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SELECCIONES MATEMÁTICAS
Universidad Nacional de Trujillo
ISSN: 2411-1783 (Online)
2023; Vol. 10(1): 147-157.

Fundamental response in the vibration control of buildings subject to seismic excitation with ATMD

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Received, Feb. 22, 2023; *Accepted, Apr. 30, 2023;* *Published, Jul. 27, 2023*

How to cite this article:
Jara F, Rubio O, Claeyseen J. *Fundamental response in the vibration control of buildings subject to seismic excitation with ATMD*. Selecciones Matemáticas. 2023;10(1):147–157. http://dx.doi.org/10.17268/sel.mat.2023.01.13

Abstract
The linear quadratic regulator for vibration systems subject to seismic excitations is discussed in his own physical newtonian space as a second-order linear differential system with matrix coefficients. The linear quadratic regulator leads to a fourth-order system and second-order transversality conditions. Those systems are studied with a matrix basis generated by a fundamental matrix solution.
Keywords . Earthquake, control, vibrating system, fundamental matrix solution, LQR problem.

1. Introduction. In this work we discuss the linear quadratic regulator for vibration systems subject to seismic excitations in his own physical newtonian formulation. The second-order linear differential system with matrix coefficients is studied using a basis of solutions generated its fundamental matrix response that is given in closed-form.

The seismic waves of body and surface cause the movement of the ground on which buildings rest. It is of interest to capture the size and know the origin of the earthquake and measure ground displacements even at great distances from the epicenter. The data obtained with the use of an accelerometer, that is, variations on the acceleration on the ground with the time that is recorded at a point of the ground during an earthquake, are recorded simultaneously along three perpendicular directions to capture the complete oscillation of the floor in one place. Of these three records of the corresponding to the width of the horizontal directions and one to the width of the vertical direction. The types of rupture, the geology off the track of the displacement sector and the geotechnical strata below the building are three critical factors that determine the ground movement characteristics in the acceleration of a station.

Dynamic actions in buildings are mainly caused by wind and earthquakes. In the structural design, the wind exerts a pressure on the surface area exposed. However, in the design of the building the earthquake, it is a random movement that acts on the ground of its base, which induces forces of inertia in the building, which in turn introduce tensions. The force of the wind over the building has a component of average value in null superimposed with a relatively small oscillatory component (Figure 1.1).

In the seismic design, the mass of the building controls the seismic design, in addition to rigidity of the construction, because the earthquake induces forces of inertia that are proportional to the mass of the building. The designs of buildings that make them behave elastically, and that it does not suffer any damage during earthquakes can do it economically unfeasible project. That's why it is necessary for the structure to suffer some damage and therefore dissipate the incoming energy during the earthquake. The seismic design makes a balance between costs and degrees of damage acceptable with the objective of obtaining a design that is resistant to earthquakes and does not test earthquakes, [1]

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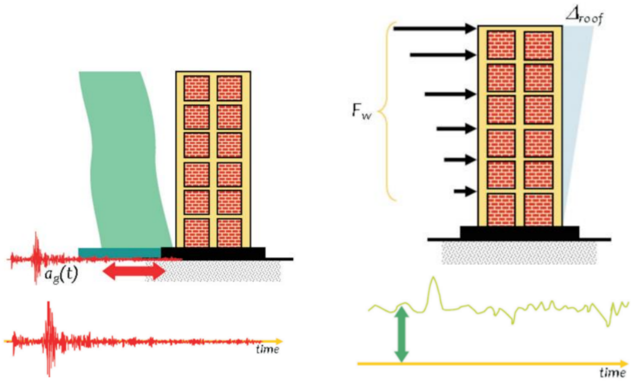

(a) Movement in the base due to the earthquake.

(b) Wind pressure in the exposed area.

Figure 1.1: Movement in the base due to the earthquake and wind pressure in the exposed area.

Active and semi-active vibration control of civil engineering structures has attracted growing worldwide interest as an innovative technology in the earthquake engineering field [2]. Various control systems have been developed to attenuate excessive vibration in an earthquake or wind excitation. These systems include passive control systems, semi-active control systems, active control systems, and hybrid control systems. The active mass damper that is not tuned in to a certain natural frequency of an objective structure is called AMD while TMD means a tuned mass damper. The active tuned mass damper denoted by ATMD has comprised most mass damper applications [3].

Active control systems are designed on the Linear Quadratic Regulator (LQR) theory or H-infinite control theory. The LQR and the instantaneous optimal control methods are two of the most popular algorithms in research and real applications for linear and nonlinear structures relatively. [4] proposed a Wilson-θ numerical method-based nonlinear instantaneous optimal control algorithm. They showed that this algorithm is computationally efficient and suitable for online implementation of control systems to nonlinear tall buildings under earthquake excitations [5].

One of the advantages of the active control system of ATMD over passive control systems such as TMD is its remarkable adaptability and performance for various excitation frequencies. It is also efficient for transitive vibrations and effectively minimizes the responses resulting from strong earthquakes.

