $\mathbb{S}^3$. We say that M and $\widetilde{M}$ are associated by a congruence of geodesic spheres, if there exists a diffeomorfism $\psi : M \to \widetilde{M}$ such that, at the corresponding points p and $\psi(p)$, M and $\widetilde{M}$ are tangent to a same geodesic sphere of the congruence of geodesic spheres.

An important special case of item (iii) is when $d\psi$ maps n principal vector fields of M to n principal vector fields of $\widetilde{M}$.

Let M be an orientable surface in $\mathbb{S}^3$ without umbilic points, with Gauss map we denote by N. Suppose that there exist 2 orthonormal principal vector fields e_1 and e_2 defined on M. We say that $\widetilde{M} \subset \mathbb{S}^3$ is associated to M by a Ribaucour transformation with respect to e_1 and e_2, if there exist a differentiable function h defined on M and a diffeomorphism $\psi : M \to \widetilde{M}$ such that

(a) for all $p \in M$, $exp_p h(p)N(p) = exp_{\psi(p)} h(p)\widetilde{N}(\psi(p))$, where $\widetilde{N}$ is the Gauss map of $\widetilde{M}$ and exp is the exponential map of $\mathbb{S}^3$;

(b) the subset $\{exp_p h(p)N(p), p \in M\}$, is a two-dimensional submanifold of $\mathbb{S}^3$;

(c) $d\psi(e_i)$ $1 \leq i \leq 2$ are orthogonal principal directions of $\widetilde{M}$.

Remark 2.1. Let M and $\widetilde{M}$ be surfaces of $\mathbb{S}^3$ as in definition above, we can rewrite condition (a) above as

$$p + \tan(\phi(p))N(p) \;=\; \psi(p) + \tan(\phi(p))\widetilde{N}(p),$$

where $\phi : M \to \left(0\,,\,\frac{\pi}{2}\right)$.

The following result gives a characterization of Ribaucour transfomations in terms of differencial equations, when the ambient space is $\mathbb{S}^3$. (see [25] for a proof and more details)

Theorem 2.1. *Let M be an orientable surface of $\mathbb{S}^3$ parametrized by $X : U \subseteq \mathbb{R}^2 \to M$, without umbilic points. Assume $e_i = \frac{X_{,i}}{a_i}$, $1 \leq i \leq 2$ where $a_i = \sqrt{g_{ii}}$ are orthogonal principal directions, $-\lambda_i$ the corresponding principal curvatures, and N is a unit vector field normal to M. A surface $\widetilde{M}$ is locally associated to M by a Ribaucour transformation if and only if there is differentiable functions $W, \Omega, \Omega_i : V \subseteq U \to R$ which satisfy*

$$\begin{aligned} \Omega_{i,j} &= \Omega_j \frac{a_{j,i}}{a_i}, \qquad for \;\; i \neq j, \\ \Omega_{,i} &= a_i \Omega_i, \\ W_{,i} &= -a_i \Omega_i \lambda_i. \end{aligned} \tag{2.1}$$

$W(W + \lambda_i\Omega) \neq 0$ and $\widetilde{X} : V \subseteq U \to \widetilde{M}$, is a parametrization of $\widetilde{M}$ given by

$$\widetilde{X} = \left(1 - \frac{2\Omega^2}{S}\right)X - \frac{2\Omega}{S}\left(\sum_{i=1}^{2} \Omega_i e_i - WN\right), \tag{2.2}$$

where

$$S = \sum_{i=1}^{2} (\Omega_i)^2 + W^2 + \Omega^2. \tag{2.3}$$

Moreover, the normal map of $\widetilde{X}$ is given by

$$\widetilde{N} = N + \frac{2W}{S}\left(\sum_{i=1}^{2} \Omega_i e_i - WN + \Omega X\right), \tag{2.4}$$

and the principal curvatures and coefficients of the first fundamental form of $\widetilde{X}$, are given by

$$\widetilde{\lambda}_i = \frac{WT_i + \lambda_i S}{S - \Omega T_i}, \qquad \widetilde{g}_{ii} = \left(\frac{S - \Omega T_i}{S}\right)^2 g_{ii}, \tag{2.5}$$

where Ω_i, Ω and W satisfy (2.1), S is given by (2.3), g_{ii}, $1 \leq i \leq 2$ are coefficients of the first fundamental form of X, and

$$T_1 = 2\left(\frac{\Omega_{1,1}}{a_1} + \frac{a_{1,2}}{a_1 a_2}\Omega_2 - W\lambda_1 + \Omega\right),\ T_2 = 2\left(\frac{\Omega_{2,2}}{a_2} + \frac{a_{2,1}}{a_1 a_2}\Omega_1 - W\lambda_2 + \Omega\right). \tag{2.6}$$

Remark 2.2. Let $Y : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a surface parametrized by lines of curvature, with principal curvatures, $-k_i$, $1 \leq i \leq 2$, first fundamental form I_Y and $N_Y : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a unit normal vector field of Y. Consider the reverse application of stereographic projection $\pi^{-1} : \mathbb{R}^3 \to \mathbb{S}^3$, given by

$$\pi^{-1}(x) = \frac{2}{1 + |x|^2}\left(x + \frac{|x|^2 - 1}{2}e_4\right),$$

where $x = (x_1, x_2, x_3, 0) \in \mathbb{R}^3$ and $e_4 = (0, 0, 0, 1)$.
Let $X = \pi^{-1} \circ Y : U \subset \mathbb{R}^2 \to \mathbb{S}^3$. Then the first fundamental form and the principal curvatures, $-\lambda_i$, $1 \leq i \leq 2$ of the X, are given by

$$I_X = \frac{1}{\Gamma^2} I_Y, \tag{2.7}$$

$$\lambda_i = \Gamma k_i - < Y, N_Y >, \tag{2.8}$$

where $\Gamma = \dfrac{1 + |Y|^2}{2}$.

A surface is called isothermic if it admits parametrization by lines of curvature and the first fundamental form is conformal.

In [4], the author define the Calapso equation by

$$\left(\frac{\omega_{,u_1u_2}}{w}\right)_{,u_1u_1} + \left(\frac{\omega_{,u_1u_2}}{w}\right)_{,u_2u_2} + \left(\omega^2\right)_{,u_1u_2} = 0. \tag{2.9}$$

Such equation describes isothermic surfaces in $\mathbb{R}^3$.

The following result gives a solutions of the Calapso equation. (see [14] for a proof and more details)

Theorem 2.2. *Let $X(u_1, u_2)$ be an isothermic surface in $\mathbb{R}^3$ with the first fundamental form givem by*

$$I = e^{2\varphi}\left(du_1^2 + du_2^2\right).$$

Then the functions $\omega = \sqrt{2}e^{\varphi}H$ and $\Omega = \sqrt{2}e^{\varphi}H'$ are solutions of the Calapso equation, where H is the mean curvature of X and H' is the skew curvature of M.

3. Ribaucour transformation for isothermic surface of $\mathbb{S}^3$. In this section we provides a sufficient condition for a Ribaucour transformation to transform a isothermic surface into another such surface.

Theorem 3.1. *Let M be a surfaces of $\mathbb{S}^3$ parametrized by $X : U \subseteq \mathbb{R}^2 \to M$, without umbilic points and let $\widetilde{M}$ parametrized by (2.2) be associated to M by a Ribaucour transformation, such that the normal lines intersect at a distance function h. Assume that $h = \frac{\Omega}{W}$ is not constant along the lines of curvature and the function Ω, Ω_i and W satisfy one of the additional relation*

$$T_1 + T_2 = \frac{2S}{\Omega} \quad \textit{or} \quad T_1 - T_2 = 0 \tag{3.1}$$

where S is given by (2.3) and T_i, $1 \leq i \leq 2$ are defined by (2.6). Then $\widetilde{M}$ parameterized by (2.2) is a isothermic surface, if and only if M is isothermic surface.

