Journal homepage http://revistas.unitru.edu.pe/index.php/SSMM

SELECCIONES MATEMÁTICAS
Universidad Nacional de Trujillo
ISSN: 2411-1783 (Online)
2022; Vol. 9(2): 370-380.

A new conformable fractional derivative and applications

Una nueva derivada fraccionaria conforme y aplicaciones

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Received, Nov. 02, 2022 *Accepted, Dec. 25, 2022*

How to cite this article:
Stojiljković V. *A new conformable fractional derivative and applications*. Selecciones Matemáticas. 2022;9(2):370–380. http://dx.doi.org/10.17268/sel.mat.2022.02.12

Abstract
The motivation for this paper comes from other papers treating the fractional derivatives. We introduce a new definition of fractional derivative which obeys classical properties including linearity, product rule, quotient rule, power rule, chain rule, Rolle's theorem, mean value theorem and Taylor series. Usage of the defined derivative is given in the example section which shows how our derivative can be used in solving differential equations. Comparison of our derivative with the derivative defined by Abdejjawad and overall conclusions are given in the conclusion section.
Keywords. Fractional derivatives, fractional calculus.

Resumen
La motivación de este artículo proviene de otros artículos que tratan las derivadas fraccionarias. Introducimos una nueva definición de derivada fraccionaria que obedece a propiedades clásicas que incluyen la linealidad, la regla del producto, la regla del cociente, la regla de la potencia, la regla de la cadena, el teorema de Rolle, teorema del valor medio y series de Taylor. El uso de esta derivada definida se proporciona en la sección de ejemplo donde se muestra cómo se puede usar nuestra derivada para resolver ecuaciones diferenciales. La comparación de nuestra derivada con la derivada definida por Abdejjawad y las conclusiones generales se dan en la sección de conclusiones.
Palabras clave. Derivadas fraccionarias, cálculo fraccionario.

1. Introduction. The idea of fractional calculus is as old as traditional calculus. The history of fractional calculus dates back to now famous letter in which L'Hopital sent a letter to Leibniz. In his message an important question about the order of the derivative emerged. What might be the derivative of $\frac{1}{2}$? Leibniz said that it is nonsense, and therefore he does not anticipate the beginning of the area known today as fractional calculus(FC). The first mathematician to suggest an integral representation was Fourier 1822 [19, 17] suggested an integral representation in order to define the derivative and his version can be considered the first definition of the arbitrary positive order. Abel in 1826 solved an integral equation associated with tautochrone problem, which was the first application of FC(fractional calculus). After Abel, many mathematicians proceeded to work in the field, some of the names are: Riemann, Grünwald and Letnikov [7, 15], Hadamard, Weyl [29], Riesz [24, 25] and many more. In the late upper half of the 20th century, Caputo [5] formulated a definition, more restrictive than the Riemann-Liouville but more appropriate to discuss problems involving fractional differential equations with initial conditions. Fractional calculus was found to be useful in physics as well, for example Whatcraft and Meerschaert (2008) described a fractional conservation of mass, Fractional Schrödinger equation in quantum theory, and many others. Until recently, research on FC was confised to mathematicians but, in the last two decades many applications

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of fractional calculus in various fields were found [6, 11, 9, 10, 13, 20, 16, 18]. As a result, fractional calculus became an important field of study for various fields. Some recent work on FC can be found here [27, 26, 28, 2, 3, 4, 21, 23, 12, 8].

Among the inconsistencies of the existing fractional derivatives D^α are:

1. Most of the fractional derivatives except Caputo-type, do not satisfy $D^\alpha(1) = 0$, if α is not a natural number.
2. All fractional derivatives do not satisfy the familiar Product Rule for two functions $D^\alpha(fg) = gD^\alpha(f) + fD^\alpha(g)$.
3. All fractional derivatives do not satisfy the familiar Quotient rule for two functions $D^\alpha(fg) = \frac{gD^\alpha(f)-fD^\alpha(g)}{g^2}$ with $g \neq 0$.
4. All fractional derivatives do not satisfy the Chain Rule for composite functions $D^\alpha(f \circ g)(t) = D^\alpha(f(g))D^\alpha g(t)$.
5. All fractional derivatives do not satisfy the Indices rule $D^\alpha D^\beta(f) = D^{\alpha+\beta}(f)$.

However, in [1] the authors define a new well-behaved simple fractional derivative called the conformable fractional derivative, depending just on the basic limit definition of the derivative (see also [22, 14]). Namely, for a function $f : [0, +\infty) \to \mathbb{R}$ the conformable fractional(if the conformable fractional derivative of f of order α exists, then we simply say f is α-differentiable) derivative of order $0 < \alpha \leqslant 1$ of f at $t > 0$ was defined by

$$T_\alpha f(t) = \lim_{\varepsilon\to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

If f is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t\to 0+} T_\alpha f(t)$ exists, then define $T_\alpha f(0) = \lim_{t\to 0+} T_\alpha f(t)$.