Robust adaptive controller has been employed in the active tuned mass damper (ATMD) system to overcome undesirable vibrations in multistory buildings under seismic excitations and with all system parameters taken as unknowns. The equation of motion for an active controlled linear structure usually consider the ground acceleration and a control force whose location depends on the location of actuators in the structure. The ATMD is usually located on the top story [6].

The linear regulator in optimal control has been presented in state-space form, transforming all the differential equations into first order. The optimal feedback matrices can then be obtained by solving the non-linear matrix Riccati equation. In mechanical vibration control problems,applicable to suppressing oscillations, the equations of motion resulting from Newton's second law or the principle least action are presented naturally in a second-order form. The system matrices coefficients have special properties (i.e., symmetry, definiteness, and/or sparsity) that with any transformation to the state formulation, these properties can not be exploited.

The control analysis based on the natural second-order form of the equation of motion has been worked with the modal equations [7] and with the minimisation of a functional depending on second derivatives. The second-order motion equations represented by the control force were then introduced into the index functional, which transformed the problem into an unconstrained variational problem that lead to a set of linear fourth-order differential equations that determines the optimal control. Due to external seismic excitations this set turns out to be nonhomogeneous or a forced problem.

In this work we shall consider ground acceleration and velocity and that the solution of the controlled equation is formulated in terms of the fundamental basis. This approach, besides modal analysis, is the first one in the seismic literature and eliminates completely the need of transforming newtonian second-order vibration systems into first-order ones in the state formulation.

2. Control Modelling. The motion equations of the structural model equipped with ATMD on the top story of a building with n floors subject to a seismic excitation are described as follows.

For the first $n-1$ floors, we have

$$m_j\ddot{x}_j + c_j\big(\dot{x}_j - \dot{x}_{j-1}\big) - c_{j+1}\big(\dot{x}_{j+1} - \dot{x}_j\big) + k_j\big(x_j - x_{j-1}\big) \\ - k_{j+1}\big(x_{j+1} - x_j\big) + \beta_j\dot{x}_j = 0, j = 1, 2, \ldots, n-1, \quad (2.1)$$

and for the top floor

$$m_n\ddot{x}_n + c_n\big(\dot{x}_n - \dot{x}_{n-1}\big) + k_n\big(x_n - x_{n-1}\big) - k_d\big(x_d - x_n\big) - c_d\big(\dot{x}_d - \dot{x}_n\big) + \beta_n\dot{x}_n = -u_d, \quad (2.2)$$

$$m_d\ddot{x}_d + c_d\big(\dot{x}_d - \dot{x}_n\big) + k_d\big(x_d - x_n\big) = u_d, \quad (2.3)$$

where m_j, c_j, β_j and k_j are the mass, internal and external damping and stiffness, respectively, u_d is the active force control, x_d is the displacement of the damping mass and m_d, c_d y k_d are the mass, damping and the stifness of the damped massa, respectively.

We introduce y_j as the relative displacement $y_j = x_j - x_0$ of the j-floor where x_0 is the ground displacemnet due to the earthquake and $y_d = -x_d - x_0$ is the relative displacement of the damped mass with respect to the ground. The equations of motion in terms of the relative displacements can be written in matrix terms as follows:

$$M\ddot{y} + C\dot{y} + Ky = Hu_d + F\ddot{x}_0(t) + G\dot{x}_0(t), \quad (2.4)$$

where

- $y = y(t) = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ y_d \end{pmatrix}$, is the relative displacement vector with respect to the ground,

  $F = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \\ m_d \end{pmatrix}$, $G = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ 0 \end{pmatrix}$ are the location matrices for the excitation forces;
- M is the mass matrix

$$M = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & \ddots & & \\ & & & m_n & \\ & & & & m_d \end{bmatrix},$$

  where m_i is the mass of the ith-floor, $i = 1, \ldots, n$, and m_d is damped masa;
- C is the $(n+1)\times(n+1)$ internal damping matrix which is symmetric and positive semi-definite

$$C = \begin{bmatrix} c_1 + c_2 + \beta_1 & -c_2 & & & & \\ -c_2 & c_2 + c_3 + \beta_2 & -c_3 & & & \\ & -c_3 & c_3 + c_4 + \beta_3 & -c_4 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -c_n & c_n + c_d + \beta_n & -c_d \\ & & & & -c_d & c_d \end{bmatrix},$$

  with c_i internal dampimg of the ith-floor, $i = 1, \ldots, n$, and c_d is the damping of the ATMD;

- K is the $(n+1)\times(n+1)$ lateral stiffness matrix which is positive semi-definite

$$K = \begin{bmatrix} k_1+k_2 & -k_2 & & & & \\ -k_2 & k_2+k_3 & -k_3 & & & \\ & -k_3 & k_3+k_4 & -k_4 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -k_n & k_n+k_d & -k_d \\ & & & & -k_d & k_d \end{bmatrix},$$

where k_i is the lateral stiffness of the ith-floor, $i = 1,\ldots,n$, and k_d is the stiffness of the ATMD;

- $H = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 1 \end{bmatrix}$ is the $(n+1)\times 1$ localization vector of control force above the top floor;
- u_d is the controller active force;
- $\dot{x}_0(t)$ is the recorded earthquake velocity.
- $\ddot{x}_0(t)$ is the recorded earthquake aceleration.