Proof: Suppose that $\widetilde{M}$ is a isothermic surface, then the coefficients of the first fundamental form of $\widetilde{X}$ satisfy, $\widetilde{g}_{11} = \widetilde{g}_{22}$. So, using (2.5), we have

$$\left(\frac{S - \Omega T_1}{S}\right)^2 g_{11} = \left(\frac{S - \Omega T_2}{S}\right)^2 g_{22}, \tag{3.2}$$

where g_{ii}, $1 \leq i \leq 2$ are the coefficients of the first fundamental form of X.
If $T_1 + T_2 = \frac{2S}{\Omega}$, then isolating T_1 and substituting in (3.2), we get $g_{11} = g_{22}$.

On the other hand, if $T_1 - T_2 = 0$, then we have from (3.2) that $g_{11} = g_{22}$. Therefore, M is a isothermic surface. $\square$

Conversely, suppose that M is a isothermic surface, then using (3.1), immediately from (2.5), we obtain that $\widetilde{M}$ is a isothermic surface.

Remark 3.1. Let $X : U \subseteq \mathbb{R}^2 \to M$ a isothermic parametrized for $M \subseteq \mathbb{S}^3$. So, the first fundamental form of X is given by $I = e^{2\varphi}\big(du_1^2 + du_2^2\big)$. Thus, the additional relations given by (3.1) are, respectively, equivalent to

$$\Delta\Omega - (\lambda_1 + \lambda_2)We^{2\varphi} + 2\Omega e^{2\varphi} = \frac{Se^{2\varphi}}{\Omega} \tag{3.3}$$

$$\Omega_{,11} - \Omega_{,22} - 2\varphi_{,1}\Omega_{,1} + 2\varphi_{,2}\Omega_{,2} - e^{2\varphi}(\lambda_1 - \lambda_2)W = 0 \tag{3.4}$$

In fact, under these conditions using (2.1), T_i $1 \leq i \leq 2$ given by (2.6), can be rewritten as

$$T_1 = \frac{2}{e^{2\varphi}}\Big(\Omega_{,11} - \varphi_{,1}\Omega_{,1} + \varphi_{,2}\Omega_{,2} - W\lambda_1 e^{2\varphi} + \Omega e^{2\varphi}\Big), \tag{3.5}$$

$$T_2 = \frac{2}{e^{2\varphi}}\Big(\Omega_{,22} - \varphi_{,2}\Omega_{,2} + \varphi_{,1}\Omega_{,1} - W\lambda_2 e^{2\varphi} + \Omega e^{2\varphi}\Big). \tag{3.6}$$

Therefore, $T_1 + T_2 = \frac{2}{e^{2\varphi}}\big(\Delta\Omega - e^{2\varphi}W(\lambda_1 + \lambda_2) + 2\Omega e^{2\varphi}\big)$ and the first additional relation of (3.1) is equivalent to (3.3). Substituting (3.5) in $T_1 - T_2 = 0$, we obtain (3.4).

Remark 3.2. Let X as in the previous remark. Then the parameterization $\widetilde{X}$ of $\widetilde{M}$, locally associated to X by a Ribaucour transformation, given by (2.2), is defined on

$$V = \{(u_1, u_2) \in U;\ \Omega T_1 - S \neq 0\}.$$

4. Families of isothermic surfaces associated to the Flat Torus in $\mathbb{S}^3$. In this section, by applying Theorem 3.1 to the Flat Torus in $\mathbb{S}^3$, we obtain two families of complete isothermic surfaces associated to the Torus, in $\mathbb{S}^3$. One of the families obtained depends on four parameters. The outher family is made up of Dupin surfaces.

Theorem 4.1. *Consider the Flat Torus in $\mathbb{S}^3$ parametrized by*

$$X(u_1, u_2) = (r_1\cos(r_2u_1), r_1\sin(r_2u_1), r_2\cos(r_1u_2), r_2\sin(r_1u_2)), \quad (u_1, u_2) \in \mathbb{R}^2 \tag{4.1}$$

r_i, $1 \leq i \leq 2$ are positive constants satisfying $r_1^2 + r_2^2 = 1$, as isothermic surface in $\mathbb{S}^3$ where the first fundamental form is $I = r_1^2r_2^2\big(du_1^2 + du_2^2\big)$. A parametrized surface $\widetilde{X}(u_1, u_2)$ is isothermic surface locally associated to X by a Ribaucour transformation as in Theorem 3.1 with additional relation given by (3.3), if and only if, up to an isometries of $\mathbb{S}^3$, it is given by

$$\widetilde{X} = \frac{1}{M}\begin{bmatrix} \big(M - 2r_2^2f\big)r_1\cos(r_2u_1) + 2r_1r_2f'\sin(r_2u_1), \\ \big(M - 2r_2^2f\big)r_1\sin(r_2u_1) - 2r_1r_2f'\cos(r_2u_1), \\ \big(M - 2r_1^2g\big)r_2\cos(r_1u_2) + 2r_1r_2g'\sin(r_1u_2), \\ \big(M - 2r_1^2g\big)r_2\sin(r_1u_2) - 2r_1r_2g'\cos(r_1u_2) \end{bmatrix} \tag{4.2}$$

defined on $V = \{(u_1, u_2) \in \mathbb{R}^2;\ f + g \neq 0\}$ where $M = 2b + (r_2^2 - c)(f - g)$ $c \neq 0$, b is real constant, and $f(u_1)$, $g(u_2)$ are solutions of the equations

$$\begin{aligned} f'' + \big(r_2^2 - r_1^2r_2^2c\big)f &= b, \\ g'' + \big(r_1^2 + r_1^2r_2^2c\big)g &= b \end{aligned} \tag{4.3}$$

with initial conditions satisfying

$$\big[(f')^2 + \big(r_2^2 - r_1^2r_2^2c\big)f^2 - 2bf + (g')^2 + \big(r_1^2 + r_1^2r_2^2c\big)g^2 - 2bg\big](u_1^0, u_2^0) = 0. \tag{4.4}$$

Moreover, the normal map of $\widetilde{X}$ is given by

$$\widetilde{N} = \frac{1}{M(f+g)}\begin{bmatrix} \big(2r_2^2f^2 - 2r_1^2fg - M(f+g)\big)r_2\cos(r_2u_1) - 2(r_2^2f - r_1^2g)f'\sin(r_2u_1), \\ \big(2r_2^2f^2 - 2r_1^2fg - M(f+g)\big)r_2\sin(r_2u_1) + 2(r_2^2f - r_1^2g)f'\cos(r_2u_1), \\ \big(-2r_1^2g^2 + 2r_2^2fg + M(f+g)\big)r_1\cos(r_1u_2) - 2(r_2^2f - r_1^2g)g'\sin(r_1u_2), \\ \big(-2r_1^2g^2 + 2r_2^2fg + M(f+g)\big)r_1\sin(r_1u_2) + 2(r_2^2f - r_1^2g)g'\cos(r_1u_2) \end{bmatrix}.$$

Proof: Consider the first fundamental form of the Flat Torus $ds^2 = r_1^2 r_2^2(du_1^2 + du_2^2)$ and the principal curvatures $-\lambda_i$ $1 \le i \le 2$ given by $\lambda_1 = \frac{-r_2}{r_1}$, $\lambda_2 = \frac{r_1}{r_2}$. Using (2.1), to obtain the Ribaucour transformations, we need to solve the following of equations

$$\Omega_{i,j} = 0, \qquad \Omega_{,i} = r_1 r_2 \Omega_i, \qquad W_{,i} = -r_1 r_2 \Omega_i \lambda_i, \;\; 1 \le i \ne j \le 2. \tag{4.5}$$

