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On the index of stability of (r,s)-linear Weingarten Clifford tori

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Dedicated to Ruth Noemi Noriega Sagástegui, in memory.

Abstract

For entire numbers r and s satisfying $0 \leq r \leq s \leq n-2$, we showed that the index of (r,s)-stability of a (r,s)-linear Weingarten Clifford torus immersed into the $(n+1)$-dimensional unit Euclidean sphere, that has a linear combination of their higher order mean curvatures H_{r+1} and H_{s+1} being null, is exactly equal to $n+3$ provided that a geometric condition involving H_{r+2} and H_{s+2} is satisfied.

Keywords . Unit Euclidean sphere, higher order mean curvatures, (r,s)-linear Weingarten Clifford torus, Jacobi operator, index of (r,s)-stability.

1. Introduction and statement of the main results. As is well known in the literature, the notion of *index of stability* for minimal compact hypersurfaces $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ immersed into the unit Euclidean sphere $\mathbb{S}^{n+1}$ has its origin in the variational problem of minimizing the area functional

$$\mathcal{A} = \int_{\Sigma^n} dv$$

of a given compact oriented hypersurface $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ for all possible variations. We will denote this index by $\mathrm{Ind}(\Sigma^n)$. In 1968, in a famous article due to J. Simons [22], it was shown that the index of stability $\mathrm{Ind}(\Sigma^n)$ of a compact oriented minimal hypersurface $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ is such that $\mathrm{Ind}(\Sigma^n) \geq 1$, with equality only for fully geodesic spheres. It is known that the index of stability $\mathrm{Ind}(\mathbb{T}^{k,n})$ of a minimal Clifford torus $\mathbb{T}^{k,n} = \mathbb{S}^k(k/n) \times \mathbb{S}^{n-k}(\sqrt{(n-k)/n}) \looparrowright \mathbb{S}^{n+1}$ immersed into Euclidean sphere $\mathbb{S}^{n+1}$ is $\mathrm{Ind}(\mathbb{T}^{k,n}) = n+3$ (see, for instance, [19]) and they stand out for admitting the lower index. In 2009 A. Barros and P. Sousa proved in [8, Theorem 1] that non-totally geodesic compact oriented minimal hypersurfaces $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ have index of stability $\mathrm{Ind}(\Sigma^n) \geq n+3$ with equality occurring at only Clifford torus $\mathbb{T}^{k,n} = \mathbb{S}^k(k/n) \times \mathbb{S}^{n-k}(\sqrt{(n-k)/n}) \looparrowright \mathbb{S}^{n+1}$ provided their square of the norms of the second fundamental forms $|A|^2$ are bounded from below by n.

An extension of the variational problem described above is that of minimizing the r-area functional

$$\mathcal{A}_r = \int_{\Sigma^n} F_r dv$$

of a compact oriented hypersurface $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ for all possible variations, where F_r is a suitable function that depends on the higher order mean curvatures H_r of $\Sigma^n \looparrowright \mathbb{S}^{n+1}$, $r \in \{0, 1, \ldots, n\}$. The concept of higher order mean curvatures of a oriented hypersurface $\Sigma^n \looparrowright \mathbb{S}^{n+1}$, studied initially by R. Reilly [20] in 1973, are such that $H_0 = 1$, H_1 is just the mean curvature H of $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ and H_2 defines a geometric

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quantity which is related to the scalar curvature of $\Sigma^n \looparrowright \mathbb{S}^{n+1}$. We have that r-*minimal* compact hypersurfaces $\Sigma^n \looparrowright \mathbb{S}^{n+1}$, namely those with $H_{r+1} = 0$, are characterized as critical points of $\mathcal{A}_r$ and, therefore, we naturally have the notion *index of* r-*stability* for r-minimal compact oriented hypersurfaces $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ immersed into $\mathbb{S}^{n+1}$, which we will denote here by $\mathrm{Ind}_r(\Sigma^n)$. In this setting, in 2010 A. Barros and P. Sousa proved in [9, Theorem 2] that the index of r-stability $\mathrm{Ind}_r(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$ of an r-minimal Clifford torus $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} = \mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2) \looparrowright \mathbb{S}^{n+1}$ is exactly $\mathrm{Ind}_r(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}) = n+3$ whenever its higher order mean curvature H_{r+2} is strictly negative. Here, $n_1, n_2 \in \mathbb{N}$ and $\rho_1, \rho_2 \in (0, +\infty)$ are chosen such that $n_1 + n_2 = n$ and $\rho_1^2 + \rho_2^2 = 1$. Furthermore, taking into account that the condition $|A|^2 \geq n$ (equivalently, $-n(n-1)H_2 \geq n$) for the minimal case admits the natural extension $H_r > 0$ and $-b_{r+1}H_{r+2} \geq b_r H_r$ for the r-minimal case, where $b_j = (n-j)\binom{n}{j}$ for all $j \in \{0, \ldots, n-2\}$, A. Barros and P. Sousa showed in [9, Theorem 3] that compact oriented r-minimal hypersurfaces $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ have index of r-stability $\mathrm{Ind}_r(\Sigma^n) \geq n+3$ with equality occurring at only Clifford torus $\mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2) \looparrowright \mathbb{S}^{n+1}$ provided their higher order mean curvatures H_r and H_{r+2} satisfy the additional condition $-b_{r+1}H_{r+2} \geq b_r H_r > 0$, thus extending [8, Theorem 1].

On the other hand, a natural extension of the hypersurfaces $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ with constant mean curvature H or constant second mean curvature H_2 are those ones whose curvatures H and H_2 obey a linear relation of the type $a_0 H + a_1 H_2 = \text{constant}$, for some real constants a_0 and a_1. These hypersurfaces are called in the literature as *linear Weingarten hypersurfaces* (see, for instance, [3, 4, 5, 11, 16, 17, 18]). A class that extends such hypersurfaces is given by the so-called generalized linear Weingarten hypersurfaces, namely, those hypersurfaces whose higher order mean curvatures H_{r+1} and H_{s+1} (for entire numbers r and s such that $0 \leq r \leq s \leq n-1$) satisfy the linear condition $a_r H_{r+1} + \cdots + a_s H_{s+1} = \text{constant}$, for some real numbers $a_r, \ldots, a_s$. For simplicity, we have named these hypersurfaces as (r,s)-*linear Weingarten*. It is not difficult to observe that geodesic spheres and Clifford torus of $\mathbb{S}^{n+1}$ are examples of (r,s)-linear Weingarten hypersurfaces in $\mathbb{S}^{n+1}$. We also observe that $(0,1)$-linear Weingarten hypersurfaces are simply linear Weingarten hypersurfaces and (r,r)-linear Weingarten hypersurfaces with $r \in \{0 \ldots, n-1\}$ are just the hypersurfaces having constant $(r+1)$-th mean curvature H_{r+1}. In recent years, several papers have been published showing the interest in understanding the geometry of the (r,s)-linear Weingarten hypersurfaces (see [1, 2, 13, 14, 15, 23]). For instance, we can highlight that the author jointly with H. de Lima and A. de Sousa showed in [23, Section 3] that (r,s)-linear Weingarten compact hypersurfaces compact are critical points of the variational problem of minimizing a suitable linear combination $a_r\mathcal{A}_r + \cdots + a_s\mathcal{A}_s$ of the j-area functionals $\mathcal{A}_j$ of a given compact oriented hypersurface $\Sigma^n \looparrowright \mathbb{S}^{n+1}$, $j \in \{r, \ldots, s\}$, for volume-preserving variations. Furthermore, they established that geodesic spheres of $\mathbb{S}^{n+1}$ are the only stable critical points of $a_r\mathcal{A}_r + \cdots + a_s\mathcal{A}_s$ for volume-preserving variations (cf. [23, Theorem 4.3]).

