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REVIEW

Fourier analysis with R

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Abstract

Fourier analysis is currently one of the important parts of mathematical analysis. Fourier analysis methods are used in physics, electrical engineering, and other sciences to solve various problems. In this article we will give the main concepts of Fourier analysis and its applications in spectral analysis in solving problems using R, as shown in the examples.

Keywords . Fourier series, fast Fourier transform, flexible form of Fourier.

1. Introduction. This work is devoted to the theory of Fourier analisis [1, 2, 3, 4, 5, 6] in R. Although the treatment can be extensive, the exposition of the theory here will be concise, but sufficient for its application to problems of applied mathematics [7] and mathematical physics [8][9].

The Fourier theory [10, 11, 12]of trigonometric series is of great practical importance because certain types of discontinuous functions which cannot be expanded in power series can be expanded in Fourier series [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. More importantly, a wide class of problems in physics and engineering possesses periodic phenomena and, as a consequence, Fourier's trigonometric series become an indispensable tool in the analysis of these problems.

2. Spectral analysis. The basic idea of spectral analysis is that every stationary stochastic process admits a single decomposition of its variance, in the contribution that harmonics of different frequencies make to it. A harmonic of frequency ω is a function of the form

$$a_\omega \cos(\omega t) + b_\omega \sin(\omega t).$$

In harmonic analysis, time series are not considered continuous functions as such, but are obtained from a sum of n cycles with a given amplitude and period, or what is the same n of different harmonics

$$x(t) = \sum_{i=1}^{n} \left[a_i \cos(\omega_i t) + b_i \sin(\omega_i t)\right], \tag{2.1}$$

where $0 < \omega_1 < \omega_2 < ... < \omega_n \leq \pi$. Being a_i and b_i random variables with [25]

$$E(a_i) = E(b_i) = 0, \tag{2.2}$$

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$$E\left(a_i a_j\right) = E\left(b_i b_j\right) = \begin{cases} \sigma^2, & i = j, \\ 0, & i \neq j, \end{cases} \tag{2.3}$$

$$E\left(a_i b_j\right) = 0.\ \forall i, j \tag{2.4}$$

In this type of process, the autocovariance function $\gamma(\tau)$ is obtained through the equation:

$$\gamma(\tau) = \sum_{i=1}^{n} \sigma_i^2 \cos\left(\omega_i \tau\right), \tag{2.5}$$

where σ_i is the variance of the ith harmonic, so that $\gamma(0) = \sum_{i=1}^{n} \sigma_i^2$ shows that the total variance of process is the sum of the variances of each harmonic.

3. Fourier Coefficients. A Fourier series is an infinite series that converges pointwise to a continuous and periodic function, that is,

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(n\omega_0 t\right) + b_n \sin\left(n\omega_0 t\right)\right], \tag{3.1}$$

with

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(t)\, dt, \tag{3.2}$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos\left(n\omega_0 t\right) dt, \tag{3.3}$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin\left(n\omega_0 t\right) dt, \tag{3.4}$$

where $\omega_0 = \frac{2\pi}{T}$ is called the fundamental frequency; a_n and b_n are called Fourier coefficients. The coefficients of a fourier series can be calculated thanks to the orthogonality of the sine and cosine functions.

An alternative way to present a Fourier series is

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(n\omega_0 t - \theta_n\right), \tag{3.5}$$

where $C_0 = a_0/2$, $C_n = \sqrt{a_n^2 + b_n^2}$ and $\theta_n = \arctan\left(\frac{b_n}{a_n}\right)$.

4. Complex form of Fourier series. We consider the Fourier series for a periodic function $f(t)$, with period $T = \frac{2\pi}{\omega_0}$

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(n\omega_0 t\right) + b_n \sin\left(n\omega_0 t\right)\right].$$

An alternative way using Euler's formulas would look like:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \tag{4.1}$$

expression that is known as complex fourier series, and its coefficients c_n can be obtained from the coefficients a_n, b_n, or else:

$$c_n = \frac{1}{T} \int_{0}^{T} f(t)\, e^{-in\omega_0 t} dt. \tag{4.2}$$

5. Fourier transform. The Fourier transform, $F(\omega)$, is defined for a continuous function of real variable, $f(t)$, by the following formula:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i\omega t} dt. \tag{5.1}$$

The Fourier transform is a complex function with a real part and an imaginary part, that is:

$$F(\omega) = R(\omega) + I(\omega), \tag{5.2}$$

where $R(\omega)$ is the real part and $I(\omega)$ is the imaginary part.