Therefore we obtain

$$\Omega = r_1 r_2\big(f_1(u_1) + f_2(u_2)\big),\; W = -r_1 r_2\big(\lambda_1 f_1 + \lambda_2 f_2\big) + \overline{c},\; \Omega_i = f_i',\; 1 \le i \ne j \le 2, \tag{4.6}$$

where $\overline{c}$ is a real constant. Thus, from (2.3), $S = (f_1')^2 + (f_2')^2 + (W)^2 + \Omega^2$.
Using (3.3), the associated surface will be isothermic when

$$\frac{\Delta\Omega}{r_1^2 r_2^2} - (\lambda_1 + \lambda_2)W + 2\Omega = \frac{S}{\Omega}.$$

Therefore, we obtain that the functions f_1 and f_2 satisfy

$$\frac{f_1'' + f_2''}{r_1 r_2} - (\lambda_1 + \lambda_2)W + \Omega = \frac{(f_1')^2 + (f_2')^2 + (W)^2}{\Omega}. \tag{4.7}$$

Differentiate this last equation with respect x_1 and x_2, using (4.5) and (4.6) we get

$$\frac{f_1'''}{r_1^2 r_2^2} + r_1 r_2\big(1 + \lambda_1^2\big)f_1' = \frac{-f_1'}{\Omega}\left(\frac{-f_1'' + f_2''}{r_1 r_2} - \big(\lambda_2 - \lambda_1\big)W\right), \tag{4.8}$$

$$\frac{f_2'''}{r_1^2 r_2^2} + r_1 r_2\big(1 + \lambda_2^2\big)f_2' = \frac{f_2'}{\Omega}\left(\frac{-f_1'' + f_2''}{r_1 r_2} - \big(\lambda_2 - \lambda_1\big)W\right). \tag{4.9}$$

From this last equation, we obtain

$$\frac{f_1'' - f_2''}{r_1 r_2} + \big(\lambda_2 - \lambda_1\big)W = c\,\Omega. \tag{4.10}$$

where c is a real constant. Thus from (4.6), we have that f_1 and g_2 satisfy

$$f_1'' + r_1^2 r_2^2\big(\lambda_1^2 + 1 - c\big)f_1 = f_2'' + r_1^2 r_2^2\big(\lambda_2^2 + 1 + c\big)f_2 - \overline{c} r_1 r_2(\lambda_2 - \lambda_1).$$

Now defining $f(u_1) = f_1(u_1) - \dfrac{\overline{c}}{r_1 r_2(\lambda_1 - \lambda_2)}$ and $g(u_2) = f_2(u_2) + \dfrac{\overline{c}}{r_1 r_2(\lambda_1 - \lambda_2)}$, we obtain that f and g satisfy (4.3) and

$$\Omega = r_1 r_2\big(f(u_1) + g(u_2)\big), \qquad W = -r_1 r_2\big(\lambda_1 f + \lambda_2 g\big). \tag{4.11}$$

Moreover, using (4.7) we get that the initial conditions satisfying (4.4) and using the Theorem 2.1, $\widetilde{X}$ is give by (4.2) and from Remark 3.2 is defined in $V = \{(u_1, u_2) \in \mathbb{R}^2;\; f + g \ne 0\}$. □

Remark 4.1. Each isothermic surfaces associated to the flat torus as in Theorem 4.1, is parametrized by lines of curvature and from (2.5), the metric is given by $ds^2 = \psi^2(du_1^2 + du_2^2)$, where

$$\psi = \frac{|c r_1^2 r_2^2(f + g)|}{|2b + c r_1^2 r_2^2(f - g)|}. \tag{4.12}$$

Moreover, from (2.5), the principal curvatures of the $\widetilde{X}$ are given by

$$\widetilde{\lambda}_1 = \frac{\big(2b + 2c r_1^2 r_2^2 f\big)g - c r_1^2 r_2^4\big(f + g\big)^2}{c r_1^3 r_2^3\big(f + g\big)^2},\;\; \widetilde{\lambda}_2 = \frac{\big(2b - 2c r_1^2 r_2^2 g\big)f + c r_1^4 r_2^2\big(f + g\big)^2}{c r_1^3 r_2^3\big(f + g\big)^2}. \tag{4.13}$$

Using (4.13) we obtain immediately

Proposition 4.1. *Consider the isothermic surfaces associated to the flat torus parametrized by (4.2). Then the mean curvature of the $\widetilde{X}$ is given by*

$$\widetilde{H} = \frac{r_2^2 - r_1^2}{2 r_1 r_2} - \frac{b}{c r_1^3 r_2^3\big(f + g\big)}. \tag{4.14}$$

Proposition 4.2. *Consider the isothermic surfaces associated to the flat torus parametrized by (4.2). Then* $\widetilde{X}$ *is* $\frac{r_2^2-r_1^2}{2r_1r_2}-cmc$*, if and only if,* $b=0$.

Proposition 4.3. *Consider the isothermic surfaces associated to the flat torus parametrized by (4.2). Then the surface* $\widetilde{X}$ *is is determined by the functions*

(i) If $c>\frac{1}{r_1^2}$*, then*

$$f=\sqrt{A_1}\cosh\left(r_2\sqrt{-1+r_1^2c}\,u_1\right)+\frac{b}{r_2^2(1-r_1^2c)},$$
$$g=\sqrt{B_1}\sin\left(r_1\sqrt{1+r_2^2c}\,u_2\right)+\frac{b}{r_1^2(1+r_2^2c)}, \tag{4.15}$$

where $\dfrac{b^2}{r_2^2r_1^2(1+r_2^2c)(-1+r_1^2c)}=r_2^2(-1+r_1^2c)A_1-r_1^2(1+r_2^2c)B_1$*, with* $B_1>0$.

(ii) If $\dfrac{-1}{r_2^2}<c<\dfrac{1}{r_1^2}$*, then*

$$f=\sqrt{A_1}\sin\left(r_2\sqrt{1-r_1^2c}\,u_1\right)+\frac{b}{r_2^2(1-r_1^2c)},$$
$$g=\sqrt{B_1}\sin\left(r_1\sqrt{1+r_2^2c}\,u_2\right)+\frac{b}{r_1^2(1+r_2^2c)}, \tag{4.16}$$

where $\dfrac{b^2}{r_2^2r_1^2(1+r_2^2c)(1-r_1^2c)}=r_2^2(1-r_1^2c)A_1+r_1^2(1+r_2^2c)B_1$*, with* $B_1>0$ *and* $A_1>0$.

(iii) If $c<\dfrac{-1}{r_2^2}$*, then*

$$f=\sqrt{A_1}\sin\left(r_2\sqrt{1-r_1^2c}\,u_1\right)+\frac{b}{r_2^2(1-r_1^2c)},$$
$$g=\sqrt{B_1}\cosh\left(r_1\sqrt{-1-r_2^2c}\,u_2\right)+\frac{b}{r_1^2(1+r_2^2c)}, \tag{4.17}$$

where $\dfrac{b^2}{r_2^2r_1^2(1+r_2^2c)(1-r_1^2c)}=r_2^2(1-r_1^2c)A_1+r_1^2(1+r_2^2c)B_1$*, with* $A_1>0$.

(iv) If $c=\dfrac{-1}{r_2^2}$*, then*

$$f=\sqrt{A_1}\sin(u_1)+b,\qquad g=\frac{b}{2}u_2^2+a_2u_2+b_2, \tag{4.18}$$

where $A_1+a_2^2+b_2^2-\left(b+b_2\right)^2=0$*, with* $A_1>0$.

(v) If $c=\dfrac{1}{r_1^2}$*, then*

$$f=\frac{b}{2}u_1^2+a_2u_1+b_2,\qquad g=\sqrt{B_1}\sin(u_2)+b, \tag{4.19}$$

where $B_1+a_2^2+b_2^2-\left(b+b_2\right)^2=0$*, with* $B_1>0$.