In this paper, motivated by the fact that Clifford tori are critical points that are not stable for the variational problem of minimizing the functional $a_r\mathcal{A}_r + \cdots + a_s\mathcal{A}_s$ for volume-preserving variations, we establish a notion of *index of* (r,s)-*stability* $\mathrm{Ind}_{r,s}(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$ for a (r,s)-linear Weingarten Clifford torus $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} = \mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2) \looparrowright \mathbb{S}^{n+1}$ whose higher order mean curvatures $H_{r+1}, \ldots, H_{s+1}$ satisfying the relation $a_r b_r H_{r+1} + \cdots + a_s b_s H_{s+1} = 0$ (see Section 3), where $a_r, \ldots, a_s$ are some nonnegative real numbers (with at least one nonzero) and $b_j = (n-j)\binom{n}{j}$ for $j \in \{r, \ldots, s\}$. All details about the meaning of (r,s)-linear Weingarten Clifford tori, which we are formulating here, are given in Section 2. Hence, arises a fundamental question for such type of hypersurfaces: what will be the value of $\mathrm{Ind}_{r,s}(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$? An answer to our question is given in the following result.

Theorem 1.1. *Let r and s be entire numbers satisfying the inequalities $0 \leq r \leq s \leq n-2$, and let $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} = \mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2) \looparrowright \mathbb{S}^{n+1}$ be a (r,s)-linear Weingarten Clifford torus whose higher order mean curvatures $H_{r+1}, \ldots, H_{s+1}$ satisfying the relation $a_r b_r H_{r+1} + \cdots + a_s b_s H_{s+1} = 0$, for some nonnegative real numbers $a_r, \ldots, a_s$ (with at least one nonzero), where $n_1, n_2 \in \mathbb{N}$ and $\rho_1, \rho_2 \in (0, +\infty)$ are such that $n_1 + n_2 = n$ and $\rho_1^2 + \rho_2^2 = 1$, respectively, and $b_j = (n-j)\binom{n}{j}$ for $j \in \{r, \ldots, s\}$. If*

$$\sum_{j=r}^{s} (j+1) a_j b_{j+1} H_{j+2} < 0 \tag{1.1}$$

on $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}$, then $\mathrm{Ind}_{r,s}(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}) = n+3$.

When $r = s$ in Theorem 1.1, we observe that the condition (1.1) reduces to $H_{r+2} < 0$, which is the main constraint in [9, Theorem 2]. Thus, the statement of Theorem 1.1 is a kind of extension of the results contained in [8, Theorem 1] and [9, Theorem 2].

For the case of linear Weingarten Clifford tori, we can establish the following result.

Theorem 1.2. *Let a_0 and a_1 be nonnegative real numbers, with at least one of them being nonzero, and let $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} = \mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2) \looparrowright \mathbb{S}^{n+1}$ be a linear Weingarten Clifford torus with mean and normalized scalar curvatures H and R satisfying $a_0 H + (n-1)a_1 R = (n-1)a_1$, where $n_1, n_2 \in \mathbb{N}$ and $\rho_1, \rho_2 \in (0, +\infty)$ are such that $n_1 + n_2 = n$ and $\rho_1^2 + \rho_2^2 = 1$, respectively. If the third mean curvature H_3*

of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ *is such that* $a_0 R + (n-2)a_1 H_3 < a_0$ *then the index of stability of* $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ *is exactly equal to* $n+3$.

The details of the proofs of Theorems 1.1 and 1.2 are recorded in Section 4. Finally, in Remark 4.1 we register some final considerations about the next steps of this study.

2. (r,s)**-linear Weingarten Clifford tori.** Let

$$\mathbb{S}^{n+1}(\rho) = \left\{(x_1,\ldots,x_{n+2}) \in \mathbb{R}^{n+2} \,:\, x_1^2 + \cdots + x_{n+2}^2 = \rho^2\right\}$$

be the $(n+1)$-dimensional Euclidean sphere of radius $\rho \in (0,+\infty)$. As an abbreviation option, when $\rho = 1$, we denote the unit Euclidean sphere simply by $\mathbb{S}^{n+1}$. We denote by $\langle\,,\rangle$ the standard metric tensor of $\mathbb{S}^{n+1}(\rho)$ (induced from $\mathbb{R}^{n+2}$) and let $\overline{\nabla}$ the Levi-Civita connection of $\mathbb{S}^{n+1}(\rho)$ with respect to $\langle\,,\rangle$. Throughout this work, $C^\infty(\mathbb{S}^{n+1}(\rho))$ denotes the commutative ring of smooth real functions on $\mathbb{S}^{n+1}(\rho)$ and $\mathfrak{X}(\mathbb{S}^{n+1}(\rho))$ stands for the $C^\infty(\mathbb{S}^{n+1}(\rho))$-module of vector fields of class C^∞ on $\mathbb{S}^{n+1}(\rho)$.