The graphical representation of the magnitude function $|F(\omega)|$ is called the Fourier Spectrum and is expressed in terms of the modulus of the complex number:

$$|F(\omega)| = \sqrt{R^2(\omega) + I^2(\omega)}, \tag{5.3}$$

and the square of $|F(\omega)|^2$ is called the Power Spectrum. The plot of squared modules versus frequency is the periodogram or empirical spectrum of the sequence $f(t)$.

The periodogram collects the contribution that each harmonic has when it comes to explaining the variance of each series, and each harmonic is characterized by the frequency in which the cycles take place. Cycles that have a high period (from when one peak occurs to the next peak) will have a low frequency and viceversa.

6. Periodogram calculation. Consider the time series X_t for which we have a discrete and finite set of T observations, generated by a random process $x(t)$. Since we are looking for a representation of X_t that fits T observations, we fit the data to a trigonometric polygon that resembles a Fourier series, choosing $\omega_i = \frac{2\pi i}{T}$, that is,

$$X_t = \frac{1}{2}a_0 + \sum_{i=1}^{n}\left[a_i \cos\left(\frac{2\pi i}{T}t\right) + b_i \sin\left(\frac{2\pi i}{T}t\right)\right], \tag{6.1}$$

or[1]

$$x_t = X_t - \widehat{\mu} = \sum_{i=1}^{n}\left[a_i \cos\left(\frac{2\pi i}{T}t\right) + b_i \sin\left(\frac{2\pi i}{T}t\right)\right]. \tag{6.2}$$

The usual way to obtain the periodogram is to estimate by least squares the coefficients a_i and b_i for each $k = T/2$ harmonics if the number of observations is even T, or $k = (T-1)/2$ if odd, in the model specified as follows:

$$x = a\cos\omega t + b\sin\omega t + v_t. \tag{6.3}$$

Where x_t would be the harmonic series; $\omega = \omega_p = \frac{2\pi p}{T}$; T is the size of the series and coincides with the period with the largest cycle that can be estimated with the size of the series; p indicates the harmonic order of the $T/2$ cycles; v_t is an unexplained residual that can be considered irrelevant (deterministic case) or that verifies the classical properties of the econometric model perturbation. The periodogram or estimator of the spectrum would then be obtained from the representation of

$$I(\omega_i) = \frac{1}{4\pi}\left[\frac{T}{2}\left(a_p^2 + b_p^2\right)\right]$$

against the p harmonics, while the variance contribution for each harmonic would be $\frac{1}{2}\left(a_p^2 + b_p^2\right)$. If a time series of empirical cycles presents in its periodogram a few cycles that explain a significant percentage of its variance, the theoretical cycle of time series can be obtained from the $|\omega_i|$ and of the harmonics corresponding to cycles.

[1]Note that $a_0/2 = \frac{1}{T}\sum_{t=1}^{T} X_t$, which implies that $a_0 = \frac{2}{T}\sum_{t=1}^{T} X_t$.