Proof: Consider the isothermic surfaces associated to the flat torus parametrized by (4.2) that is not cmc.

If $c>\frac{1}{r_1^2}$, then using (4.3), we get

$$f=a_1\cosh\left(r_2\sqrt{-1+r_1^2c}u_1\right)+a_2\sinh\left(r_2\sqrt{-1+r_1^2c}u_1\right)+\frac{b}{r_2^2(1-r_1^2c)},$$
$$g=b_1\sin\left(r_1\sqrt{1+r_2^2c}u_2\right)+b_2\cos\left(r_1\sqrt{1+r_2^2c}u_2\right)+\frac{b}{r_1^2(1+r_2^2c)},$$

Where from (4.4) we have

$$\frac{b^2}{r_2^2r_1^2(1+r_2^2c)(-1+r_1^2c)}+r_1^2(1+r_2^2c)(b_1^2+b_2^2)=r_2^2(-1+r_1^2c)(a_1^2-a_2^2).$$

This last equation, we have $a_1^2 - a_2^2 > 0$ and we can rewrite f and g as

$$f = \sqrt{A_1}\cosh\left(r_2\sqrt{-1+r_1^2c}u_1 + A_2\right) + \frac{b}{r_2^2(1-r_1^2c)},$$
$$g = \sqrt{B_1}\sin\left(r_1\sqrt{1+r_2^2c}u_2 + B_2\right) + \frac{b}{r_1^2(1+r_2^2c)},$$

where $A_1 = a_1^2 - a_2^2$ and $B_1 = b_1^2 + b_2^2$.
The constants A_2 and B_2, without loss of generality, my be considered to be zero. One can verify that the surfaces with different values of A_2 and B_2 are congruent. In fact, using the notation $\widetilde{X}_{A_2B_2}$ for the surface $\widetilde{X}$ with fixed constants A_2 and B_2, we have

$$\widetilde{X}_{A_2B_2} = R_{(\frac{-A_2}{c_1}, \frac{-B_2}{c_2})}\widetilde{X}_{00} \circ h$$

where $h(u_1, u_2) = \left(u_1 + \dfrac{A_2}{c_1}, u_1 + \dfrac{B_2}{c_2}\right)$ with $c_1 = r_2\sqrt{-1+r_1^2c}$, $c_2 = r_1\sqrt{1+r_2^2c}$ and

$$R_{(\theta,\phi)}(x_1, x_2, x_3, x_4) = (x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta, x_3\cos\phi - x_4\sin\phi, x_3\sin\phi + x_4\cos\phi).$$

We conclude that f and g are given by (4.15). Where from (4.4) we obtain
$\dfrac{b^2}{r_2^2r_1^2(1+r_2^2c)(-1+r_1^2c)} = r_2^2(-1+r_1^2c)A_1 - r_1^2(1+r_2^2c)B_1$, with $B_1 > 0$.

Finally, with analogous argument, we obtain (4.16)-(4.19). □

Proposition 4.4. *Any isothermic surfaces associated to the flat torus $\widetilde{X}$, given by Theorem 4.1 is complete.*

Proof: For divergent curves $\gamma(t) = (u_1(t), u_2(t))$, such that $\lim\limits_{t\to\infty}\left(u_1^2 + u_2^2\right) = \infty$, we have $l(\widetilde{X}\circ\gamma) = \infty$.

In fact, if $c \geq \frac{1}{r_1^2}$, then the functions f and g are given by (4.15), and from (4.12) the coefficients of the first fundamental form is

$$\psi = \frac{|cr_1^2r_2^2(f+g)|}{|2b + cr_1^2r_2^2(f-g)|}.$$

Therefore, $\lim\limits_{|u_1|\to\infty}\psi = 1$ uniformly in u_2. Hence, there exist $k > 0$ such that $|\psi(u_1, u_2)| > \frac{1}{2}$ for $(u_1, u_2) \in \mathbb{R}^2$ with $|u_1| > k$. Let

$$m = min\left\{|\psi(u_1, u_2)|;\ (u_1, u_2) \in [-k, k] \times \left[0, \frac{2\pi}{r_1\sqrt{1+r_2^2c}}\right]\right\}. \tag{4.20}$$

Note that, $g(u_2) = g(u_2 + \frac{2\pi}{r_1\sqrt{1+r_2^2c}})$, therefore $|\psi(u_1, u_2)| > m$ in $[-k, k] \times R$. Consider $m_0 = min\{m, \frac{1}{2}\}$, then $|\psi(u_1, u_2)| > m_0$ in $\mathbb{R}^2$. Thus $l(\widetilde{X}\circ\gamma) = \infty$. The case $c \leq \dfrac{-1}{r_2^2}$ is analogous.

Finally if $\dfrac{-1}{r_2^2} < c < \dfrac{1}{r_1^2}$, then the functions f and g are given by (4.16), and from (4.12) the coefficients of the first fundamental form is $\psi = \dfrac{|cr_1^2r_2^2(f+g)|}{|2b + cr_1^2r_2^2(f-g)|}$. In this case, let

$$m_0 = min\left\{|\psi(u_1, u_2)|;\ (u_1, u_2) \in \left[0, \frac{2\pi}{r_2\sqrt{1-r_1^2c}}\right] \times \left[0, \frac{2\pi}{r_1\sqrt{1+r_2^2c}}\right]\right\}. \tag{4.21}$$

Note that, $g(u_2) = g(u_2 + \frac{2\pi}{r_1\sqrt{1+r_2^2c}})$ and $f(u_1) = f(u_1 + \frac{2\pi}{r_2\sqrt{1-r_1^2c}})$, therefore $|\psi(u_1, u_2)| > m_0$ in $\mathbb{R}^2$. Thus $l(\widetilde{X}\circ\gamma) = \infty$ and we conclude that $\widetilde{X}$ is a complete surface. □

Remark 4.2. Consider the isothermic surfaces associated to the flat torus parametrized by (4.2).

I) If $c > \frac{1}{r_1^2}$, then f and g are given by (4.15) and if $\sqrt{1+cr_2^2} = \frac{n}{m}$ irreducible rational number, then $\widetilde{X}(u_1, u_2) = \widetilde{X}(u_1, u_2 + \frac{2m\pi}{r_1})$, i.e, $\widetilde{X}$ is periodic in the variable u_2 with period $\frac{2m\pi}{r_1}$.
Let $\pi : \mathbb{S}^3 \to \mathbb{R}^3$ be the stereographic projection. Therefore we get an n-bubble surface into $\mathbb{R}^3$ parametrized by $\widetilde{Y} = \pi\circ\widetilde{X}$. Moreover, if $b < 0$, then we have n-bubble outside $\widetilde{Y}$, and if $b > 0$, we have n-bubble inside $\widetilde{Y}$.