In this setting, let $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} = \mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2) \looparrowright \mathbb{S}^{n+1}$ be a n*-dimensional Clifford torus* immersed into $\mathbb{S}^{n+1}$, with $n_1, n_2 \in \mathbb{N}$ satisfying $n = n_1 + n_2$ and $\rho_1, \rho_2 \in (0,+\infty)$ such that $\rho_1^2 + \rho_2^2 = 1$. We have that the shape operator $A : \mathfrak{X}\left(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}\right) \to \mathfrak{X}\left(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}\right)$ of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ with respect to the Gauss map

$$\begin{array}{rccl} N \;:\; & \mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} & \to & \mathbb{S}^n \\ & (p,q) & \mapsto & N(p,q)) = \left(-\dfrac{\rho_2}{\rho_1}\,p\,,\dfrac{\rho_1}{\rho_2}\,q\right) \end{array}$$

is given by

$$A = \begin{bmatrix} \dfrac{\rho_2}{\rho_1} I_{n_1} & 0 \\ 0 & -\dfrac{\rho_1}{\rho_2} I_{n_2} \end{bmatrix}, \tag{2.1}$$

where $I_{n_1} : \mathfrak{X}(\mathbb{S}^{n_1}(\rho_1)) \to \mathfrak{X}(\mathbb{S}^{n_1}(\rho_1))$ and $I_{n_2} : \mathfrak{X}(\mathbb{S}^{n_2}(\rho_2)) \to \mathfrak{X}(\mathbb{S}^{n_2}(\rho_2))$ denote the identity operators. Thus, the *principal curvatures* $\kappa_1,\ldots,\kappa_n$ of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ are such that

$$\kappa_1 = \cdots = \kappa_{n_1} = \frac{\rho_1}{\rho_2}\,, \qquad \kappa_{n_1+1} = \cdots = \kappa_n = -\frac{\rho_1}{\rho_2}\,. \tag{2.2}$$

For $j \in \{0,\ldots,n\}$, if $\sigma_j \in \mathbb{R}[X_1,\ldots,X_n]$ is the j-th elementary symmetric polynomial on the indeterminates $X_1,\ldots,X_n$, from (2.2) and the fundamental counting principle we can get that the j*-th elementary symmetric function*

$$S_j = \sigma_j(\kappa_1,\ldots,\kappa_n)$$

on the principal curvatures $\kappa_1,\ldots,\kappa_n$ of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ is given by

$$S_j = \sum_{0\le k\le j} (-1)^{j-k}\binom{n_1}{k}\binom{n_2}{j-k}\left(\frac{\rho_2}{\rho_1}\right)^k\left(\frac{\rho_1}{\rho_2}\right)^{j-k}, \tag{2.3}$$

where $S_0 = 1$ by definition. Furthermore, one defines the *higher order mean curvature* (or the j*-th mean curvature*) H_j of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ by

$$\binom{n}{j} H_j = S_j = \sum_{0\le k\le j} (-1)^{j-k}\binom{n_1}{k}\binom{n_2}{j-k}\left(\frac{\rho_2}{\rho_1}\right)^k\left(\frac{\rho_1}{\rho_2}\right)^{j-k}, \tag{2.4}$$

$j \in \{0,\ldots,n\}$.

In particular, $H_0 = 1$, for $j = 1$ we have that

$$H_1 = \frac{1}{n}\sum_{i=1}^n \kappa_i = \frac{n_1 - n\rho_1^2}{n\rho_1\rho_2} = H$$

is the *mean curvature* of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$, which is the main extrinsic curvature of the hypersurface, and for $r = n$,

$$H_n = S_n = \kappa_1\kappa_1\cdots\kappa_n = (-1)^{n_2}\left(\frac{\rho_2}{\rho_1}\right)^{n_1}\left(\frac{\rho_1}{\rho_2}\right)^{n_2}$$

is the Gauss-Kronecker curvature of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$. On the order hand, the second mean curvature

$$H_2 = \frac{2}{n(n-1)}\sum_{i<j}\kappa_i\kappa_j = \frac{n_2(n_2-1)\rho_1^4 - 2n_1n_2\rho_1^2\rho_2^2 + n_1(n_1-1)\rho_2^4}{n(n-1)\rho_1^2\rho_2^2}$$

defines a geometric quantity which is related to the (intrinsic) *normalized scalar curvature* R of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$. More precisely, it follows from the Gauss equation of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ that

$$R = 1 + H_2. \tag{2.5}$$

We also define, for $j \in \{0, \ldots, n\}$, the *j-th Newton transformation*

$$P_j : \mathfrak{X}(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}) \to \mathfrak{X}(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$$

associated to $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ by setting $P_0 = I$ (the identity operator) and, for $j \in \{1, \ldots, n\}$, via the recurrence relation

$$P_j = S_j\, I - AP_{j-1}.$$

A trivial induction shows that

$$P_j = \left(S_j\, I - S_{j-1}A + S_{j-2}A^2 - \cdots + jA^j\right), \tag{2.6}$$

so that Cayley-Hamilton Theorem gives $P_n = 0$. Moreover, since P_j is a polynomial in A for every j, it is also self-adjoint whose eigenvalues are $\frac{\partial S_{j+1}}{\partial \kappa_i}$ (where the κ_i's are the eigenvalues of A) and commutes with A. Therefore, all bases of of tangent space $T_{(p,q)}\left(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}\right)$ diagonalizing A at $(p,q) \in \mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}$ also diagonalize all of the P_j at (p,q). Let $\{e_1, \ldots, e_n\}$ be such a basis. Denoting by A_i the restriction of A to $\langle e_i\rangle^{\perp} \subset T_{p.q}\left(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}\right)$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1}(-1)^k S_k(A_i)t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1\le j_1<\ldots<j_m\le n\\ j_1,\ldots,j_m\neq i}} \lambda_{j_1}\cdots\lambda_{j_m}.$$

With the above notations, it is also immediate to check that $P_j(e_i) = S_j(A_i)e_i$, and hence (cf. [6, Lemma 2.1])

$$\left\{\begin{array}{rcl} \mathrm{tr}(P_j) &=& (n-j)S_j \;=\; b_jH_j;\\ \mathrm{tr}(A\circ P_j) &=& (j+1)S_{j+1} \;=\; b_jH_{j+1};\\ \mathrm{tr}(A^2\circ P_j) &=& S_1S_{j+1} - (j+2)S_{j+2} \;=\; n\dfrac{b_j}{j+1}HH_{j+1} - b_{j+1}H_{j+2}, \end{array}\right. \tag{2.7}$$

where $b_j = (j+1)\binom{n}{j+1} = (n-j)\binom{n}{j}$.

Associated to each Newton Transformation P_j, $j \in \{0, \ldots, n\}$, one has the *second order linear differential operator*

$$\begin{array}{rcccl} L_j &:& C^\infty(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}) &\to& C^\infty(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})\\ && f &\mapsto& L_j(f) = \mathrm{tr}(P_j \circ \mathrm{Hess}\, f). \end{array} \tag{2.8}$$

We remark that L_0 is the Laplacian operator Δ and L_1 is the Cheng-Yau's square operator $\square$ defined in [12]. According to [21], since $\mathbb{S}^{n+1}$ has constant sectional curvatures, P_j is a divergence-free and; consequently,

$$L_j(f) = \mathrm{div}(P_j(\nabla f)) \tag{2.9}$$

for all $f \in C^\infty(\Sigma^n)$.