As a consequence of the above definition, the authors proved that many of the previous issues are overcome. In the derivative definition above given by authors the angle derivative tends to the standard derivative as $\alpha \to 1$, that is $T_\alpha f(t) \to f'(t)$ and therefore the angle of the tangent line is preserved. Which is not the case with our derivative.

In the following section, we give our new definition of a non-conformable fractional derivative of a function at a point t and obtain several results that are close resemblance of those found in classical calculus. The purpose of this paper is to introduce a new definition of fractional non-conformable derivative as a natural extension of the well-known definition of derivative of a function in a point, in particular show that the inadequacies 1) to 4) are overcome. Thing to note is that our derivative gives a closer approximation to the ordinary derivative than the one investigated by Miguel [23], which can be seen from the graph given in the example section. In future works we will complete the study of this new non-conformable fractional derivative and apply it to more physical problems as to show the validity of our results in a practical manner.

2. Main results.

We will define the non-conformable fractional derivative in the following way.

Definition 2.1. *Given a function $f : (0, +\infty) \to \mathbb{R}$. Then the $\mathfrak{D}$-derivative of f of order α is defined by*

$${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(t) = \lim_{\varepsilon\to 0} \frac{f(t + \frac{\varepsilon}{\ln(e+t^{-\alpha})}) - f(t)}{\varepsilon},$$

for all $t > 0, \alpha \in (0, 1)$. If f is α-differentiable in some $(0, a)$, and $\lim_{t\to 0^+} {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(t)$ exists, then define

$${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(0) = \lim_{t\to 0^+} {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(t).$$

As a consequence, we obtain the following result similar to the one in the classical analysis.

Theorem 2.1. *If a function $f : (0, +\infty) \to \mathbb{R}$ is $\mathfrak{D}$-differentiable at $t_0 > 0$,$\alpha \in (0, 1)$ then f is continuous at t_0.*

Proof: Let t_0 be an arbitrary point greater than zero. Since

$$f\left(t_0 + \frac{\varepsilon}{\ln(e + t_0^{-\alpha})}\right) - f(t_0) = \frac{f(t_0 + \frac{\varepsilon}{\ln(e+t_0^{-\alpha})}) - f(t_0)}{\varepsilon}\varepsilon.$$

Then

$$\lim_{\varepsilon\to 0}\left(f(t_0 + \frac{\varepsilon}{\ln(e + t_0^{-\alpha})}) - f(t_0)\right) = \lim_{\varepsilon\to 0} \frac{f(t_0 + \frac{\varepsilon}{\ln(e+t_0^{-\alpha})}) - f(t_0)}{\varepsilon} \cdot \lim_{\varepsilon\to 0} \varepsilon.$$

Let $k = \frac{\varepsilon}{\ln(e+t_0^{-\alpha})}$, then $k \to 0$ if $\varepsilon \to 0$, so we have

$$\lim_{\varepsilon\to 0} \frac{f(t_0 + \frac{\varepsilon}{\ln(e+t_0^{-\alpha})}) - f(t_0)}{\varepsilon} \cdot \lim_{\varepsilon\to 0} \varepsilon = \frac{1}{\ln(e+t_0^{-\alpha})} \lim_{k\to 0} \frac{f(t_0+k) - f(t_0)}{k} \cdot \lim_{\varepsilon\to 0} \varepsilon = 0.$$

□

Theorem 2.2. *Let f and g be $\mathfrak{D}$-differentiable at a point $t > 0$ and $\alpha \in (0, 1]$. Then*
a) ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(af + bg)(t) = a\,{}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(f)(t) + b\,{}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(g)(t)$.
b) ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(t^p) = \frac{1}{\ln(e+t^{-\alpha})} pt^{p-1}, p \in \mathbb{R}$.
c) ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(\lambda) = 0, \lambda \in \mathbb{R}$.
d) ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(fg)(t) = f\,{}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(g)(t) + g\,{}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(f)(t)$.
e) ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}\left(\frac{f}{g}\right)(t) = \frac{g\,{}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(f)(t) - f\,{}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(g)(t)}{g^2(t)}$.
h) $If\ in\ addition\ f\ is\ differentiable\ then\ {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(f) = \frac{1}{\ln(e+t^{-\alpha})} f'(t)$.