For example, consider $r_1 = \frac{12}{13}$, $r_2 = \frac{5}{13}$ $b = -8\sqrt{165}$, $c = \frac{5408}{45}$, $A_1 = 4$. $B_1 = 1$. In this case, $\sqrt{1+cr_2^2} = \frac{13}{3}$. Let $\pi : \mathbb{S}^3 \to \mathbb{R}^3$ be the stereographic projection. Therefore we have 13-bubble surface into $\mathbb{R}^3$ parametrized by $\widetilde{Y} = \pi \circ \widetilde{X}$. (See Figure 4.1)

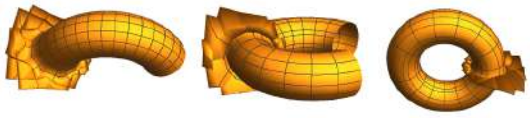

Figure 4.1

Another example, consider $r_1 = \frac{12}{13}$, $r_2 = \frac{5}{13}$ $b = \frac{-504\sqrt{18469}}{2197}$, $c = \frac{1352}{25}$, $A_1 = 4$. $B_1 = 1$. In this case, $\sqrt{1+cr_2^2} = \frac{3}{1}$. Let $\pi : \mathbb{S}^3 \to \mathbb{R}^3$ be the stereographic projection. Therefore wee have 3-bubble surface into $\mathbb{R}^3$ parametrized by $\widetilde{Y} = \pi \circ \widetilde{X}$. (See Figure 4.2)

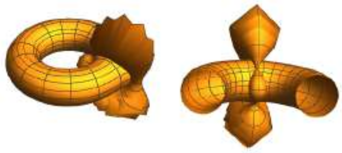

Figure 4.2

The next example, consider $r_1 = \frac{12}{13}$, $r_2 = \frac{5}{13}$ $b = \frac{48\sqrt{107041}}{2197}$, $c = \frac{507}{25}$, $A_1 = 4$. $B_1 = 1$. In this case, $\sqrt{1+cr_2^2} = \frac{2}{1}$. Let $\pi : \mathbb{S}^3 \to R^4$ be the stereographic projection. Therefore we have 2-bubble surface into $\mathbb{R}^3$ parametrized by $\widetilde{Y} = \pi \circ \widetilde{X}$. In this case, $b > 0$ thus we have 2-bubble inside torus. (See Figure 4.3)

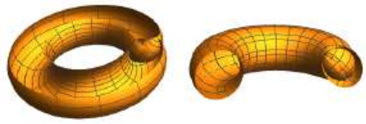

Figure 4.3

II) If $-\frac{1}{r_2^2} < c < \frac{1}{r_1^2}$, then f and g are given by (4.16) and if $\sqrt{1+cr_2^2} = \frac{n}{m}$ or $\sqrt{1-cr_2^2} = \frac{n}{m}$ irreducible rational number, then $\widetilde{X}(u_1,u_2) = \widetilde{X}(u_1, u_2 + \frac{2m\pi}{r_1})$ or $\widetilde{X}(u_1,u_2) = \widetilde{X}(u_1 + \frac{2m\pi}{r_2}, u_2)$, i.e, $\widetilde{X}$ can be periodic in the variable u_2 with period $\frac{2m\pi}{r_1}$ or in the variable u_1 with period $\frac{2m\pi}{r_2}$ or in both variables.

Theorem 4.2. *Consider the Flat Torus in $\mathbb{S}^3$ parametrized by*

$$X(u_1,u_2) = (r_1\cos(r_2u_1), r_1\sin(r_2u_1), r_2\cos(r_1u_2), r_2\sin(r_1u_2)), \quad (u_1,u_2) \in \mathbb{R}^2 \tag{4.22}$$

r_i, $1 \le i \le 2$ are positive constants satisfying $r_1^2 + r_2^2 = 1$, as isothermic surface in $\mathbb{S}^3$ where the first fundamental form is $I = r_1^2r_2^2(du_1^2 + du_2^2)$. A parametrized surface $\widetilde{X}(u_1,u_2)$ is isothermic surface locally associated to X by a Ribaucour transformation as in Theorem 3.1 with additional relation given by (3.4), if and only if, up to an isometries of $\mathbb{S}^3$, it is given by

$$\widetilde{X} = \frac{1}{S}\begin{bmatrix} \big(r_1S - 2r_1\Omega^2 - 2r_2\Omega W\big)\cos(r_2u_1) + r_2\sqrt{A_1}\Omega\sin(2r_2u_1), \\ \big(r_1S - 2r_1\Omega^2 - 2r_2\Omega W\big)\sin(r_2u_1) - 2r_2\sqrt{A_1}\Omega\cos^2(r_2u_1), \\ \big(r_2S - 2r_2\Omega^2 + 2r_1\Omega W\big)\cos(r_1u_2) + r_1\sqrt{B_1}\Omega\sin(2r_1u_2), \\ \big(r_2S - 2r_2\Omega^2 + 2r_1\Omega W\big)\sin(r_1u_2) - 2r_1\sqrt{B_1}\Omega\cos^2(r_1u_2) \end{bmatrix} \tag{4.23}$$

defined on $V = \{(u_1,u_2) \in \mathbb{R}^2;\ f + g \neq 0\}$ where

$$S = \frac{b^2}{r_1^2r_2^2} + 2b\sqrt{A_1}\sin(r_2u_1) + 2b\sqrt{B_1}\sin(r_1u_2) + B_1r_1^2 + A_1r_2^2 \tag{4.24}$$

$$\Omega = r_1r_2\big(\sqrt{A_1}\sin(r_2u_1) + \sqrt{B_1}\sin(r_1u_2)\big) + \frac{b}{r_1r_2} \tag{4.25}$$

$$W = r_2^2\sqrt{A_1}\sin(r_2u_1) - r_1^2\sqrt{B_1}\sin(r_1u_2), \tag{4.26}$$

with the constants satisfying $r_1^2 r_2^2\big(B_1 r_1^2 + A_1 r_2^2\big) - b^2 \neq 0$ *Moreover, the normal map of* $\widetilde{X}$ *is given by*

$$\widetilde{N} = \frac{1}{S}\begin{bmatrix} \big(-r_2 S - 2r_2 W^2 + 2r_1\Omega W\big)\cos(r_2 u_1) + r_2\sqrt{A_1}\,\Omega\sin(2r_2u_1), \\ \big(-r_2 S - 2r_2 W^2 + 2r_1\Omega W\big)\sin(r_2 u_1) - 2r_2\sqrt{A_1}\,\Omega\cos^2(r_2u_1), \\ \big(r_1 S + 2r_1 W^2 - 2r_2\Omega W\big)\cos(r_1 u_2) + r_1\sqrt{B_1}\,\Omega\sin(2r_1u_2), \\ \big(r_1 S + 2r_1 W^2 - 2r_2\Omega W\big)\sin(r_1 u_2) - 2r_1\sqrt{B_1}\,\Omega\cos^2(r_1u_2) \end{bmatrix}. \tag{4.27}$$

Proof: Consider the first fundamental form of the Flat Torus $ds^2 = r_1^2 r_2^2(du_1^2 + du_2^2)$ and the principal curvatures $\lambda_1 = \frac{-r_2}{r_1}$, $\lambda_2 = \frac{r_1}{r_2}$. Using (2.1), to obtain the Ribaucour transformations, we need to solve the following of equations

$$\Omega_{i,j} = 0, \qquad \Omega_{,i} = r_1 r_2 \Omega_i, \qquad W_{,i} = -r_1 r_2 \Omega_i \lambda_i, \ \ 1 \le i \neq j \le 2. \tag{4.28}$$

Therefore we obtain

$$\Omega = r_1 r_2\big(f_1(u_1) + f_2(u_2)\big), \ W = -r_1 r_2\big(\lambda_1 f_1 + \lambda_2 f_2\big) + \bar{c}, \ \Omega_i = f_i', \ 1 \le i \neq j \le 2,$$

where $\bar{c}$ is a real constant. Thus, from (2.3), $S = (f_1')^2 + (f_2')^2 + (W)^2 + \Omega^2$.
Using (3.4), the associated surface will be isothermic when $\dfrac{\Omega_{,11} - \Omega_{,22}}{r_1^2 r_2^2} - (\lambda_1 - \lambda_2)W = 0$. Therefore, using (4.6) we obtain that the functions f_1 and g_2 satisfy

$$f_1'' - f_2'' = r_1^2 f_2 - r_2^2 f_1 - \bar{c}.$$

Now defining $f = f_1 + \bar{c}$ and $g = f_2 - \bar{c}$, we obtain that

$$\Omega = r_1 r_2\big(f(u_1) + g(u_2)\big), \qquad W = -r_1 r_2\big(\lambda_1 f + \lambda_2 g\big). \tag{4.29}$$

and f and g satisfy

$$f'' + r_2^2 f = g'' + r_1^2 g = b.$$

Therefore, f and g are given by

$$f = \sqrt{A_1}\sin\big(r_2 u_1 + A_2\big) + \frac{b}{r_2^2} \quad g = \sqrt{B_1}\sin\big(r_1 u_2 + B_2\big) + \frac{b}{r_1^2}.$$