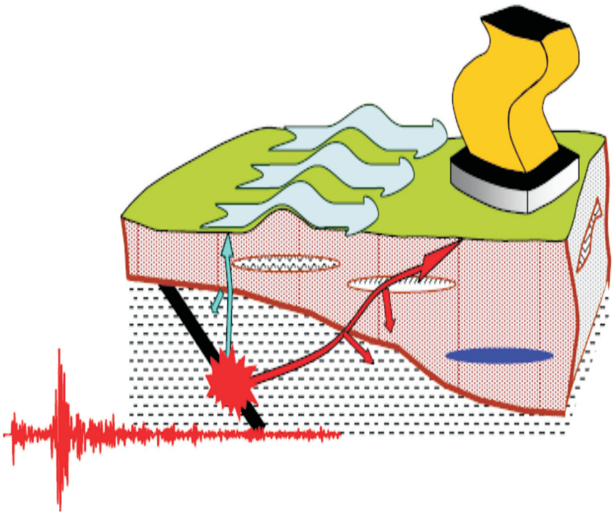

Figure 2.1: Reflections and refractions during the earthquake and frequency seepage under the building.

The ground acceleration $\ddot{x}_0(t) = \phi(t)\ddot{x}(t)$ is usually simulated as a non stationary random process uniformely modulated, in which $\phi(t)$ is a nonnegative deteriministic involving function and $\ddot{x}$ is a stationary process with null mean value and a spectral density with parameters depending upon the earthquake intensity and, in particular, the geological localization.

3. The linear regulator problem in optimal control of vibrating systems. Modern linear system theory requires that the mathematical model of the system be formulated in a as a first-order form or state formulation. This later formulation gives rise to a unified approach for different phenomena such as heat transfer, fluid flow and wave propagation, among others. Thus a direct application of modern control theory makes it necessary to transform the original second-order model formulation to an equivalent first-order system. In vibration problems, the equations of motion resulting from Newton's second law or the principle of stationary action are presented naturally in a second-order form Eq.(2.4) where the system matrices coefficients have special properties (i.e., symmetry, definiteness, and/or sparsity). With any transformation to the state formulation these properties can not be exploited. [8]

When a linear regulator problem for mechanical vibrating systems is studied in the second order formulation given in Eq.(2.4), in this context we willreformulate it as follows:

$$M\ddot{y} + C\dot{y} + Ky = BU_d + F\ddot{x}_0(t) + G\dot{x}_0(t), \tag{3.1}$$

where B and U_d are suitable matrices

$$B = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & -1 & \\ & & & & 1 \end{bmatrix}, \quad U_d = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -u_d \\ u_d \end{bmatrix}$$

we face two problems [9]: subject to Eq.(3.1) and initial conditions, to determine the family of solutions $U_d(t)$ that make stationary the quadratic performance functional

$$J(\mathsf{y}) = \int_0^t \mathbf{F}(t, y, \dot{y})\, dt = \int_0^t (y^T U y + \dot{y}^T V \dot{y} + U_d^T W U_d)\, dt, \tag{3.2}$$

that is, $J'(\mathsf{y}+\epsilon\mathsf{h})|_{\epsilon=0} = 0$, $\mathsf{y} = (y, y')$ and to determine a closed-loop feedback realisation $U_d = Py + Qy'$. Here U, V, W are positive semidefinite weighted matrices and $y(t)$ is subject to $M\ddot{y} + C\dot{y} + Ky = BU_d + F\ddot{x}_0(t) + G\dot{x}_0(t)$.

Writing $U_d(t) = B^{-1}g$, $\widehat{W} = B^{-T}WB^{-1}$ and the integrand as $\mathbf{F} = y^T U y + \dot{y}^T V \dot{y} + g^T \widehat{W} g$, where

$$g = M\ddot{y} + C\dot{y} + Ky - (F\ddot{x}_0(t) + G\dot{x}_0(t)), \tag{3.3}$$

it follows that the Euler-Lagrange stationary equation and its associate transversality condition at certain time $t = t_f$, are

$$\frac{d^2}{dt^2}\frac{\partial \mathbf{F}}{\partial \ddot{y}} - \frac{d}{dt}\frac{\partial \mathbf{F}}{\partial \dot{y}} + \frac{\partial \mathbf{F}}{\partial y} = 0 \tag{3.4}$$

$$\frac{\partial \mathbf{F}}{\partial \ddot{y}} = 0, \;\; \frac{d}{dt}\frac{\partial \mathbf{F}}{\partial \ddot{y}} - \frac{\partial \mathbf{F}}{\partial \dot{y}} = 0, \;\; t = t_f. \tag{3.5}$$