Proof: Starting with a). Let $H(t) = (af + bg)(t)$ then

$${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} H(t) = \lim_{\varepsilon\to 0} \frac{H(t + \frac{\varepsilon}{\ln(e+t^{-\alpha})}) - H(t)}{\varepsilon}$$

and from this we have the desired result.

b)It is sufficient to expand $(t + \frac{\varepsilon}{\ln(e+t^{-\alpha})})^p$ in power series, in this way we have two cases. First case is when $p \in \mathbb{N}$. Then we have the following

$$\begin{aligned}
{}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(t^p) &= \lim_{\varepsilon\to 0} \frac{\left(t + \frac{\varepsilon}{\ln(e+t^{-\alpha})}\right)^n - t^n}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} t^{n-k} \left(\frac{\varepsilon}{\ln(e+t^{-\alpha})}\right)^k - t^n}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} t^{n-k} \left(\frac{\varepsilon}{\ln(e+t^{-\alpha})}\right)^k - t^n}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{t^n + nt^{n-1}\frac{\varepsilon}{\ln(e+t^{-\alpha})} + \binom{n}{2} t^{n-2} \left(\frac{\varepsilon}{\ln(e+t^{-\alpha})}\right)^2 + ... - t^n}{\varepsilon} \\
&= \frac{nt^{n-1}}{\ln(e+t^{-\alpha})}.
\end{aligned}$$

Similar approach gives us the same result when $p \in \mathbb{R}$.
c) Easily follows from the definition.
d)From the definition we have

$$\begin{aligned}
{}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(fg)(t) &= \lim_{\varepsilon\to 0} \frac{f\left(t + \frac{\varepsilon}{\ln(e+t^{-\alpha})}\right) g\left(t + \frac{\varepsilon}{\ln(e+t^{-\alpha})}\right) - f(t)g(t)}{\varepsilon} \\
&\quad + \lim_{\varepsilon\to 0} \frac{f(t)g\left(t + \frac{\varepsilon}{\ln(e+t^{-\alpha})}\right) - f(t)g\left(t + \frac{\varepsilon}{\ln(e+t^{-\alpha})}\right)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\left(f\left(t + \frac{\varepsilon}{\ln(e+t^{-\alpha})}\right) - f(t)\right) g\left(t + \frac{\varepsilon}{\ln(e+t^{-\alpha})}\right)}{\varepsilon} + \lim_{\varepsilon\to 0} \frac{\left((g\left(t + \frac{\varepsilon}{\ln(e+t^{-\alpha})}\right) - g(t)\right) f(t)}{\varepsilon} \\
&= f\,{}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(g)(t) + g\,{}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(f)(t).
\end{aligned}$$

e)

$${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}\left(\frac{f}{g}\right)(t) = \lim_{\varepsilon\to 0} \frac{\frac{f\left(t+\frac{\varepsilon}{\ln(e+t^{-\alpha})}\right)}{g\left(t+\frac{\varepsilon}{\ln(e+t^{-\alpha})}\right)} - \frac{f(t)}{g(t)}}{\varepsilon}.$$

Let us focus on the numerator. Multiplying the second term with $\frac{g(t+\frac{\varepsilon}{\ln(e+t^{-\alpha})})}{g(t+\frac{\varepsilon}{\ln(e+t^{-\alpha})})}$ and then adding the two terms, we get

$$\frac{f(t + \frac{\varepsilon}{\ln(e+t^{-\alpha})})g(t) - f(t)g(t + \frac{\varepsilon}{\ln(e+t^{-\alpha})})}{g(t)g(t + \frac{\varepsilon}{\ln(e+t^{-\alpha})})} - \frac{f(t)g(t)}{g(t)g(t + \frac{\varepsilon}{\ln(e+t^{-\alpha})})} + \frac{f(t)g(t)}{g(t)g(t + \frac{\varepsilon}{\ln(e+t^{-\alpha})})}$$

$$= \frac{\left(f(t+\frac{\varepsilon}{\ln(e+t^{-\alpha})}) - f(t)\right) g(t)}{g(t)g(t+\frac{\varepsilon}{\ln(e+t^{-\alpha})})} - \frac{\left(g(t+\frac{\varepsilon}{\ln(e+t^{-\alpha})}) - g(t)\right) f(t)}{g(t)g(t+\frac{\varepsilon}{\ln(e+t^{-\alpha})})}.$$

Which when put back into our expression with limits, we get the desired result, namely

$$^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}\left(\frac{f}{g}\right)(t) = \frac{g\ ^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(f)(t) - f\ ^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(g)(t)}{g^2(t)}.$$

h) From the definition, we have

$$^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(f) = \lim_{\varepsilon\to 0} \frac{f(t+\frac{\varepsilon}{\ln(e+t^{-\alpha})}) - f(t)}{\varepsilon}.$$

Introducing a substitution $k = \frac{\varepsilon}{\ln(e+t^{-\alpha})}$ we get

$$\lim_{\varepsilon\to 0} \frac{f(t+\frac{\varepsilon}{\ln(e+t^{-\alpha})}) - f(t)}{\varepsilon} = \frac{1}{\ln(e+t^{-\alpha})}\lim_{k\to 0}\frac{f(t+k)-f(t)}{k} = \frac{1}{\ln(e+t^{-\alpha})} f'(t).$$