The constants A_2 and B_2, without loss of generality, my be considered to be zero. One can verify that the surfaces with different values of A_2 and B_2 are congruent. In fact, using the notation $\widetilde{X}_{A2B2}$ for the surface $\widetilde{X}$ with fixed constants A_2 and B_2, we have

$$\widetilde{X}_{A2B2} = R_{(\frac{-A_2}{r_2}, \frac{-B_2}{r_1})}\widetilde{X}_{00} \circ h$$

where $h(u_1, u_2) = \left(u_1 + \dfrac{A_2}{r_2}, u_1 + \dfrac{B_2}{r_1}\right)$ and

$R_{(\theta,\phi)}(x_1, x_2, x_3, x_4) = (x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta, x_3\cos\phi - x_4\sin\phi, x_3\sin\phi + x_4\cos\phi)$.

We conclude that f and g are given by

$$f = \sqrt{A_1}\sin\big(r_2 u_1\big) + \frac{b}{r_2^2}, \qquad g = \sqrt{B_1}\sin\big(r_1 u_2\big) + \frac{b}{r_1^2}.$$

Therefore substituting in (4.29) and $S = (f_1')^2 + (f_2')^2 + (W)^2 + \Omega^2$, we have (4.25)-(4.26) and from (2.2) and (2.4) we have (4.23) and (4.27). From the condition of regularity we obtain that the constants satisfy $r_1^2 r_2^2\big(B_1 r_1^2 + A_1 r_2^2\big) - b^2 \neq 0$. □

Proposition 4.5. *Let* $\widetilde{X}$ *be a isothermic surface locally associated by a Ribaucour transformation to flat torus given by Theorem 4.2. Then* $\widetilde{X}$ *is a Dupin surfaces.*

Proof: From (2.5) the principal curvatures of the $\widetilde{X}$ are given

$$\widetilde{\lambda}_1 = \frac{-r_2\big(2br_1^2\sqrt{B_1}\sin(r_1u_2) + r_1^2r_2^2(r_1^2B_1 + r_2^2A_1) + b^2\big)}{r_1(r_1^4r_2^2B_1 + r_2^4r_1^2A_1 - b^2)}, \tag{4.30}$$

$$\widetilde{\lambda}_2 = \frac{r_1\big(2br_2^2\sqrt{A_1}\sin(r_2u_1) + r_1^2r_2^2(r_1^2B_1 + r_2^2A_1) + b^2\big)}{r_2(r_1^4r_2^2B_1 + r_2^4r_1^2A_1 - b^2)}. \tag{4.31}$$

Therefore, $\widetilde{X}$ is a Dupin surfaces. □

Remark 4.3. Let $\widetilde{X}$ be a isothermic surface locally associated by a Ribaucour transformation to flat torus given by Theorem 4.2 Using (2.5) we obtain that the first fundamental form of the $\widetilde{X}$ is given by

$$I = \left(\frac{1}{\widetilde{\lambda_1} - \widetilde{\lambda_1}}\right)^2 (du_1^2 + du_2^2), \tag{4.32}$$

where $-\widetilde{\lambda}_i$, $1 \leq i \leq 2$, are the principal curvature of the $\widetilde{X}$.

5. Solution of the Calapso Equation.. In the section, we obtain solutions of the Calapso Equation (2.9).

Proposition 5.1. *Consider the isothermic surfaces associated to the flat torus parametrized by (4.2), whose first fundamental form is given by*

$$I = \left(\frac{cr_1^2r_2^2(f+g)}{M}\right)^2 (du_1^2 + du_2^2),$$

where the functions f and g are given by (4.15)-(4.18) and $M = 2b + cr_1^2r_2^2(f-g)$. Then the functions

$$\omega = \frac{\epsilon\sqrt{2}}{2}\left(\frac{(r_2^2 - r_1^2)cr_1^2r_2^2(f+g) - 2b - 2cr_1^3r_2^3\Gamma\widetilde{N}_4(f+g)}{r_1r_2M}\right) \text{ and } \Omega = \frac{\epsilon\sqrt{2}(f-g)}{2r_1r_2(f+g)},$$

with $\epsilon = 1$ if $c > 0$ and $\epsilon = -1$ if $c < 0$, are solutions of the Calapso equation, where $\Gamma = \frac{1}{1-\widetilde{X}_4}$, $\widetilde{X}_4 =< \widetilde{X}, e_4 >$, $\widetilde{N}_4 =< \widetilde{N}, e_4 >$, $e_4 = (0,0,0,1)$ and $\widetilde{X}$, $\widetilde{N}$ are given by Theorem 4.1

Proof: Consider stereographic projection $\pi : \mathbb{S}^3 \to \mathbb{R}^3$

$$\pi(x_1, x_2, x_3, x_4) = \frac{1}{1 - x_4}\Big(x_1, x_2, x_3\Big) \tag{5.1}$$

where $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$.

Let $Y = \pi \circ \widetilde{X} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, where $\widetilde{X}$ is an isothermic surface locally associated by a Ribaucour transformation to flat torus, given by (4.2). Thus Y is an isothermic surface in $\mathbb{R}^3$, where by (2.7) and (4.12), the first fundamental form is given by

$$I_Y = \left(\frac{cr_1^2r_2^2(f+g)\Gamma}{M}\right)^2 (du_1^2 + du_2^2), \tag{5.2}$$

where $\Gamma = \frac{1}{1-\widetilde{X}_4}$, $\widetilde{X}_4 =< \widetilde{X}, e_4 >$, $e_4 = (0,0,0,1)$, $M = 2b + cr_1^2r_2^2(f-g)$ and the functions f and g are given by Proposition 4.3.

From stereographic projection, we obtain that the unit normal vector field of Y, is given by $N_Y = \dfrac{d\pi_{\widetilde{X}}(\widetilde{N})}{\Gamma}$, where $\widetilde{N}$ is given by Theorem 4.1 Therefore

$$N_Y = \widetilde{N}_4 Y + \widetilde{N} - \widetilde{N}_4 e_4, \tag{5.3}$$

where $\widetilde{N}_4 =< \widetilde{N}, e_4 >$. Hence we obtain that $< Y, N_Y >= \Gamma\widetilde{N}_4$ and from (2.8) we get that principal curvatures $-k_i$, $1 \leq i \leq 2$ of Y are given by

$$k_i = \widetilde{\lambda}_i(1 - \widetilde{X}_4) + \widetilde{N}_4, \tag{5.4}$$

where $-\widetilde{\lambda}_i$, $1 \leq i \leq 2$ are the principal curvature of the $\widetilde{X}$ given by (4.13).
Using (4.14) and this last equation, we obtain the mean and skew curvatures of Y,

$$H = \frac{(r_2^2 - r_1^2)cr_1^2 r_2^2(f+g) - 2b - 2\Gamma r_1^3 r2^3(f+g)\widetilde{N}_4}{2c\Gamma r_1^3 r_2^3(f+g)}, \tag{5.5}$$

$$H' = \frac{M(f-g)}{2c\Gamma r_1^3 r_2^3 (f+g)^2}. \tag{5.6}$$

Therefore, using the Theorem 2.2, we obtain that

$$\omega = \frac{\epsilon\sqrt{2}}{2}\left(\frac{(r_2^2 - r_1^2)cr_1^2 r_2^2(f+g) - 2b - 2cr_1^3 r_2^3 \Gamma \widetilde{N}_4(f+g)}{r_1 r_2 M}\right) \ \textit{and} \ \Omega = \frac{\epsilon\sqrt{2}(f-g)}{2r_1 r_2 (f+g)},$$

with $\epsilon = 1$ if $c > 0$ and $\epsilon = -1$ if $c < 0$, are solutions of the Calapso equation. $\square$