From (2.4) we can observe that there is a considerable amount of Clifford tori $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ satisfying

$$a_0H + a_1R \in \mathbb{R}, \tag{2.10}$$

for some real constants a_0 and a_1(at least one of them nonzero). On account of (2.5) we can understand the linear condition (2.10) as

$$a_0 H + a_1 H_2 = 0, \tag{2.11}$$

for some real constants a_0 and a_1(at least one of them nonzero). The Clifford toi $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ that satisfy (2.11) belongs to a class of hypersurfaces called the *linear Weingarten*, and there is a vast recent literature treating the problem of characterizing these hypersurfaces (see, for instance, [3, 4, 5, 11, 16, 17, 18]). This class of Clifford tori contains those that are *minimal* (when $a_1 = 0$ in (2.11)), as well as those that are 1-minimal (when $a_0 = 0$ in (2.11)). This will motivate us to establish the following notion.

Definition 2.1. *Let r and s be any entire numbers satisfying the inequalities $0 \leq r \leq s \leq n-1$ and choose $n_1, n_2 \in \mathbb{N}$ and $\rho_1, \rho_2 \in (0, +\infty)$ such that $n_1 + n_2 = n$ and $\rho_1^2 + \rho_2^2 = 1$, respectively. We say that $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} = \mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2) \looparrowright \mathbb{S}^{n+1}$ is a (r,s)-linear Weingarten Clifford torus if there exist real numbers $a_r, \ldots, a_s$ (at least one of them nonzero) such that the following linear relation occurs on $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}$:*

$$a_r H_{r+1} + \cdots + a_s H_{s+1} = 0, \tag{2.12}$$

where H_j is the j-th mean curvature of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$, with $j \in \{r, \ldots, s\}$. In the case of $(0,1)$-linear Weingarten Clifford tori, we will simply call them linear Weingarten Clifford tori.

Observe that our Definition 2.1 is recorded in such a way that it contains those Clifford tori that are r-*minimal*, namely, when $r = s \in \{0, \ldots, n-1\}$ in (2.12), $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ must be r-minimal.

3. The notion of index of stability for a (r,s)-linear Weingarten Clifford torus. According to [7, Lemma 2.2] and [23, Proposition 3.6], any real valued smooth function defined on a compact orientable hypersurface $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ satisfying

$$\int_{\Sigma^n} f dv = 0 \tag{3.1}$$

induces a normal variation $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{S}^{n+1}$ with variational normal field $\frac{\partial X}{\partial t}|_{t=0} = fN$, and with first variation

$$\delta_f \widetilde{\mathcal{A}}_{r,s} = \frac{d}{dt}\widetilde{\mathcal{A}}_{r,s}(t)\Big|_{t=0}$$

of the functional

$$\begin{array}{rcccl} \widetilde{\mathcal{A}}_{r,s}: & (-\epsilon,\epsilon) & \to & \mathbb{R} & \\ & t & \mapsto & \widetilde{\mathcal{A}}_{r,s}(t) & = & a_r\mathcal{A}_r(t) + \cdots + a_s\mathcal{A}_s(t) \end{array}$$

given by

$$\delta_f \widetilde{\mathcal{A}}_{r,s} = -\int_M \Big\{ \sum_{j=r}^{s} a_j b_j H_{j+1} \Big\} f dv, \tag{3.2}$$

where N is the Gauss map of $\Sigma^n \looparrowright \mathbb{S}^{n+1}$, r and s are entire numbers satisfying the inequalities $0 \leq r \leq s \leq n-1$, $a_r, \ldots, a_s$ are nonnegative real numbers (with at least one nonzero), $\mathcal{A}_j$ is the j-th area functional of $\Sigma^n \looparrowright \mathbb{S}^{n+1}$, $j \in \{r, \ldots, s\}$, H_j is the j-th mean curvature of $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ with respect to N and $b_j = (j+1)\binom{n}{j+1}$, $j \in \{r, \ldots, s\}$. Here, $\mathcal{A}_j$ is given by (cf. [6])

$$\begin{array}{rcccl} \mathcal{A}_j: & (-\epsilon,\epsilon) & \to & \mathbb{R} & \\ & t & \mapsto & \mathcal{A}_j(t) & = & \displaystyle\int_{\Sigma^n} F_j\left(S_1(t), S_2(t), \ldots, S_j(t)\right) dv_t, \end{array}$$

where $S_j(t) = S_j(t, \cdot)$ is the j-th elementary symmetric fuunction of Σ^n via the immersion

$$\begin{array}{rcccl} X_t: & \Sigma^n & \to & \mathbb{S}^{n+1} & \\ & p & \mapsto & X_t(p) & = & X(t,p) \end{array}$$

and F_j is recursively defined by setting $F_0 = 1$, $F_1 = S_1(t)$ and, for $2 \leq j \leq n-1$,

$$F_j = S_j(t) + \frac{(n-j+1)}{j-1} F_{j-2}.$$

As a consequence of (3.2), all compact orientable hypersurface $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ with higher order mean curvatures $H_{r+1}, \ldots, H_{s+1}$ verifying

$$a_r b_r H_{r+1} + \cdots + a_s b_s H_{s+1} = c \in \mathbb{R}$$

is a critical point of $\widetilde{\mathcal{A}}_{r,s}$ restricted to to smooth functions defined on Σ^n that obey the condition (3.1), where $b_j = (n-j)\binom{n}{j}$ for $j \in \{r, \ldots, s\}$. Geometrically, this request means that the variations under consideration preserve a certain volume function (for more details, see [23, Section 3]). Here, we can observe that geodesic spheres of $\mathbb{S}^{n+1}$ and (r,s)-linear Weingarten Clifford tori $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} = \mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2) \looparrowright \mathbb{S}^{n+1}$ (see Definition 2.1) satisfying the relation

$$a_r b_r H_{r+1} + \cdots + a_s b_s H_{s+1} = 0$$

are critical points for this geometric variational problem.