□

Corollary 2.1. *If* $^{\alpha}\mathfrak{D}_{\frac{1}{\ln}} f(t)$ *exists for* $t > 0$ *then* f *is differentiable at* t *and*

$$f'(t) = \ln(e+t^{-\alpha})\ ^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(t).$$

Corollary 2.2. *Let* α, β *be positive constants such that* $0 < \alpha, \beta \leqslant 1$ *and* f *be a function (non-constant) twice differentiable on an interval* $(0, +\infty)$*. Then*

$$^{\alpha+\beta}\mathfrak{D}_{\frac{1}{\ln}} f(t) \neq\ ^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}\left(^{\beta}\mathfrak{D}_{\frac{1}{\ln}} f(t)\right).$$

Proof: Follows easily from definition.

Theorem 2.3. *We have the following according to the Theorem 2, part h.*

i) $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(1) = 0.$

ii) $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(e^{ct}) = \frac{ce^{ct}}{\ln(e+t^{-\alpha})}.$

iii) $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(\sin(bt)) = \frac{b\cos(bt)}{\ln(e+t^{-\alpha})}.$

iv) $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(\cos(bt)) = -\frac{b\sin(bt)}{\ln(e+t^{-\alpha})}.$

Now we will present the equivalent result for $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}$, of the well known chain rule of classical calculus, and therefore we overcome the problem 4(which is the problem of fractional derivatives not satisfying the Chain rule).

Theorem 2.4. *Let* $\alpha \in (0,1]$*,* $g\ ^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}$ *at* $t > 0$ *and* f *differentiable at* $g(t)$ *then* $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(f\circ g)(t) = f'(g(t))\ ^{\alpha}D_{\frac{1}{\ln}} g(t).$

Proof: We prove the result following a standard limit-approach. First case, if the function g is constant in a neighborhood of $a > 0$ then $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(f\circ g)(t) = 0$. If g is not constant in a neighbourhood of $a > 0$ we can find a $t_0 > 0$ such that $g(x_1) \neq g(x_2)$ for any $x_1, x_2 \in (a - t_0, a + t_0)$. Now since g is continuous at a, for ε small, we have

$$\begin{aligned} ^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(f\circ g)(a) &= \lim_{\varepsilon\to 0}\frac{f(g(t+\frac{\varepsilon}{\ln(e+a^{-\alpha})})) - f(g(a))}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{f(g(t+\frac{\varepsilon}{\ln(e+a^{-\alpha})})) - f(g(a))}{g(a+\frac{\varepsilon}{\ln(e+a^{-\alpha})}) - g(a)}\frac{g(a+\frac{\varepsilon}{\ln(e+a^{-\alpha})}) - g(a)}{\varepsilon}. \end{aligned}$$

Introducing a substitution $k = g(a+\frac{\varepsilon}{\ln(e+a^{-\alpha})}) - g(a)$ in the first fraction and realizing that the second part is part h of Theorem 2, we get that

$$(f\circ g)(t) = f'(g(t))\ ^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} g(t).$$

□

Theorem 2.5. *Let* $f, h : [0, +\infty) \to \mathbb{R}$ *be functions such that* $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}$ *exists for* $t > 0$*, if* f *is differentiable on* $(0, +\infty)$ *and* $^{\alpha}\mathfrak{D}_{\frac{1}{\ln}} f(t) = \frac{1}{\ln(e+t^{-\alpha})} h(t)$*. Then* $h(t) = f'(t)$ *for all* $t > 0$*.* Following the same procedure of the ordinary calculus, we can prove the following result.

Theorem 2.6. *Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a given function that satisfies*
i) f is continuous on $[a, b]$,
ii) f is $\mathfrak{D}$ differentiable for some $\alpha \in (0, 1)$.
Then, we have that if ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(t) \geqslant 0$ $(\leqslant 0)$, then f is a nondecreasing (increasing) function.

Theorem 2.7. *(Racetrack Type Principle). Let $a > 0$ and $f, g : [a, b] \to \mathbb{R}$ be given functions satisfying*

i) f and g are continuous on $[a, b]$

ii) f and g are ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}$ differentiable for some $\alpha \in (0, 1)$,

iii) ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(t) \geqslant {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} g(t)$ for all $t \in (a, b)$.

Then we have the following:
I) If $f(a) = g(a)$, then $f(t) \geqslant g(t)$ for all $t \in (a, b)$.
II) If $f(b) = g(b)$, then $f(t) \leqslant g(t)$ for all $t \in (a, b)$.

Proof: Consider the auxiliary function $h(t) = f(t) - g(t)$. Then h is continuous on $[a, b]$ and ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}$-differentiable for some $\alpha \in (0, 1)$. From here, we obtain that ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} h(t) \geqslant 0$ for all $t \in (a, b)$, so by the previous theorem, h is a nonincreasing function. Hence for any $t \in (a, b)$ we have $h(a) \leqslant h(t)$ which gives us $g(t) \leqslant f(t)$. Second part is similar. □

Now we will discuss the occurrence of local maxima and local minima of a function. In fact, these points are crucial to many questions related to application problems.