Example 5.1. *Consider the isothermic surfaces associated to the flat torus given by (4.2). We have $b = \frac{4}{3}$, $c = 3$, $r_1 = \frac{\sqrt{2}}{\sqrt{3}}$, $r_2 = \frac{1}{\sqrt{3}}$. In this case, $c = 3 > \frac{1}{r_1^2} = \frac{3}{2}$. Hence, from Proposition 4.3 f and g are given by (4.15). Choosing $A_1 = 16$ and $B_1 = 1$ we have $f(u_1) = 4\cosh\left(\frac{\sqrt{3}}{3}u_1\right) - 4$ and $g(u_2) = \sin\left(\frac{2\sqrt{3}}{3}u_2\right) + 1$. Using the Proposition 5.1, we have the solutions of the Calapso equation*

$$\omega = \frac{-\epsilon\big(f + g + 12 + 2\sqrt{2}\Gamma\widetilde{N}_4(f+g)\big)}{3M}, \quad \Omega = \frac{3\epsilon\big(f-g\big)}{2\big(f+g\big)},$$

where $\Gamma = \frac{1}{1-\widetilde{X}_4}$ and

$$\widetilde{X}_4 \quad = \quad \frac{\sqrt{3}\big(3M - 4g\big)\sin\left(\frac{\sqrt{6}}{3}u_2\right) - 6\sqrt{2}g'\cos\left(\frac{\sqrt{6}}{3}u_2\right)}{9M}, \tag{5.7}$$

$$\widetilde{N}_4 \quad = \quad \frac{-\sqrt{6}\big(4g^2 - 2fg - 3M(f+g)\big)\sin\left(\frac{\sqrt{6}}{3}u_2\right) + 3g'\big(2f - 4g\big)\cos\left(\frac{\sqrt{6}}{3}u_2\right)}{9M(f+g)}. \tag{5.8}$$

The isothermic surface of this example is given in Figure 5.1.

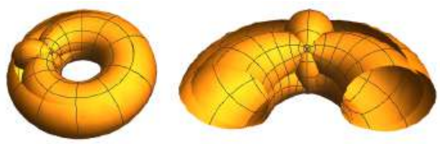

Figure 5.1: In the figure above we have an isothermic surface $Y = \pi \circ \widetilde{X}$ in $\mathbb{R}^3$ used in the Example 5.1.

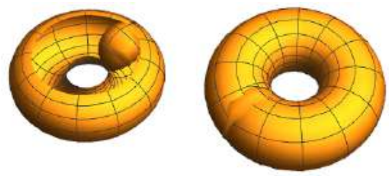

Figure 5.2: In the figure above we have the isothermic surface $Y = \pi \circ \widetilde{X}$ in $\mathbb{R}^3$ where $\widetilde{X}$ is an isothermic associated to the flat torus in $\mathbb{S}^3$ with $r_1 = \frac{\sqrt{2}}{\sqrt{3}}$, $r_2 = \frac{1}{\sqrt{3}}$, $b = 2$, $c = 15$, $A_1 = \frac{4}{9}$ and $B_1 = \frac{1}{4}$.

Example 5.2. *Consider the isothermic surfaces associated to the flat torus given by (4.2). We have $b = \frac{24}{25}$, $c = \frac{-69}{50}$, $r_1 = \frac{\sqrt{2}}{\sqrt{3}}$, $r_2 = \frac{1}{\sqrt{3}}$. In this case, $-3 = \frac{-1}{r_2^2} < c = \frac{-69}{50} < \frac{1}{r_1^2} = \frac{3}{2}$. Hence, from Proposition 4.3 f and g are given by (4.16). Choosing $A_1 = 4$ and $B_1 = 4$ we have $f(u_1) = 2\sin\left(\frac{4}{5}u_1\right) + \frac{3}{2}$ and $g(u_2) = 2\sin\left(\frac{3}{5}u_2\right) + \frac{8}{3}$. Using the Proposition 5.1, we have the solutions of the Calapso equation*

$$\omega = \frac{\epsilon\big(23(f+g) - 432 + 46\sqrt{2}\Gamma\widetilde{N}_4(f+g)\big)}{150M}, \quad \Omega = \frac{3\epsilon\big(f-g\big)}{2\big(f+g\big)},$$

where $\Gamma = \frac{1}{1-\widetilde{X}_4}$, $\widetilde{X}_4$ and $\widetilde{N}_4$ are given by (5.7) and (5.8). The isothermic surface of this example is given in Figure 5.3.

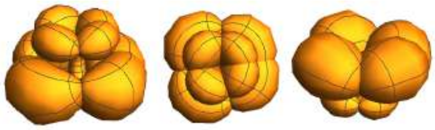

Figure 5.3: In the figure above we have an isothermic surface $Y = \pi \circ \widetilde{X}$ in $\mathbb{R}^3$ used in the Example 5.2.

Example 5.3. *Consider the isothermic surfaces associated to the flat torus given by (4.2). We have $b = 4\sqrt{6}$, $c = \frac{-33}{2}$, $r_1 = \frac{\sqrt{2}}{\sqrt{3}}$, $r_2 = \frac{1}{\sqrt{3}}$. In this case, $-3 = \frac{-1}{r_2^2} > c = \frac{-33}{2}$. Hence, from Proposition 4.3 f and g are given by (4.17). Choosing $A_1 = 1$ and $B_1 = 4$ we have $f(u_1) = \sin\left(2\,u_1\right) + \sqrt{6}$ and $g(u_2) = 2\cosh\left(\sqrt{3}u_2\right) - \frac{4\sqrt{6}}{3}$. Using the Proposition 5.1, we have the solutions of the Calapso equation*

$$\omega = \frac{\epsilon\big(11(f+g) - 72\sqrt{6} + 22\sqrt{2}\Gamma\widetilde{N}_4(f+g)\big)}{6M}, \quad \Omega = \frac{3\epsilon\big(f-g\big)}{2\big(f+g\big)},$$

where $\Gamma = \frac{1}{1-\widetilde{X}_4}$, $\widetilde{X}_4$ and $\widetilde{N}_4$ are given by (5.7) and (5.8). The isothermic surface of this example is given in Figure 5.4.

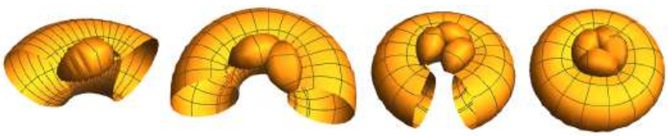

Figure 5.4: In the figure above we have an isothermic surface $Y = \pi \circ \widetilde{X}$ in $\mathbb{R}^3$ used in the Example 5.3.

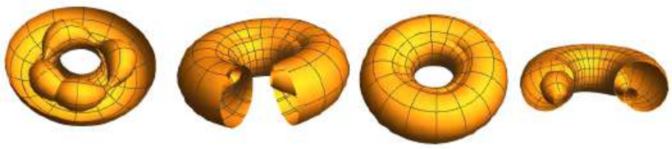

Figure 5.5: In the figure above we have the isothermic surface $Y = \pi \circ \widetilde{X}$ in $\mathbb{R}^3$ where $\widetilde{X}$ is an isothermic surface associated to the flat torus in $\mathbb{S}^3$ with $r_1 = \frac{\sqrt{2}}{\sqrt{3}}$, $r_2 = \frac{1}{\sqrt{3}}$, $b = -4\sqrt{6}$, $c = \frac{-33}{2}$, $A_1 = 1$ and $B_1 = 4$.