For these critical points, [23, Proposition 3.9] asserts that the stability of the corresponding variational problem of minimizing the functional $\widetilde{\mathcal{A}}_{r,s}$ for all variations that preserve the volume is given by the second variation

$$\delta_f^2 \widetilde{\mathcal{A}}_{r,s} = \frac{d^2}{dt^2} \widetilde{\mathcal{A}}_{r,s}(t)\Big|_{t=0} = -\int_{\Sigma^n} f \mathcal{J}_{r,s}(f) dv$$

of $\widetilde{\mathcal{A}}_{r,s}$, where

$$\begin{aligned} \mathcal{J}_{r,s} : C^\infty(\Sigma^n) &\rightarrow C^\infty(\Sigma^n) \\ f &\mapsto \mathcal{J}_{r,s}(f) = \mathcal{L}_{r,s}(f) + \sum_{j=r}^{s} (j+1) a_j \{\mathrm{tr}(P_j) + \mathrm{tr}(A^2 \circ P_j)\} f \end{aligned} \tag{3.3}$$

is the *Jacobi operator* associated with $\widetilde{\mathcal{A}}_{r,s}$. Here, $\mathcal{L}_{r,s}$ is the second order linear differential operator on Σ^n given by

$$\mathcal{L}_{r,s}(f) = \sum_{j=r}^{s} (j+1) a_j L_j(f), \tag{3.4}$$

A and P_j are the shape operator and the j-th Newton transformation of $\Sigma^n \looparrowright \mathbb{S}^{n+1}$, respectively, and L_j is the differential operator on Σ^n given in (2.8). On the notion of stability for the critical points described above, let us remember that such a critical point is called *stable* when $\delta_f^2 \widetilde{\mathcal{A}}_{r,s} \geq 0$ for any function f that satisfies (3.1) (cf. [23, Remark 3.8]). In this context, in [23, Theorem 4.3] it was established that geodesic spheres of $\mathbb{S}^{n+1}$ are the only stable critical points of the functional $\widetilde{\mathcal{A}}_{r,s}$ for volume-preserving variations.

Regarding the critical points associated with a geometric variational problem that are not stable, it is interesting to develop a study to try to classify them through the so-called stability index. Thinking in this way, here we focus our attention on studying the stability index associated with the variational problem described at the beginning of this section of (r,s)-linear Weingarten Clifford tori $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} = \mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2)$ immersed into $\mathbb{S}^{n+1}$ satisfying

$$a_r b_r H_{r+1} + \cdots + a_s b_s H_{s+1} = 0.$$

For that, from the compactness of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$, from (2.9) and Divergence Theorem we observe that the Jacobi operator $\mathcal{J}_{r,s}$ satisfies

$$\int_{\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}} f \mathcal{J}_{r,s}(g)\, dv = \int_{\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}} g \mathcal{J}_{r,s}(f)\, dv$$

on $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}$, for any $f, g \in C^\infty(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$.

Taking into account all that has been studied in this section, we are motivated to establish the following notions.

Definition 3.1. *Let r and s be any entire numbers satisfying the inequalities $0 \leq r \leq s \leq n-1$ and choose $n_1, n_2 \in \mathbb{N}$ and $\rho_1, \rho_2 \in (0, +\infty)$ such that $n_1 + n_2 = n$ and $\rho_1^2 + \rho_2^2 = 1$, respectively. Let $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} = \mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2) \looparrowright \mathbb{S}^{n+1}$ be a (r,s)-linear Weingarten Clifford torus whose higher order mean curvatures $H_{r+1}, \ldots, H_{s+1}$ satisfying the relation*

$$a_r b_r H_{r+1} + \cdots + a_s b_s H_{s+1} = 0,$$

for some nonnegative real numbers $a_r, \ldots, a_s$ (with at least one nonzero), where $b_j = (n-j)\binom{n}{j}$ for $j \in \{r, \ldots, s\}$.

(a) *The* index form $\mathcal{Q}_{r,s} : C^\infty(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}) \to \mathbb{R}$ *of* $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ *is the quadratic form associated to the symmetric bilinear form*

$$\begin{array}{rcl} \mathcal{B}_{r,s} \;:\; C^\infty(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}) \times C^\infty(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}) & \to & \mathbb{R} \\ (f,g) & \mapsto & \mathcal{B}_{r,s}(f,g) = -\displaystyle\int_{\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}} f\,\mathcal{J}_{r,s}(g)\,dv, \end{array} \tag{3.5}$$

where $\mathcal{J}_{r,s}$ *is the Jacobi operator on* $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}$ *defined in* (3.3).

(b) *The* index of (r,s)-stability *of* $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$*, denoted by* $\mathrm{Ind}_{r,s}(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$*, is given by the maximal dimension of the subspace* $\{f \in C^\infty(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}) : \mathcal{Q}_{r,s}(f) < 0\}$.

(c) *Regarding the index of* $(0,1)$*-stability of a linear Weingarten Clifford torus* $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ *with its first two mean curvatures* H *and* H_2 *satisfying*

$$na_0H + n(n-1)a_1H_2 = 0,$$

we simply call it the index of stability.

Remark 3.1. *Let* $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ *be a* (r,s)*-linear Weingarten Clifford torus as described in Definition 3.1. From* (3.5)*, we have that* $\mathrm{Ind}_{r,s}(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$ *is equivalent to the number of negative eigenvalues (counted with multiplicity) of the Jacobi operator* $\mathcal{J}_{r,s}$ *of* $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ *given in* (3.4)*. We remember that, with our notations, a real number* ϱ *is an eigenvalue of* $\mathcal{J}_{r,s}$ *if and only if* $\mathcal{J}_{r,s}(f) + \varrho f = 0$ *for some function* $f \in C^\infty(\Sigma^n)$*. Throughout this work, a similar definition is assumed for the eigenvalues of any differential operator.*

4. Proof of main results. Let $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} = \mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2) \looparrowright \mathbb{S}^{n+1}$ be a Clifford torus as described at the beginning of Section 2. We have that its shape operator (2.1) admits the expression

$$A = \begin{bmatrix} \lambda I_{n_1} & 0 \\ 0 & \mu I_{n_2} \end{bmatrix}, \tag{4.1}$$

where $\lambda = \rho_2/\rho_1$ and $\mu = -\rho_1/\rho_2$. For $j \in \{0,\dots,n\}$, let S_j be the j-th elementary symmetric function of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ given in (2.3). From (2.6), we observe that the j-th Newton transformation P_j of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ can be written as $P_j = \varphi^j(A)$, where

$$\varphi_j(t) = \sum_{k=0}^{j} (-1)^k S_{j-k} t^k = S_j - S_{j-1}t + S_{j-2}t^2 - \cdots + j\,t^j, \tag{4.2}$$

So, from (4.1),

$$P_j = \begin{bmatrix} \varphi_j(\lambda) I_{n_1} & 0 \\ 0 & \varphi_j(\mu) I_{n_2} \end{bmatrix}. \tag{4.3}$$

This enable us to write the differential operator $L_j : C^\infty(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}) \to C^\infty(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$ (see (2.8)) of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ on the following way

$$L_j = \varphi_j(\lambda)\Delta_{\mathbb{S}^{n_1}(\rho_1)} \oplus \varphi_j(\mu)\Delta_{\mathbb{S}^{n_2}(\rho_2)}. \tag{4.4}$$

In particular, if we denote the eigenfunctions and the eigenvalues of $\mathbb{S}^{n_1}(\rho_1)$ and $\mathbb{S}^{n_2}(\rho_2)$ by $\{(f_k,\lambda_k)\}_{k=0}^\infty$ and $\{(g_l,\mu_l)\}_{l=0}^\infty$, respectively, we have the eigenfunctions and eigenvalues of L_j are given by

$$L_j(f_{k,l}) + (\lambda_k\varphi_j(\lambda) + \mu_l\varphi_j(\mu))\, f_{k,l} = 0 \tag{4.5}$$

where $f_{k,l} = f_k \otimes g_l$ is the product function on $\mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2)$ defined by

$$f_{k,l}(p,q) = f_k \otimes g_l(p,q) = f_k(p)g_l(q) \tag{4.6}$$

for all $(p,q) \in \mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2)$.