Thus from Eq.(3.4), we have

$$M\widehat{W}\ddot{g} - C\widehat{W}\dot{g} + K\widehat{W}g = V\ddot{y} - Uy. \tag{3.6}$$

Substituting Eq.(3.3) into Eq.(3.6), it turns out the fourth-order matrix differential equation

$$A_0 y^{(iv)} + A_1 y^{(iii)} + A_2 y'' + A_3 y' + A_4 y = f \tag{3.7}$$

where

$$\begin{aligned} A_0 &= M\widehat{W}M,\; A_1 = (M\widehat{W}C - C\widehat{W}M), A_2 = (M\widehat{W}K + K\widehat{W}M - C\widehat{W}C - V), \\ A_3 &= (K\widehat{W}C - C\widehat{W}K), A_4 = (K\widehat{W}K + U) \quad (3.8) \\ f &= -M\widehat{W}F\ddddot{x}_o + (M\widehat{W}G - C\widehat{W}F)\dddot{x}_o + (-C\widehat{W} + k\widehat{W}F)\ddot{x}_o + K\widehat{W}G\dot{x}_o). \quad (3.9) \end{aligned}$$

The transversality conditions without seismic excitations are

$$MW(M\ddot{y}(t_f) + C\dot{y}(t_f) + Ky(t_f)) = 0 \tag{3.10}$$

$$V\dot{y}(t_f) + (C - M)W\,(M\ddot{y}(t_f) + C\dot{y}(t_f) + Ky(t_f)) = 0 \tag{3.11}$$

We observe that when using the second-order newtonian formulation, the nonlinear Ricatti equation which has a central role in the state formulation of the regulator problem, is substituted by a fourth-order linear differential equation with matrix coefficients.

4. Fundamental basis. In order to determine the matrices P and Q independent of the initial conditions of an unforced system, we shall introduce a fundamental matrix solution for the motion equation Eq.(3.1).

The study of higher-order equations Eq.(3.1) and Eq.(3.7) can be carried out using a fundamental matrix basis instead of the Euler basis that seeks solutions as linear superposition with exponential solutions of the type $e^{\lambda t}\mathbf{v}$. This basis was introduced for second-order systems [10] and extended to equations of arbitrary order in [11]. We have that the general solution of the system

$$\mathbf{A}_0\mathbf{y}^{(m)}(t) + \mathbf{A}_1\mathbf{y}^{(m-1)}(t) + \cdots + \mathbf{A}_m\mathbf{y}(t) = \mathbf{f}(t) \tag{4.1}$$

where the coefficients $\mathbf{A}_j's$ are $n \times n$ constant matrices with $\mathbf{A}_0$ nonsingular, $\mathbf{y}(t)$ and $\mathbf{f}(t)$ are $n \times 1$, can be written in terms of the fundamental matrix basis

$$\left\{h(t)\; h'(t)\; \cdots\; h^{(m-1)}(t)\right\} \tag{4.2}$$

as

$$\mathbf{y}(t) = \sum_{j=0}^{m-1} \mathbf{h}^{(j)}(t)c_j + \int_0^t \mathbf{h}(t-\tau)A_o^{-1}f(\tau)d\tau \tag{4.3}$$

where $\mathbf{h}(t)$ is an $n \times n$ matrix solution with impulsive initial conditions

$$\mathbf{A}_0\mathbf{h}^{(m)}(t) + \mathbf{A}_1\mathbf{h}^{(m-1)}(t) + \cdots + \mathbf{A}_m\mathbf{h}(t) = \mathbf{0},$$
$$\mathbf{h}(0) = \mathbf{0}, \mathbf{h}'(0) = \mathbf{0}, \cdots, \mathbf{h}^{(m-2)}(0) = \mathbf{0}, \mathbf{A}_0\mathbf{h}^{(m-1)}(0) = \mathrm{I}.$$

Moreover, the fundamental solution $h(t)$ was given in closed-form as

$$\mathbf{h}(t) = \sum_{j=1}^{mn}\sum_{i=0}^{j-1} b_i d^{(j-i-1)}(t)\mathbf{h}_{mn-j}, \tag{4.4}$$

where b_i, $i = 0, 1, ..., mN$ are the coefficients of the characteristic polynomial

$$P(\eta) = \det\left(\sum_{k=0}^{m} \mathbf{A}_k\eta^{m-k}\right) = \sum_{k=0}^{mN} b_k\eta^{mN-k}, \tag{4.5}$$