Example 5.4. *Consider the isothermic surfaces associated to the flat torus given by (4.2). We have $b = 2$, $c = -3$, $r_1 = \frac{\sqrt{2}}{\sqrt{3}}$, $r_2 = \frac{1}{\sqrt{3}}$. In this case, $-3 = \frac{-1}{r_2^2} = c$. Hence, from Proposition 4.3 f and g are given by (4.18). Choosing $A_1 = 16$ and $b_2 = 3$ we have $f(u_1) = 4\sin\left(u_1\right) + 2$ and $g(u_2) = u_2^2 + 3$. Using the Proposition 5.1, we have the solutions of the Calapso equation*

$$\omega = \frac{\epsilon\big(f + g - 18 + 2\sqrt{2}\Gamma\widetilde{N}_4(f+g)\big)}{3M}, \quad \Omega = \frac{3\epsilon\big(f-g\big)}{2\big(f+g\big)},$$

where $\Gamma = \frac{1}{1-\widetilde{X}_4}$, $\widetilde{X}_4$ and $\widetilde{N}_4$ are given by (5.7) and (5.8). The isothermic surface of this example is given in Figure 5.6.

Example 5.5. *Consider the isothermic surfaces associated to the flat torus given by (4.2). We have $b = 2$, $c = \frac{3}{2}$, $r_1 = \frac{\sqrt{2}}{\sqrt{3}}$, $r_2 = \frac{1}{\sqrt{3}}$. In this case, $c = \frac{3}{2} = \frac{1}{r_1^2}$. Hence, from Proposition 4.3 f and g are given by (4.19). Choosing $B_1 = 16$ and $b_2 = 3$ we have $f(u_1) = u_1^2 + 3$ and $g(u_2) = 4\sin\left(u_2\right) + 2$. Using the Proposition 5.1, we have the solutions of the Calapso equation*

$$\omega = \frac{-\epsilon\big(f + g + 36 + 2\sqrt{2}\Gamma\widetilde{N}_4(f+g)\big)}{6M}, \quad \Omega = \frac{3\epsilon\big(f-g\big)}{2\big(f+g\big)},$$

where $\Gamma = \frac{1}{1-\widetilde{X}_4}$, $\widetilde{X}_4$ and $\widetilde{N}_4$ are given by (5.7) and (5.8). The isothermic surface of this example is given in Figure 5.8.

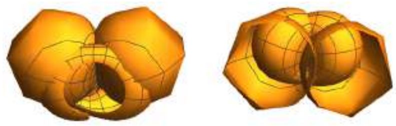

Figure 5.6: In the figure above we have an isothermic surface $Y = \pi \circ \widetilde{X}$ in $\mathbb{R}^3$ used in the Example 5.4.

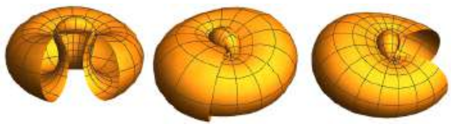

Figure 5.7: In the figure above we have the isothermic surface $Y = \pi \circ \widetilde{X}$ in $\mathbb{R}^3$ where $\widetilde{X}$ is an isothermic surface associated to the flat torus in $\mathbb{S}^3$ with $r_1 = \frac{\sqrt{2}}{\sqrt{3}}$, $r_2 = \frac{1}{\sqrt{3}}$, $b = -8$, $c = -3$, $A_1 = 16$ and $b_2 = 3$.

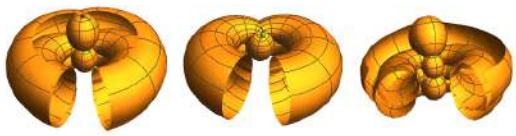

Figure 5.8: In the figure above we have an isothermic surface $Y = \pi \circ \widetilde{X}$ in $\mathbb{R}^3$ used in the Example 5.5.

Proposition 5.2. *Consider the isothermic surfaces associated to the flat torus parametrized by (4.23), whose first fundamental form is given by*

$$I = \left(\frac{1}{\widetilde{\lambda_1} - \widetilde{\lambda_2}}\right)^2 (du_1^2 + du_2^2),$$

where $-\widetilde{\lambda}_i$, $1 \leq i \leq 2$, *are the principal curvature of the (4.23), given by (4.30) and (4.31). Then the functions* $\omega = \dfrac{\epsilon\sqrt{2}(\widetilde{\lambda_1} + \widetilde{\lambda_2})(1 - \widetilde{X}_4) + 2\widetilde{N}_4}{2(\widetilde{\lambda_1} - \widetilde{\lambda_2})(1 - \widetilde{X}_4)}$ *and* $\Omega = \dfrac{\epsilon\sqrt{2}}{2}$, *with* $\epsilon^2 = 1$, *are solutions of the Calapso equation, where* $\widetilde{X}_4 =< \widetilde{X}, e_4 >$, $\widetilde{N}_4 =< \widetilde{N}, e_4 >$, $e_4 = (0, 0, 0, 1)$ *and* $\widetilde{X}$, $\widetilde{N}$, *respectively, are given by (4.23) and (4.27).*

Proof: Consider stereographic projection $\pi : \mathbb{S}^3 \to \mathbb{R}^3$ given by (5.1). Let $Y = \pi \circ \widetilde{X} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, where $\widetilde{X}$ is an isothermic surface locally associated by a Ribaucour transformation to flat torus, given by (4.23). Thus Y is an isothermic surface in $\mathbb{R}^3$, where by (2.7), the first fundamental form is given by

$$I_Y = \left(\frac{1}{k_1 - k_2}\right)^2 (du_1^2 + du_2^2), \tag{5.9}$$

where k_i, $1 \leq i \leq 2$, are given by (5.4). Therefore, using the Theorem 2.2, we conclude the proof. □

Example 5.6. *Consider the isothermic surfaces associated to the flat torus given by (4.23).*

We have $b = 1$, $r_1 = \frac{\sqrt{6}}{3}$, $r_2 = \frac{\sqrt{3}}{3}$, $A_1 = 1$ *and* $B_1 = 1$. *Using the Proposition 5.2, we have the solutions of the Calapso equation*

$$\omega = \frac{\epsilon\sqrt{2}\left(-33 - 12\sqrt{3} + (36 + 29\sqrt{3})\sin\left(\frac{\sqrt{6}}{3}u_2\right) - (36 + 8\sqrt{3})\sin\left(\frac{\sqrt{3}}{3}u_1\right)\right)}{99 + 36\sqrt{3} + (36 + 29\sqrt{3})\sin\left(\frac{\sqrt{6}}{3}u_2\right) + (36 + 8\sqrt{3})\sin\left(\frac{\sqrt{3}}{3}u_1\right)}, \quad \Omega = \frac{\epsilon\sqrt{2}}{2}.$$

The isothermic surface of this example is given in first surface in the Figure 5.9.

Figure 5.9: In the figures above we have isothermic surfaces $Y = \pi \circ \widetilde{X}$ in $\mathbb{R}^3$. The first surface is that of the Example 5.6. On the second surface we have, $r_1 = \frac{3}{5}$, $r_2 = \frac{4}{5}$, $b = 30$, $A_1 = B_1 = 1$. On the third surface we have, $r_1 = \frac{3}{5}$, $r_2 = \frac{4}{5}$, $b = A_1 = B_1 = 1$.

6. Conclusions. From the results obtained in this work we can make the following conclusions: All isothermic surfaces in $\mathbb{S}^3$ locally associated to the flat torus by a Ribaucour transformation are complete surfaces. Moreover, these surfaces has no umbilic points.

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