Definition 2.2. *A function f is said to have a local maximum at c iff there exists an interval I around c such that $f(c) \geqslant f(x)$ for all $x \in \mathbb{I}$. Analogously, f is said to have a local minimum at c iff there exists an interval I around c such that $f(c) \leqslant f(x)$ for all $x \in \mathbb{I}$. A local extremum is a local maximum or a local minimum.*

Theorem 2.8. *(Rolle's Theorem). Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a given function that satisfies*

- *$f \in C[a, b]$*
- *f is ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}$-differentiable on (a, b) for some $\alpha \in (0, 1)$,*
- *$f(a) = f(b)$*

Then, there exists $\zeta \in (a, b)$ such that ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(\zeta) = 0$.

Proof: Since f is continuous in $[a, b]$ and $f(a) = f(b)$, there is $\zeta \in (a, b)$, at least one, which is a point of local extreme. Let us assume that point ζ is the point of local maximum. On the other hand, since f is ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}$-differentiable in (a, b) for some α, we have

$$\begin{aligned} {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(\zeta) = {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(\zeta^+) = \lim_{\varepsilon \to 0^+} \frac{f(\zeta + \frac{\varepsilon}{\ln(e+\zeta^{-\alpha})}) - f(\zeta)}{\varepsilon} &= {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(\zeta^-) \\ = \lim_{\varepsilon \to 0^-} \frac{f(\zeta + \frac{\varepsilon}{\ln(e+\zeta^{-\alpha})}) - f(\zeta)}{\varepsilon}&, \end{aligned}$$

but ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(\zeta^+)$ and ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(\zeta^-)$ have opposite signs, therefore ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(\zeta) = 0$. If ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(\zeta^+)$ and ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(\zeta^-)$ have the same sign then as $f(a) = f(b)$, we have that f is constant and the result is trivially followed. This concludes the proof. □

Theorem 2.9. *(Mean Value Theorem). Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a function that satisfies:*

- *f is continuous in $[a, b]$,*
- *f is ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}$-differentiable on (a, b) for some $\alpha \in (0, 1]$.*

Then there exists $\zeta \in (a, b)$ such that

$${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}[f(\zeta)] = \frac{f(b) - f(a)}{b - a} \frac{1}{\ln(e + \zeta^{-\alpha})}.$$

Proof: Consider the function

$$g(t) = f(t) - f(a) - \left[\frac{f(b) - f(a)}{b - a}\right](t - a).$$

The function g satisfies all the conditions of the Rolle's Theorem, and therefore there exists $\zeta \in (a, b)$ such that ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} g(\zeta) = 0$.

$${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} g(t) = {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} (f(t) - f(a)) - \frac{f(b) - f(a)}{b - a} {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} (t - a),$$

and from here it follows that

$$^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} g(\zeta) = {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(\zeta) - \frac{f(b)-f(a)}{b-a}\frac{1}{\ln(e+\zeta^{-\alpha})} = 0,$$

from where

$$^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} [f(\zeta)] = \frac{f(b)-f(a)}{b-a}\frac{1}{\ln(e+\zeta^{-\alpha})}.$$

This concludes the proof. □

Theorem 2.10. *Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a given function that satisfies*
$i) f \; is \; continuous \; on \; [a,b],$
$ii) f \; is \; {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} \; differentiable \; for \; some \; \alpha \in (0,1).$
If $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(t) = 0$ for all $t \in (a,b)$, then f is constant on $[a,b]$.

Proof: Let us consider any interval contained in (a,b). For example let us consider $a < c < d < b$. Using Rolle's Theorem on the interval $(c,d) \subset (a,b)$, we get that there exists $\zeta \in (c,d)$ such that $f'(\zeta) = \frac{f(d)-f(c)}{d-c}\frac{1}{\ln(e+t^{-\alpha})}$ but since the derivative is zero across the entire interval (a,b) it must follow that $f(d) = f(c)$. Applying the same procedure on any sub-interval of (a,b) we get that $f(a) = f(b) = k$ for some k, therefore f is constant. □

Theorem 2.11. *Let $f : [a,b] \to \mathbb{R}$ be $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}$-differentiable for some $\alpha \in (0,1)$. If*
i) $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(t)$ is bounded on $[a,b]$ with $a > 0$, then f is uniformly continuous on $[a,b]$, and hence f is bounded.
ii) $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(t)$ is bounded on $[a,b]$ and continuous at $a > 0$, then f is continuous on $[a,b]$, and hence f is bounded.