$d(t)$ is the scalar function that satisfies the initial value problem

$$b_0d^{(mn)}(t) + b_1d^{(mn-1)}(t) + \cdots + b_{mn}d(t) = 0,$$

$$d(0) = 0, d'(0) = 0, \cdots, d^{mn-2}(0) = 0, b_0d^{mn-1}(0) = 1,$$

and $\mathbf{h}_k = \mathbf{h}^{(k)}(0)$ are coupling matrices $n \times n$ that satisfy the m-th order initial-value matrix difference system

$$\mathbf{A}_0\mathbf{h}_{k+m} + \mathbf{A}_1\mathbf{h}_{k+m-1} + \cdots \mathbf{A}_m\mathbf{h}_k = \mathbf{0},$$

$$\mathbf{h}_0 = \mathbf{0}, \mathbf{h}_1 = \mathbf{0}, \cdots, \mathbf{h}_{m-2} = \mathbf{0}, \mathbf{A}_0\mathbf{h}_{m-1} = \mathrm{I}.$$

We have the Heaviside formula

$$d(t) = \sum_{k=1}^{r}\sum_{l=1}^{m_k} \frac{\psi_{kl}(\eta_k)}{(m_k-l)!(l-1)!} z^{m_k-l}e^{\eta_k t} \tag{4.6}$$

where $\psi_{kl}(\eta) = \frac{d^{l-1}}{d\eta^{l-1}}\left[\frac{1}{P_k(\eta)}\right]$, $P_k(\eta) = \frac{P(\eta)}{(\eta-\eta_k)^{m_k}}$ for $P(\eta) = c_0(\eta-\eta_1)^{m_1}\cdot\ldots\cdot(\eta-\eta_r)^{m_r}$, $\eta_i \neq \eta_k$ for $k \neq i$, being $P(\eta)$ given in (4.5). When all roots η_k are simple, formula (4.6) reduces to

$$d(t) = \sum_{k=1}^{mn} \frac{e^{\eta_k t}}{P'(\eta_k)}.$$

The coefficients of the characteristic polynomial $P(\eta)$ can be obtained by several methods [10],[11],[12] From Eq.(3.7), we have that the general solution of the unforced system

$$A_0 y^{(iv)} + A_1 y^{(iii)} + A_2 y'' + A_3 y' + A_4 y = 0 \tag{4.7}$$

is given by

$$\mathbf{y}(t) = \mathbf{h}(t)c_1 + \mathbf{h}'(t)c_2 + \mathbf{h}''(t)c_3 + \mathbf{h}'''(t)c_4 = \Phi(t)c \tag{4.8}$$

$$\Phi(t) = \left(\mathbf{h}(t)\ \mathbf{h}'(t)\ \mathbf{h}''(t)\ \mathbf{h}'''(t)\right), \quad c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \tag{4.9}$$

Here the $n+1 \times n+1$ matrix $\mathbf{h}(t)$ is the fundamental solution of Eq.(3.7), that is, satisfies the initial value problem

$$A_0 \mathbf{h}^{(iv)} + A_1 \mathbf{h}^{(iii)} + A_2 \mathbf{h}'' + A_3 \mathbf{h}' + A_4 \mathbf{h} = 0 \tag{4.10}$$
$$\mathbf{h}(0) = 0, \mathbf{h}'(0) = 0, \mathbf{h}''(0) = 0, A_0 \mathbf{h}'''(0) = I \tag{4.11}$$

where I denotes the fourth-order matrix identity.

The vector c has $4(n+1)$ components that will reduce to $2(n+1)$ components when using the transversality equations. Let us know assume that $\mathbf{y} = \Phi(t)a$, where a has $2(n+1)$ null components so that the vector c essentially reduces to a $2(n+1) \times 1$ dimensional constant vector. Substituting in $U_d = B^{-1}g$, $g = M\ddot{y} + C\dot{y} + Ky = Py + Q\dot{y}$, it follows

$$U_d = B^{-1}\left(M\ddot{\Phi}(t) + C\dot{\Phi}(t) + K\Phi\right)a. \tag{4.12}$$

We can determine matrices P y Q such that $U_d = Py + Q\dot{y} = (P\Phi(t) + Q\dot{\Phi}(t))a$ be independent of the initial conditions imbeedded in a if we assume

$$B^{-1}(M\ddot{\Phi}(t) + C\dot{\Phi}(t) + K\Phi) = P\Phi(t) + Q\dot{\Phi}(t). \tag{4.13}$$

If we denote

$$R = \begin{pmatrix} P^T \\ Q^T \end{pmatrix}, \quad \Psi = \left[\Phi^T, \dot{\Phi}^T\right]$$

then transposing Eq.(4.13), we have the system

$$\Psi R = (\ddot{\Phi}^T M + \dot{\Phi}^T C + \Phi^T K)B^{-T} \tag{4.14}$$

that can be solved whenever Ψ is nonsingular.