Furthermore, (4.1) and (4.3) ensure that

$$A \circ P_j = \begin{bmatrix} \lambda\varphi_j(\lambda) I_{n_1} & 0 \\ 0 & \mu\varphi_j(\mu) I_{n_2} \end{bmatrix}$$

and, consequently,

$$\operatorname{tr}(A \circ P_j) = \lambda\varphi_j(\lambda)n_1 + \mu\varphi_j(\mu)n_2. \tag{4.7}$$

Before to establish the index of (r, s)-stability $\operatorname{Ind}_{r,s}(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$ of a (r, s)-linear Weingarten Clifford torus $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ we need the next result which gives an explicit expression for your Jacobi functional.

Proposition 4.1. *Let r and s be any entire numbers satisfying the inequalities $0 \le r \le s \le n-1$ and choose $n_1, n_2 \in \mathbb{N}$ and let $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} = \mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2) \looparrowright \mathbb{S}^{n+1}$ be a (r, s)-linear Weingarten Clifford torus whose higher order mean curvatures $H_{r+1}, \dots, H_{s+1}$ satisfying*

$$a_r b_r H_{r+1} + \cdots + a_s b_s H_{s+1} = 0, \tag{4.8}$$

for some nonnegative real numbers $a_r, \dots, a_s$ (with at least one nonzero), where $n_1, n_2 \in \mathbb{N}$ and $\rho_1, \rho_2 \in (0, +\infty)$ are such that $n_1 + n_2 = n$ and $\rho_1^2 + \rho_2^2 = 1$, respectively, and $b_j = (n-j)\binom{n}{j}$ for $j \in \{r, \dots, s\}$.

(a) *The Jacobi operator $\mathcal{J}_{r,s} : C^\infty(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}) \to C^\infty(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$ of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ is given by*

$$\mathcal{J}_{r,s} = \sum_{j=r}^{s}(j+1)a_j\left\{\varphi_j(\lambda)\Delta_{\mathbb{S}^{n_1}(\rho_1)} \oplus \varphi_j(\mu)\Delta_{\mathbb{S}^{n_2}(\rho_2)}\right\} + 2\sum_{j=r}^{s}(j+1)a_j b_j H_j,$$

where $\lambda = \rho_2/\rho_1$, $\mu = -\rho_1/\rho_2$ and φ_j is the real value function given in (4.2).

(b) *The index of (r, s)-stability $\operatorname{Ind}_{r,s}(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$ of a (r, s)-linear Weingarten Clifford torus $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ reduces to the number of eigenvalues of the differential operator*

$$\sum_{j=r}^{s}(j+1)a_j\left\{\varphi_j(\lambda)\Delta_{\mathbb{S}^{n_1}(\rho_1)} \oplus \varphi_j(\mu)\Delta_{\mathbb{S}^{n_2}(\rho_2)}\right\}$$

(counted with multiplicity) which are strictly less than

$$2\sum_{j=r}^{s}(j+1)a_j b_j H_j.$$

Proof: First, from (2.4) and (4.8) we observe that the elementary symmetric function $S_{r+1}, \dots, S_{s+1}$ of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ satisfy

$$a_r(r+1)S_{r+1} + \cdots + a_s(s+1)S_{s+1} = 0,$$

Next, from (2.7) and (4.7) we get

$$\begin{aligned} 0 = \sum_{j=r}^{s} a_j(j+1)S_{j+1} &= \sum_{j=r}^{s} a_j(j+1)\operatorname{tr}(A \circ P_j) \\ &= n_1\lambda\sum_{j=r}^{s} a_j(j+1)\varphi_j(\lambda) - n_2\mu\sum_{j=r}^{s} a_j(j+1)\varphi_j(\mu), \end{aligned}$$

or equivalently,

$$\frac{n_1}{\rho_1^2}\sum_{j=r}^{s} a_j(j+1)\varphi_j(\lambda) = \frac{n_2}{\rho_2^2}\sum_{j=r}^{s} a_j(j+1)\varphi_j(\mu),$$

Taking into account that (see (4.3))

$$\sum_{j=r}^{s} a_j(j+1)\operatorname{tr}(Pj) = n_1\sum_{j=r}^{s} a_j(j+1)\varphi_j(\lambda) + n_2\sum_{j=r}^{s} a_j(j+1)\varphi_j(\mu)$$

we may write this in two ways:

$$\sum_{j=r}^{s} a_j(j+1)\operatorname{tr}(Pj) = \left(\frac{\rho_1^2}{\rho_2^2} + 1\right)n_2\sum_{j=r}^{s} a_j(j+1)\varphi_j(\mu) = \frac{n_2}{\rho_2^2}\sum_{j=r}^{s} a_j(j+1)\varphi_j(\mu) \tag{4.9}$$

and

$$\sum_{j=r}^{s} a_j(j+1)\operatorname{tr}(Pj) = \left(\frac{\rho_2^2}{\rho_1^2} + 1\right)n_1\sum_{j=r}^{s} a_j(j+1)\varphi_j(\lambda) = \frac{n_1}{\rho_1^2}\sum_{j=r}^{s} a_j(j+1)\varphi_j(\lambda) \tag{4.10}$$

Now from equations (4.9) and (4.10) we deduce

$$\sum_{j=r}^{s} a_j(j+1)\mathrm{tr}(Pj) = \frac{1}{2}\left(\frac{n_1}{\rho_1^2}\sum_{j=r}^{s} a_j(j+1)\varphi_j(\lambda) + \frac{n_2}{\rho_2^2}\sum_{j=r}^{s} a_j(j+1)\varphi_j(\mu)\right) \tag{4.11}$$