Proof: Since $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(t)$ is bounded on $[a,b]$ for all $x,y \in (a,b)$. We have from Lagrange's Theorem

$$^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} (f(\zeta)) = \frac{f(y)-f(x)}{y-x}\frac{1}{\ln(e+\zeta^{-\alpha})}.$$

A function is uniformly continuous if for all $\varepsilon > 0$ exists $\delta > 0$ such that for all $x,y \in [a,b]$ $|x-y| < \delta$ yields $|f(y)-f(x)| < \varepsilon$. In our case we can see that $f(y)-f(x) = {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(\zeta)\ln(e+\zeta^{-\alpha})(y-x)$, and since it is bounded, using it in our case we get that $|f(y)-f(x)| < M \cdot |y-x|\ln(e+\zeta^{-\alpha}) < M^* M'^* \delta$. Since $\ln(e+\zeta^{-\alpha})$ is a finite number, it can be bounded by some M'. Choosing $\delta = \frac{\varepsilon}{M \cdot M'}$ we get that $|f(y)-f(x)| < M \cdot M \cdot \frac{\varepsilon}{M \cdot M'} = \varepsilon$ and therefore f is uniformly continuous.
Second claim follows similarly. □

Theorem 2.12. *(Extended Mean Value Theorem). Let $\alpha \in (0,1]$ and $a > 0$. If $f,g : [a,b] \to \mathbb{R}$ are functions that satisfy*
$i) f,g \; are \; continuous \; in \; [a,b],$
$ii) f,g \; are \; {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} \; differentiable \; on \; (a,b), \; for \; some \; \alpha \in (0,1],$
$iii) \; {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} g(t) \neq 0 \; for \; all \; t \in (a,b).$ *Then, there exists $\zeta \in (a,b)$ such that*

$$\frac{^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(\zeta)}{^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} g(\zeta)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof: Consider the function

$$h(t) = f(t) - f(a) - \left[\frac{f(b)-f(a)}{g(b)-g(a)}\right](g(t)-g(a)).$$

Then the auxiliary function h satisfies the assumptions of Rolle's Theorem. Thus, there exists $\zeta \in (a,b)$ such that $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(\zeta) = 0$ for some $\alpha \in (0,1)$. From here, we have

$$^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} h(\zeta) = {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(\zeta) - \left[\frac{f(b)-f(a)}{g(b)-g(a)}\right] {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} g(\zeta) = 0.$$

From which we obtain the following

$$\frac{^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(\zeta)}{^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} g(\zeta)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

□

Integral of the $\mathfrak{D}$ type is defined as follows.

Definition 2.3. *Let $\alpha \in (0,1]$ and $a \geqslant 0$. Let f be defined on $(a,t]$. Then the fractional integral of f is*

$$I_{\alpha}^{a}(f)(t) = \int_{a}^{t} \ln(e + s^{-\alpha}) f(s) ds.$$

Theorem 2.13. *If $f : [a,+\infty) \to \mathbb{R}$ is any continuous function on the domain I_a and $0 < \alpha \leqslant 1$. Then for $t > a$ we have*

$${}^{(\alpha)}\mathfrak{D}^{a}_{\frac{1}{\ln}} I_{\alpha}^{a} f(t) = f(t)$$

Proof:

$$\begin{aligned}
{}^{(\alpha)}\mathfrak{D}^{a}_{\frac{1}{\ln}} I_{\alpha}^{a} f(t) &= \frac{1}{\ln(e+t^{-\alpha})} \frac{d}{dt} I_{a}^{\alpha} f(t) = \frac{1}{\ln(e+t^{-\alpha})} \frac{d}{dt} \int_{a}^{t} \ln(e+s^{-\alpha}) f(s) ds \\
&= \frac{1}{\ln(e+t^{-\alpha})} \ln(e+t^{-\alpha}) f(t) = f(t),
\end{aligned}$$

since

$$\frac{d}{dt} \int_{a}^{t} \ln(e+s^{-\alpha}) f(s) ds = \ln(e+t^{-\alpha}) f(t).$$

□

Our last result is a generalization of one of the most important, and oldest, theorems of mathematical analysis, the Taylor series, which establishes under what conditions a function f can be approximated in a neighborhood of a point $t = a$, by a linear combination of polynomials.

Theorem 2.14. *Let $f : [a,b] \to \mathbb{R}$ be n times continuously ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}$-differentiable and $n+1$ times ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}$-differentiable in (a,b) and let $c \in (a,b)$. Then, for each $t \in (a,b)$ with $t \neq c$, there exists a point $\zeta \in (c,t)$, respectively (t,c), such that*

$$f(t) = \sum_{j=0}^{n} \frac{f^{j}(c)}{j!} (t-c)^{j} + R_{n+1}(f)$$

with

$$f_{j+1}(\cdot) = {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(\cdot) \ln(e + (\cdot)^{-\alpha}), f^{0}(\cdot) = f(\cdot)$$