5. Illustartive Example. Let us consider the simple harmonic oscillator subject to a control force $u(t)$ as described in Fig.5.1. Using the free body diagram, we have the equation of motion

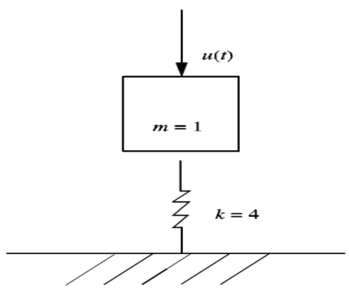

Figure 5.1: uncontrolled system

$$\frac{d^2y}{dt^2} + 4y(t) = u(t)$$

We wish to determine functions $P(t)$ and $Q(t)$ such that the cost functional

$$J = \int_{t_0}^{t_f} (\dot{y}^2 + 0,04u^2)\, dt.$$

is stationary for arbitary initial conditions $y(0)$ and $\dot{y}(0)$, where

$$u = Py + Q\dot{y}.$$

Let us consider the case wher $t_f = 2$ and having into account Eq.(3.7) and the specific values:

$$M = 1,\ C = 0,\ K = 4,\ H = 1,\ U = 0,\ V = 1 \text{ y } W = 0,04.$$

we have

$$\frac{d^4y}{dt^4} - 17\frac{d^2y}{dt^2} + 16y = 0.$$

whose general solution is:

$$y(t) = h(t)c_0 + h'(t)c_1 + h''(t)c_1 + h'''(t)c_2 \tag{5.1}$$

where

$$h(t) = \sum_{k=1}^{4} \frac{e^{\beta_k t}}{P'(\beta_k)},\ P(\beta) = \beta^4 - 17\beta^2 + 16 \tag{5.2}$$

for $\beta_1 = -1$, $\beta_2 = -4$,$\beta_3 = 1$ and $\beta_4 = 4$ the roots of $P(\beta)$. Substituting Eq.(5.2) in Eq.(5.1), the solution can be expressed in the Euler basis as

$$y(t) = C_1e^{-t} + C_2e^{-4t} + C_3e^{t} + C_4e^{4t} \tag{5.3}$$

for appropriate constants C_1, C_2, C_3, C_4. The transversality conditions (3.10) y (3.11) give us the restrictions

$$\ddot{y}(2) + 4y(2) = 5C_1e^{-2} + 20C_2e^{-8} + 5C_3e^{2} + 20C_4e^{8} = 0 \tag{5.4}$$

and

$$\dot{y}(2) - 0,04(\ddot{y}(2) + 4y(2)) = 0,8(-C_1e^{-2} - C_2e^{-8} + C_3e^{2} + C_4e^{8}) = 0. \tag{5.5}$$

Thus in natrix terms

$$\begin{pmatrix} 5e^2 & 20e^8 \\ 0,8e^2 & 0,8e^8 \end{pmatrix} \begin{pmatrix} C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} -5e^{-2} & -20e^{-8} \\ 0,8e^{-2} & 0,8e^{-8} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \tag{5.6}$$

Thus

$$\begin{pmatrix} C_3 \\ C_4 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 5C_1e^{-4} + 8C_2e^{-10} \\ -2C_1e^{-10} - 5C_2e^{-16} \end{pmatrix} \tag{5.7}$$

Considreing Eq.(5.3), we have

$$y(t) = C_1\left(e^{-t} + \frac{5}{3}e^{t-4} - \frac{2}{3}e^{4t-10}\right) + C_2\left(e^{-4t} + \frac{8}{3}e^{t-10} - \frac{5}{3}e^{4t-16}\right) \tag{5.8}$$

and Eq.(4.8) give us

$$\phi(t) = \left(e^{-t} + \frac{5}{3}e^{t-4} - \frac{2}{3}e^{4t-10}; e^{-4t} + \frac{8}{3}e^{t-10} - \frac{5}{3}e^{4t-16}\right) \tag{5.9}$$

from which

$$\dot{\phi(t)} = \left(-e^{-t} + \frac{5}{3}e^{t-4} - \frac{8}{3}e^{4t-10}; -4e^{-4t} + \frac{8}{3}e^{t-10} - \frac{20}{3}e^{4t-16}\right) \tag{5.10}$$

and

$$\ddot{\phi(t)} = \left(-e^{-t} + \frac{5}{3}e^{t-4} - \frac{32}{3}e^{4t-10}; 16e^{-4t} + \frac{8}{3}e^{t-10} - \frac{80}{3}e^{4t-16}\right). \tag{5.11}$$

Using

$$H^{-1}[M\ddot{\phi}(t) + C\dot{\phi}(t) + K\phi(t)] = P\phi(t) + Q\dot{\phi}(t) \tag{5.12}$$

and setting

$$R = \begin{pmatrix} P^T \\ Q^T \end{pmatrix} \quad \text{y} \quad \psi = \begin{pmatrix} \phi^T \\ \dot{\phi}^T \end{pmatrix} \tag{5.13}$$

we obtain from Eq.(5.12) with Eq. (5.13) that

$$\psi = \begin{pmatrix} e^{-t} + \frac{5}{3}e^{t-4} - \frac{2}{3}e^{4t-10} - e^{-t} + \frac{5}{3}e^{t-4} - \frac{8}{3}e^{4t-10} \\ e^{-4t} + \frac{8}{3}e^{t-10} - \frac{5}{3}e^{4t-16} - 4e^{-4t} + \frac{8}{3}e^{t-10} - \frac{20}{3}e^{4t-16} \end{pmatrix} \tag{5.14}$$