On the order hand, since

$$A^2 \circ P_j = \begin{bmatrix} \lambda^2\varphi_j(\lambda)I_{n_1} & 0 \\ 0 & \mu^2\varphi_j(\mu)I_{n_2} \end{bmatrix},$$

from (2.7) and (4) we have

$$\begin{aligned} &\sum_{j=r}^{s} a_j(j+1)\left(\lambda^2\varphi_j(\lambda)n_1 + \mu^2\varphi_j(\mu)n_2\right) = \sum_{j=r}^{s} a_j(j+1)\mathrm{tr}(A^2\circ P_j) = \\ &= \sum_{j=r}^{s} a_j(j+1)\left(S_1S_{j+1} - (j+2)S_{j+2}\right) \\ &= S_1\underbrace{\Big(\sum_{j=r}^{s} a_j(j+1)S_{j+1}\Big)}_{0} - \sum_{j=r}^{s} a_j(j+1)(j+2)S_{j+2}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &n_1\frac{\rho_2^2}{\rho_1^2}\sum_{j=r}^{s} a_j(j+1)\varphi_j(\lambda) + n_2\frac{\rho_1^2}{\rho_2^2}\sum_{j=r}^{s} a_j(j+1)\varphi_j(\mu) = \\ &= -\sum_{j=r}^{s} a_j(j+1)(j+2)S_{j+2}\sum_{j=r}^{s} a_j(j+1)\mathrm{tr}(A^2\circ P_j). \end{aligned} \tag{4.12}$$

Next, we combine equations (4.9), (4.10) and (4.12) to deduce

$$\begin{aligned} &\sum_{j=r}^{s} a_j(j+1)\left\{(n-j)S_j + (j+2)S_{j+2}\right\} = \\ &= \sum_{j=r}^{s} a_j(j+1)\left\{\mathrm{tr}(P_j) - \mathrm{tr}(A^2\circ P_j)\right\} \\ &= \frac{n_2}{\rho_2^2}\sum_{j=r}^{s} a_j(j+1)\varphi_j(\mu) - n_1\frac{\rho_2^2}{\rho_1^2}\sum_{j=r}^{s} a_j(j+1)\varphi_j(\lambda) \\ &\quad -n_2\frac{\rho_1^2}{\rho_2^2}\sum_{j=r}^{s} a_j(j+1)\varphi_j(\mu) \\ &= n_2\sum_{j=r}^{s} a_j(j+1)\varphi_j(\mu) - \rho_2^2\sum_{j=r}^{s} a_j(j+1)\mathrm{tr}(P_j) \\ &= n_2\sum_{j=r}^{s} a_j(j+1)\varphi_j(\mu) - \rho_2^2\frac{n_2}{\rho_2^2}\sum_{j=r}^{s} a_j(j+1)\varphi_j(\mu) = 0. \end{aligned} \tag{4.13}$$

Therefore, from (2.4), (3.3), (3.4), (4.4) and (4.13) we get item (a). For item (b), just take into account the comments in Remark 3.1 and the result of item (a). □

From (2.5), Definition 2.1 and Proposition 4.1 we get the following result that we provide a criterion to calculate the index of stability of a linear Weingarten Clifford torus.

Proposition 4.2. *Let a_0 and a_1 be nonnegative real numbers, with at least one of them being nonzero, and let $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} = \mathbb{S}^{n_1}(\rho_1)\times\mathbb{S}^{n_2}(\rho_2) \looparrowright \mathbb{S}^{n+1}$ be a linear Weingarten Clifford torus with mean and scalar curvatures H and R satisfying*

$$a_0H + (n+1)a_1R = (n+1)a_1,$$

where $n_1, n_2 \in \mathbb{N}$ and $\rho_1, \rho_2 \in (0, +\infty)$ are such that $n_1 + n_2 = n$ and $\rho_1^2 + \rho_2^2 = 1$, respectively. Then, the Jacobi operator $\mathcal{J}_{0,1} : C^\infty(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}) \to C^\infty(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$ of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ is given by

$$\mathcal{J}_{0,1} = \left\{ a_0 + \frac{2a_1\left(2n\rho_1\rho_2 H + \rho_1^2 - \rho_2^2\right)}{\rho_1\rho_2} \right\} \left\{ \Delta_{\mathbb{S}^{n_1}(\rho_0)} \oplus \Delta_{\mathbb{S}^{n_2}(\rho_2)} \right\} + 2na_0 + 4n(n-1)a_1 H.$$

Furthermore, the index of stability of a linear Weingarten Clifford torus $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ reduces to the number of eigenvalues of the differential operator

$$\left\{ a_0 + \frac{2a_1\left(2n\rho_1\rho_2 H + \rho_1^2 - \rho_2^2\right)}{\rho_1\rho_2} \right\} \left\{ \Delta_{\mathbb{S}^{n_1}(\rho_0)} \oplus \Delta_{\mathbb{S}^{n_2}(\rho_2)} \right\}$$

(counted with multiplicity) which are strictly less than $2na_0 + 4n(n-1)a_1 H$.

Proof of Theorem 1.1.

We are now in a position to provide the proof of our main result concerning the index of (r,s)-stability of a (r,s)-linear Weingarten Clifford torus $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} = \mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2) \looparrowright \mathbb{S}^{n+1}$ that admit a null linear combination of their higher order mean curvatures H_{r+1} and H_{s+1}, with $0 \le r \le s \le n-1$.

Proof:

In order to compute the index of a (r,s)-linear Weingarten Clifford torus $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ it is important to have the spectrum of each sphere $\mathbb{S}^{n_1}(\rho_1)$ and $\mathbb{S}^{n_2}(\rho_2)$. In that regard, from [10, Section 2.4], we have that the eigenvalues of $\mathbb{S}^{n_1}(\rho_1)$ are

$$\lambda_k = \frac{k}{\rho_1^2}(n_1 + k - 1), \qquad k \in \{0, 1, 2, \ldots\}, \tag{4.14}$$

with multiplicities

$$m(\lambda_k) = \frac{(n_1 + k - 2)(n_1 + k - 3)\ldots(n_1 + 1)n_1}{k!}(n_1 + 2k - 1), \tag{4.15}$$

while for $\mathbb{S}^{n_2}(\rho_2)$ the eigenvalues are

$$\mu_l = \frac{l}{\rho_2^2}(n_2 + l - 1), \qquad l \in \{0, 1, 2, \ldots\}, \tag{4.16}$$

with multiplicities

$$m(\mu_l) = \frac{(n_2 + l - 2)(n_2 + l - 3)\ldots(n_2 + 1)n_2}{l!}(n_2 + 2l - 1). \tag{4.17}$$

On the other hand, from equation (4.11) and the fact that $\lambda_1 = n_1/\rho_1^2$ and $\mu_1 = n_2/\rho_2^2$ to obtain

$$\begin{aligned} \sum_{j=r}^{s}(j+1)a_j b_j H_j &= \sum_{j=r}^{s} a_j(j+1)\mathrm{tr}(Pj) \\ &= \frac{1}{2}\left(\lambda_1 \sum_{j=r}^{s} a_j(j+1)\varphi_j(\lambda) + \mu_1 \sum_{j=r}^{s} a_j(j+1)\varphi_j(\mu)\right) \end{aligned}$$