holds, and the remainder can be written as

$$R_{n+1}(f) = \frac{f^{n+1}(\zeta)}{(n+1)!} (t-a)^{n+1}.$$

Proof: Define the following function

$$g(y) = f(t) - f(y) - f^{(1)}(t-y) - ... - f^{n}(y) \frac{(t-y)^{n}}{n!} - \frac{M}{(n+1)!} (t-y)^{n+1},$$

where M is chosen such that $g(c) = 0$.Using $g(c) = g(t) = 0$, Theorem 2.6 allows us to affirm that there exist $\zeta \in (c,t)$, respectively (t,c), with ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} g(\zeta) = 0$,

$$-\,{}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f^{n}(y) \frac{(t-y)^{n}}{n!} - \frac{M}{n!}(-1) \frac{(t-y)^{n}}{\ln(e+y^{-\alpha})} = 0.$$

From which we get the following

$$M = f^{n+1}(\zeta).$$

Setting $y = c$ in the function g, we obtain the original expression. □

3. Application of the fractional derivative. Application of our fractional derivative can be seen in the use of the MVTs as well as in solving differential equations. We begin with an application of the MVT.

Example 1
Let f be a continuous function on $[a,b]$ and differentiable on (a,b), show that the solution of the following differential equation exists

$$\frac{f'(t)}{\ln^2(e+t^{-\alpha})} + \frac{\alpha f(t)}{\ln^3(e+t^{-\alpha})(et^{\alpha+1}+t)} = 0.$$

such that $f(a) = f(b) = 0$ with $0 < a < b$ and $0 < \alpha < 1$.

Proof: Using the quotient rule (Theorem 2, h)) we recognize that the differential equation can be written in the following form

$$^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}\left[\frac{f(t)}{\ln(e+t^{-\alpha})}\right] = 0.$$

Taking $h(t) = \frac{f(t)}{\ln(e+t^{-\alpha})}$ we can see that the function satisfies Rolle's Theorem conditions, therefore there exists $\zeta \in (a,b)$ such that $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} h(\zeta) = 0$.

Example 2
Let us consider a linear first-order differential equation where $t > 0$

$$\frac{f'(t)e^t}{\ln(e+t^{-\alpha})} + \frac{f(t)e^t}{\ln(e+t^{-\alpha})} = \frac{\beta \cdot t^{\beta-1}}{\ln(e+t^{-\alpha})}.$$

Proof: Using the product rule (Theorem 2) we can rewrite the above equation as follows

$$^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}[f(t)e^t] = {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}[t^\beta]$$

We know that $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(f)(t) = \frac{1}{\ln(e+t^{-\alpha})} f'(t)$. So we get that

$$f(t) \cdot e^t = t^\beta + c$$

$$f(t) = (t^\beta + c)e^{-t}.$$

Example 3
Let us consider a linear first-order differential equation where $t > 0, \beta \in \mathbb{R}$

$$f'(t) - \frac{\alpha f(t) t^{-\alpha-1}}{(t^{-\alpha}+e)\ln(e+t^{-\alpha})} = \frac{\beta t^{\beta-1}}{\ln(e+t^{-\alpha})}.$$

Proof: It is clear that this differential equation can be written in the following way

$$^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(f(t)\ln(e+t^{-\alpha})) = {}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}}(t^\beta).$$

Which gives us

$$f(t) = \frac{t^\beta + c}{\ln(e+t^{-\alpha})}.$$

3.1. Applications to Physics..

3.1.1. Falling bodies. This problem considers the fall of a body of mass m, starting from rest, under the action of gravity. Suppose that the chosen reference system has as its origin the starting point (rest of the body) at a height A from the floor at the moment the fall begins, that is, at $t = 0$. The downward movement will be chosen as positive. At any point P on its trajectory, the distance traveled will be the function y dependent on time t, consequently, using ordinary derivatives, the instantaneous velocity and acceleration snapshot, respectively, are given by

$$v(t) = \frac{dy(t)}{dt}, a = \frac{dv(t)}{dt} = \frac{d^2y(t)}{dt^2}.$$

According to Newton's Law, we have $F = mg$, $\frac{dv(t)}{dt} = g$.

Then this problem is modeled by the following differential equation with initial condition as follows

$$\frac{dv(t)}{dt} = g, v(0) = 0, y(0) = A.$$

Now consider the problem with the conformable fractional derivative defined in Definition 1. We will set $\alpha = \frac{1}{2}$, reason for this is because the integral of $\ln(e + t^{-\alpha})$ for arbitrary α is a hypergeometric function. In the fractional setting, we have the following conditions

$$^{\frac{1}{2}}\mathfrak{D}_{\frac{1}{\ln}} v(t) = g, v(0) = 0, y(0) = A.$$

Using Theorem 2.2, part h) we get

$$\frac{v'(t)}{\ln(e + t^{-\frac{1}{2}})} = g, v'(t) = g\ln(e + t^{-\frac{1}{2}}).$$