Using

$$\Psi R = (\ddot{\Phi}^T M + \dot{\Phi}^T C + \Phi^T K)B^{-T} \tag{5.15}$$

, we have

$$\begin{aligned} &\begin{pmatrix} e^{-t} + \frac{5}{3}e^{t-4} - \frac{2}{3}e^{4t-10} - e^{-t} + \frac{5}{3}e^{t-4} - \frac{8}{3}e^{4t-10} \\ e^{-4t} + \frac{8}{3}e^{t-10} - \frac{5}{3}e^{4t-16} - 4e^{-4t} + \frac{8}{3}e^{t-10} - \frac{20}{3}e^{4t-16} \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} \\ &= \begin{pmatrix} 5e^{-t} + \frac{25}{3}e^{t-4} - \frac{40}{3}e^{4t-10} \\ 20e^{-4t} + \frac{40}{3}e^{t-10} - \frac{100}{3}e^{4t-16} \end{pmatrix} \end{aligned} \tag{5.16}$$

Therefore the solution for P and Q is:

$$P = \frac{200(-e^{3t-4} + 2e^{-10} - e^{3t-16})}{-9e^{-5t} + 32e^{-10} - 25e^{3t-16} - 25e^{-3t-4} - 9e^{5t-20}}$$

y

$$Q = \frac{-15(-3e^{-5t} - 5e^{-3t-4} + 3e^{5t-20} + 5e^{3t-16})}{-9e^{-5t} + 32e^{-10} - 25e^{3t-16} - 25e^{-3t-4} - 9e^{5t-20}}$$

With the initial conditions $y(0) = 2,0183$, $\dot{y}(0) = 4,9187$, the control force and the response are shown in Fig.5.2.

We observe that when $t_f \to +\infty$ and with the response to be bounded, is necessary that $C_3 = C_4 = 0$. Therefore,

$$y(t) = C_1 e^{-t} + C_2 e^{-4t}.$$

On the odher hand

$$\begin{aligned} \phi(t) &= \begin{pmatrix} e^{-t} & e^{-4t} \end{pmatrix} \\ \dot{\phi}(t) &= \begin{pmatrix} -e^{-t} & -4e^{-4t} \end{pmatrix} \\ \ddot{\phi}(t) &= \begin{pmatrix} e^{-t} & 16e^{-4t} \end{pmatrix} \end{aligned}$$

From Eq.(5.14)

$$\psi = \begin{pmatrix} e^{-t} & -e^{-t} \\ e^{-4t} & -4e^{-4t} \end{pmatrix}$$

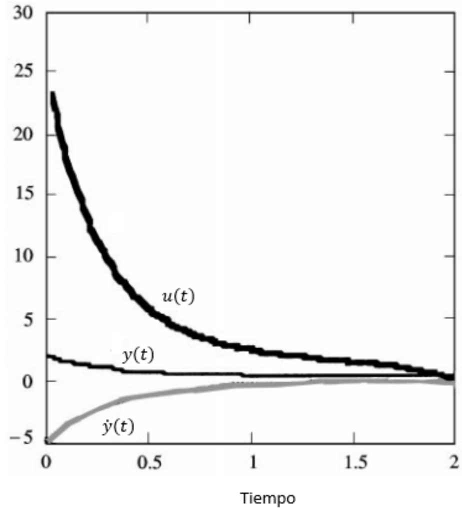

Figure 5.2: Control and response, $t_f = 2$

Eq.(5.16) give us

$$\begin{pmatrix} e^{-t} & -e^{-t} \\ e^{-4t} & -4e^{-4t} \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} 5e^{-t} \\ 20e^{-4t} \end{pmatrix}$$

or equivalently

$$\begin{pmatrix} 1 & -1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} 5 \\ 20 \end{pmatrix}.$$

In this illustrative case, the solution is $P = 0$ y $Q = -5$. Thus the control can be realised through a passive element, that is, adding a damping with constant $C = 5$, as we can observe in Fig. 5.3.