Using last equation, as well (4.5) and Proposition 4.1 we infer that the Jacobi operator $\mathcal{J}_{r,s} : C^\infty(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}) \to C^\infty(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$ of $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} \looparrowright \mathbb{S}^{n+1}$ verifies

$$\mathcal{J}_{r,s}(f_{k,l}) = \left((\lambda_1 - \lambda_k)\sum_{j=r}^{s}(j+1)a_j\varphi_j(\lambda) + (\mu_1 - \mu_l)\sum_{j=r}^{s}(j+1)a_j\varphi_j(\mu)\right) f_{k,l} = 0,$$

where $f_{k,l}$ is the product function on $\mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2)$ defined in (4.6), and, hence, the eigenvalues $\{\varrho_{k,l}\}_{k,l=0}^{\infty}$ of $\mathcal{J}_{r,s}$ are given by

$$\varrho_{k,l} = (\lambda_k - \lambda_1)\sum_{j=r}^{s}(j+1)a_j\varphi_j(\lambda) + (\mu_l - \mu_1)\sum_{j=r}^{s}(j+1)a_j\varphi_j(\mu), \tag{4.18}$$

with $k, l \in \{0, 1, 2, \ldots\}$.

Using equations (4.9) and (4.10) we have

$$\begin{aligned}\sum_{j=r}^{s} a_j(j+1)(n-j)S_j &= \sum_{j=r}^{s} a_j(j+1)\mathrm{tr}(Pj)\\ &= \lambda_1 \sum_{j=r}^{s} a_j(j+1)\varphi_j(\lambda) = \mu_1 \sum_{j=r}^{s} a_j(j+1)\varphi_j(\mu).\end{aligned}$$

Then, (4.13) yield

$$\sum_{j=r}^{s} a_j(j+1)\varphi_j(\lambda) > 0 \quad \left(\text{and, in turn,} \quad \sum_{j=r}^{s} a_j(j+1)\varphi_j(\mu) > 0\right)$$

if and only if

$$\sum_{j=r}^{s} a_j(j+1)(j+2)S_{j+2} < 0, \tag{4.19}$$

which is the hypothesis (1.1) in Theorem 1.1.

According to Remark 3.1, the only eigenvalues $\varrho_{k,l}$ of $\mathcal{J}_{r,s}$ that contribute to the calculation of the index $\mathrm{Ind}_{r,s}(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$ are those that are negative (counted with their multiplicity). In this sense, from (4.18) we immediately have that $\varrho_{1,1} = 0$ does not contribute to computing $\mathrm{Ind}_{r,s}(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$. Furthermore, from (4.14) and (4.16) we observe that

$$\lambda_2 - \lambda_1 = \frac{n_1+2}{\rho_1^2} > 0, \qquad \mu_2 - \mu_1 = \frac{n_2+2}{\rho_2^2} > 0,$$

and, more generally, $\lambda_k - \lambda_1 > 0$ and $\mu_2 - \mu_1 > 0$ for all $k, l \geq 2$. This added to the fact that $a_r, \dots, a_s$ are nonnegative real numbers (with at least one nonzero) guarantees us, from (4.18), that the eigenvalues $\varrho_{k,l}$ with $k, l \geq 2$ do not contribute to the calculation of $\mathrm{Ind}_{r,s}(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$ provided that (4.19) is true on $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}$. So, assuming this same condition and taking into account that the sequences $\{\lambda_k\}_{k=0}^{\infty}$ and $\{\mu_l\}_{l=0}^{\infty}$ are increasing we have that the unique functions which contribute to the index $\mathrm{Ind}_{r,s}(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2})$ are

$$\begin{cases} \varrho_{0,0} = -\lambda_1 \sum_{j=r}^{s}(j+1)a_j\varphi_j(\lambda) - \mu_1 \sum_{j=r}^{s}(j+1)a_j\varphi_j(\mu),\\ \varrho_{1,0} = -\mu_1 \sum_{j=r}^{s}(j+1)a_j\varphi_j(\mu),\\ \varrho_{0,1} = -\lambda_1 \sum_{j=r}^{s}(j+1)a_j\varphi_j(\lambda),\end{cases}$$

whose multiplicities are respectively, 1, n_1+1 and n_2+1 (see (4.15) and (4.17)). Therefore, we can conclude that $\mathrm{Ind}_{r,s}(\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}) = n+3$ provided that (4.19) is true on $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2}$. □

Proof of Theorem 1.2.

To end this section, we provide the proof of our last result, which summarizes our entire study on the index of stability for the case of linear Weingarten Clifford tori immersed into $\mathbb{S}^{n+1}$.

Proof: Taking into account Definition 2.1, item (c) of Definition 3.1, Proposition 4.2, the definition of the higher order mean curvatures H_r given in (2.4) and the relation (2.5) between the second mean curvature H_2 and the normalized scalar curvature R of a linear Weingarten Clifford torus $\mathbb{T}^{n_1,n_2}_{\rho_1,\rho_2} = \mathbb{S}^{n_1}(\rho_1) \times \mathbb{S}^{n_2}(\rho_2) \looparrowright \mathbb{S}^{n+1}$, the result follows from Theorem 1.1 making $r=0$ and $s=1$. □

Remark 4.1. *We observe, initially, that the notion of index of (r,s)-stability recorded in Definition 3.1 could be extended to a (r,s)-linear Weingarten compact oriented hypersurface $\Sigma^n \looparrowright \mathbb{S}^{n+1}$, but in this new situation it is not known so far some appropriate ellipticity criterion for the Jacobi operator $\mathcal{J}_{r,s}$ defined in (3.3) and, as a consequence, it is not known whether the eigenvalues $\varrho_j{}'s$ of $\mathcal{J}_{r,s}$ on Σ^n admit the behavior*

$$(-\infty <)\, \varrho_1 < \varrho_2 \leq \cdots \leq \varrho_j \cdots \to +\infty,$$

repeated according to their multiplicity, behavior that is required in the approach of our study. We believe that this difficulty can be resolved in the near future and, thus, to be able to carry out a study of index of (r,s)-stability of any (r,s)-linear Weingarten compact hypersurface $\Sigma^n \looparrowright \mathbb{S}^{n+1}$ whose higher order mean curvatures $H_{r+1}, \dots, H_{s+1}$ satisfying $a_r b_r H_{r+1} + \cdots + a_s b_s H_{s+1} = 0$ for some nonnegative real numbers $a_r, \dots, a_s$ (with at least one nonzero), where $b_j = (n-j)\binom{n}{j}$ for $j \in \{r, \dots, s\}$.

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