Integrating, we have

$$v(t) = \frac{g\left(2e\sqrt{t} + 2\left(e^2t - 1\right)\log\left(\frac{1}{\sqrt{t}} + e\right) - \log(t)\right)}{2e^2} + c.$$

Using the initial condition, namely $v(0) = 0$ we find that the constant is zero. Similarly we obtain the following

$$y'(t) = \ln(e + t^{-\frac{1}{2}})\left(\frac{g\left(2e\sqrt{t} + 2\left(e^2t - 1\right)\log\left(\frac{1}{\sqrt{t}} + e\right) - \log(t)\right)}{2e^2}\right).$$

From which we get

$$y(t) = \frac{g\left(-2e\sqrt{t} + \log(t) + \left(2 - 2e^2t\right)\log\left(\frac{1}{\sqrt{t}} + e\right)\right)^2}{8e^4} + L.$$

Using the initial conditions, we find that $L = A$. Therefore all together we have

$$y(t) = \frac{g\left(-2e\sqrt{t} + \log(t) + \left(2 - 2e^2t\right)\log\left(\frac{1}{\sqrt{t}} + e\right)\right)^2}{8e^4} + A.$$

Finding the time when the body hits the ground, we set $y(t) = 0$ and deduce t. Comparing how our derivative defined by Definition 1 behaves in comparison to the normal derivative and to the derivative investigated by Miguel is given .

4. Conclusions. In this article, a brief introduction of the new conformable derivative is made, proving its basic properties as a differential operator and from these we apply the said fractional derivative in solving differential equations and in a classic problem of physics, the problem of falling bodies. We find original solutions to these problems and the comparison of our derivative with the derivative investigated by Miguel and the standard derivative is given on a graph (Figure 3.1). Clearly from the graph we can see that the derivative defined by Definition 1 is closer to the ordinary derivative. Thing to note here is that our derivative is better in approximating when $t > 1$, when $t \in (0, 1)$ the derivative of Miguel gives a better result, as our derivative tends to $+\infty$ as $t \to 0$. It is natural to ask what relation has $^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(t)$ to the

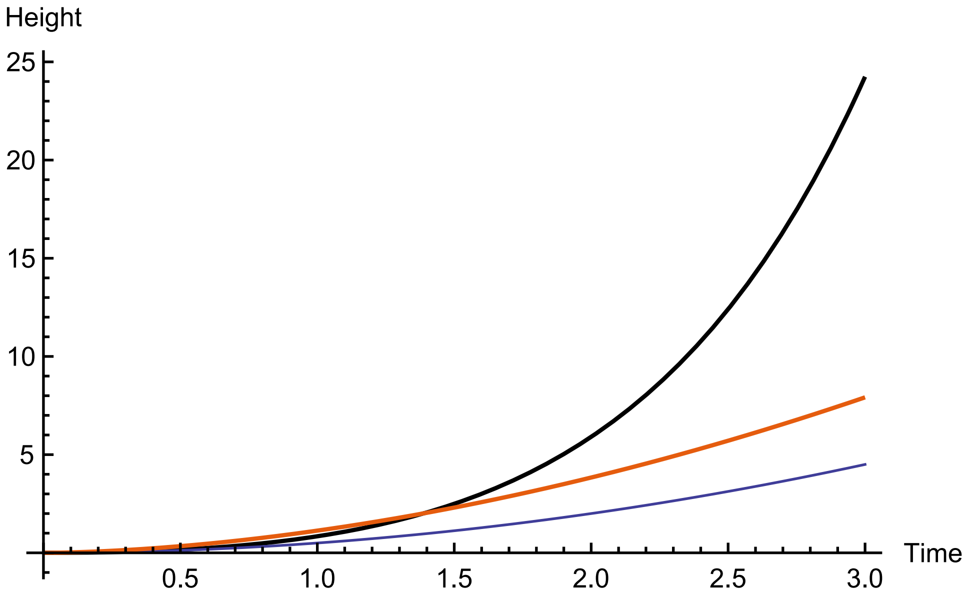

Figure 3.1: The picture shows the black line representing the Miguel's derivative, the blue line representing the ordinary derivative and an orange line which is the solution of the initial problem using derivative defined by Definition 2.1.

derivative defined by Abdejjawad in [1]. In the case that f is a derivable function, when $t = 0$ we have $T_\alpha f(0) = 0$ while ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(t) \to +\infty$, on the other hand, if $t \to +\infty$ we obtain that $T_\alpha f(t) \to +\infty$ while ${}^{(\alpha)}\mathfrak{D}_{\frac{1}{\ln}} f(t) \to f'(t)$, i.e., our derivative behaves asymptotically like a classical derivative, which ensures that if f is derivable, the asymptotic properties of f are inherited, which is of vital importance in the Qualitative Theory of Differential Equations.

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