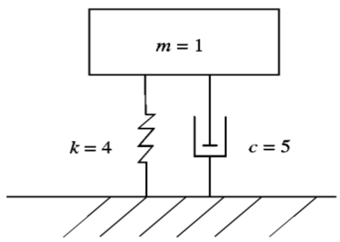

Figure 5.3: controlled system

5.1. Feedback law. When the horizontal acceleration of the earthquake $\ddot{x}_0(t)$ is zero, the feedback control law is related to the Ricatti equation or the equation on of fourth order (3.7). However, when seismic excitation is considered the optimal control law of the regulator quadratic must be modified. The history of the seismic acceleration $\ddot{x}_0(t)$, although measurable in real time with sensors installed at the base of each floor, is not known a priori. The performance index J must be modified with the inclusion of the input excitation and have, at some particular time, a record of the excited base. Using the laws of óptimal instantaneousćontrol with índex of squared performance,[4] which is a measure of control effectiveness (less effort, less displacement and speed) , of the kind

$$J(t) = y'^T Q_1 y + y''^T Q_2 y' + R\big(u_d(t)\big)^2 \tag{5.17}$$

for certain matrices Q_{21} and Q_{22} of order $(n+1)\times(n+1)$ and the scalar R are selected by certain criteria, from which it results that the control force $u_d(t)$ is given by the feedback law, [13], [14], [2]:

$$u_d(t) = -\frac{\Delta t}{2}\left(K_1 y(t) + K_1 \dot{y} + Ef(t)\right). \tag{5.18}$$

where E is the location matrix for the excitation forces. From Eq.(3.1) with $Ef(t) = F\ddot{x}_0(t) + G\dot{x}_0(t)$, and Eq.(2.4), it follows that

$$M\ddot{y}(t) + (C - HK_1)\dot{y}(t) + (K - HK_2)y(t) = (E + HE_1)f(t). \tag{5.19}$$

It is seen from above equation, that the effect of structural control is to modify the damping, the stiffness and the excitation, in such a way that the response of the system is controlled. The gain matrices K_1, K_2, E_1 can be obtained in such a way that the response, in principle, can be totally eliminated. In practice, to find the gain matrices or the control force different control algorithms have been proposed, keeping in view an objective function that reduces the structural response [2].

The study of Eq.(5.19) can be carried out in its own physical space using the fundamental basis Eq.(4.2) with solutions given as in Eq.(4.3). In [15], it is performed modal analysis with second-order in order to avoid the state formulation.

6. Conclusions. The linear quadratic regulator for vibration systems subject to seismic excitations by using the natural physical newtonian second-order formulation has been analysed by using a fundamental matrix basis. The Euler-Lagrange equation and its associate transversality conditions for a functional with second-order derivatives involves fourth and second order linear systems of ordinary differential equations. This avoids the use the nonlinear Ricatti equation for determining optimal control feedback matrices. The study of the corresponding higher-order systems can be realised using the corresponding fundamental matrix basis. The technique is illustrated with an harmonic oscillator.

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References

[1] Murty C, Goswami R, Vijayanarayanan A, Mehta V. Earthquake Behaviour of Buildings, Gujarat State Disaster Management Authority. Gujarat, India. 2002.

[2] Datta T. A state-of-the-art review on active control of structures. ISET Journal of Earthquake Technology. 2003; 40(1):1–17.

[3] Ikeda Y. Active and semi-active control of buildings in japan. Journal of Japan Association for Earthquake Engineering. 2004; 4(3).

[4] Yang J, Long F, Wong D. Optimal control of nonlinear structures. Journal of Applied Mechanics. 1988; 55(4):931–938.

[5] Mohammadi R, Ghamari H, Farsangi E. Active control of building structures under seismic load using a new nniform deformation-based control algorithm. Structures. 2021; 33:593–605.

[6] Sabetahd R, Mousavi S, Poursorkhabi R, Mohammadzadeh A, Zandi Y. Response attenuation of a structure equipped with atmd under seismic excitations using methods of online simple adaptive controller and online adaptive type-2 neural-fuzzy controller. Computational Intelligence and Neuroscience. 2022.

[7] Skelton R. Dynamic Systems Control. John Wiley. 1988.

[8] Oshman Y, Inman D, Laub A. Square-root state estimation for second order large space structures models. Guidance Control Dynamics.1989; 12(5):698–708.

[9] Ram Y, Inman D. Optimal control for vibrating systems. Mechanical Systems and Signal Processing. 1999; 13(6):879–892.

[10] Claeyssen J. On predicting the response of non-conservative linear vibrating systems by using the dynamical matrix solution. Journal of Sound and Vibration. 1990; 140(1):73–84.

[11] Claeyssen J, Canahualpa G, Jung C. A direct approach to second-order matrix non-classical vibrating equations. Applied Numerical Mathematics. 1999; 30(1):65–78.

[12] Pereslavtseva O. Calculation of the characteristic polynomial of a matrix. Discrete Mathematics and Applications. 2011; 21(1):109–129.

[13] Yang J, Akbarpour A, Ghaemmghami P. Instantaneous optimal control laws for tall buildings under seismic excitations. in: Tech. Report NCEER 87-0007, National Center for Earthquake Engineering Research. 1987

[14] Yang J, Li Z. Instantaneous optimal control for linear, nonlinear and hysteretic structures: Stable controllers. in: Tech. Report NCEER-TR-91-0026, National Center for Earthquake Engineering Research. Buffalo. 1991.

[15] Zhang J. Optimal control for mechanical vibration systems based on second-order matrix equation. Mechanical Systems and Signal Processing. 2002; 16(1):61–67.