Example 1 (In R). PIB ¡- c(
265269.03384951, 286886.212605139,
309230.943247666, 326605.313346147,
328375.812358207, 339224.607770332,
348855.309254073, 353957.416081134,
354106.129594857, 358711.564921997,
358078.846631934,363686.473128754,
371758.549723152, 377213.223988016,
387066.612651189, 399452.752857041,
421987.447279937, 443768.181241992,
464793.484167491, 482179.495293887,
493114.622150825, 496503.965509794,
490727.815979349, 501775.349195006,
515404.9785653, 527862.380457747,
548283.760750351, 572781.960184498,
599965.833392424, 630263,
653254.999999999, 670920.422982676,
691694.679133236, 714291.204243673,
740108.017228758, 769850.2298552,
797366.780908084, 804223.061862853)
Diferencias de medias
Y ¡- PIB-mean(PIB)
#series senos y cosenos
t ¡- c(0:37)
t1¡-1*2*pi*t/38
t2¡-2*2*pi*t/38
t3¡-3*2*pi*t/38
t4¡-4*2*pi*t/38
t5¡-5*2*pi*t/38
t6¡-6*2*pi*t/38
t7¡-7*2*pi*t/38
t8¡-8*2*pi*t/38
t9¡-9*2*pi*t/38
t10¡-10*2*pi*t/38
t11¡-11*2*pi*t/38
t12¡-12*2*pi*t/38
t13¡-13*2*pi*t/38
t14¡-14*2*pi*t/38
t15¡-15*2*pi*t/38
t16¡-16*2*pi*t/38
t17¡-17*2*pi*t/38
t18¡-18*2*pi*t/38
t19¡-19*2*pi*t/38
#estimaciones armonicos
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t1),cos(t1)),nrow=38)
LS1 ¡- lsfit(X,Y)
Armo1 = Y-LS1$residuals
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t2),cos(t2)),nrow=38)
LS2 ¡- lsfit(X,Y)
Armo2 = Y-LS2$residuals
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t3),cos(t3)),nrow=38)
LS3 ¡- lsfit(X,Y)
Armo3 = Y-LS3$residuals
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t4),cos(t4)),nrow=38)
LS4 ¡- lsfit(X,Y)
Armo4 = Y-LS4$residuals

```
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t5),cos(t5)),nrow=38)
LS5 ¡- lsfit(X,Y)
Armo5 = Y-LS5$residuals
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t6),cos(t6)),nrow=38)
LS6 ¡- lsfit(X,Y)
Armo6 = Y-LS6$residuals
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t7),cos(t7)),nrow=38)
LS7 ¡- lsfit(X,Y)
Armo7 = Y-LS7$residuals
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t8),cos(t8)),nrow=38)
LS8 ¡- lsfit(X,Y)
Armo8 = Y-LS8$residuals
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t9),cos(t9)),nrow=38)
LS9 ¡- lsfit(X,Y)
Armo9 = Y-LS9$residuals
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t10),cos(t10)),nrow=38)
LS10 ¡- lsfit(X,Y)
Armo10 = Y-LS10$residuals
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t11),cos(t11)),nrow=38)
LS11 ¡- lsfit(X,Y)
Armo11 = Y-LS11$residuals
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t12),cos(t12)),nrow=38)
LS12 ¡- lsfit(X,Y)
Armo12 = Y-LS12$residuals
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t13),cos(t13)),nrow=38)
LS13 ¡- lsfit(X,Y)
Armo13 = Y-LS13$residuals
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t14),cos(t14)),nrow=38)
LS14 ¡- lsfit(X,Y)
Armo14 = Y-LS14$residuals
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t15),cos(t15)),nrow=38)
LS15 ¡- lsfit(X,Y)
Armo15 = Y-LS15$residuals
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t16),cos(t16)),nrow=38)
LS16 ¡- lsfit(X,Y)
Armo16 = Y-LS16$residuals
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t17),cos(t17)),nrow=38)
LS17 ¡- lsfit(X,Y)
Armo17 = Y-LS17$residuals
Y¡-matrix(c(Y),nrow=38)
X¡-matrix(c(sin(t18),cos(t18)),nrow=38)
LS18 ¡- lsfit(X,Y)
Armo18 = Y-LS18$residuals
# Periodograma periodograma ¡-c(19*(LS1$coefficients["X1 "]^2+LS1$coefficients["X2"]^2)/4*pi,
19*(LS2$coefficients["X1"]^2+LS2$coefficients["X2"]^2)/4*pi,
19*(LS3$coefficients["X1"]^2+LS3$coefficients["X2"]^2)/4*pi,
19*(LS4$coefficients["X1"]^2+LS4$coefficients["X2"]^2)/4*pi,
```

```
19*(LS5$coefficients[“X1”]^2+LS5$coefficients[“X2”]^2)/4*pi,
19*(LS6$coefficients[“X1”]^2+LS6$coefficients[“X2”]^2)/4*pi,
19*(LS7$coefficients[“X1”]^2+LS7$coefficients[“X2”]^2)/4*pi,
19*(LS8$coefficients[“X1”]^2+LS8$coefficients[“X2”]^2)/4*pi,
19*(LS9$coefficients[“X1”]^2+LS9$coefficients[“X2”]^2)/4*p ,
19*(LS10$coefficients[“X1”]^2+LS10$coefficients[“X2”]^2)/4*pi,
19*(LS11$coefficients[“X1”]^2+LS11$coefficients[“X2”]^2)/4*pi,
19*(LS12$coefficients[“X1”]^2+LS12$coefficients[“X2”]^2)/4*pi,
19*(LS13$coefficients[“X1”]^2+LS13$coefficients[“X2”]^2)/4*pi,
19*(LS14$coefficients[“X1”]^2+LS14$coefficients[“X2”]^2)/4*pi,
19*(LS15$coefficients[“X1”]^2+LS15$coefficients[“X2”]^2)/4*pi,
19*(LS16$coefficients[“X1”]^2+LS16$coefficients[“X2”]^2)/4*pi,
19*(LS17$coefficients[“X1”]^2+LS17$coefficients[“X2”]^2)/4*pi,
19*(LS18$coefficients[“X1”]^2+LS18$coefficients[“X2”]^2)/4*pi)
Omega ¡- c( 1*2*pi/38, 2*2*pi/38,
3*2*pi/38, 4*2*pi/38,
5*2*pi/38, 6*2*pi/38,
7*2*pi/38, 8*2*pi/38,
9*2*pi/38, 10*2*pi/38,
11*2*pi/38, 12*2*pi/38,
13*2*pi/38, 14*2*pi/38,
15*2*pi/38, 16*2*pi/38,
17*2*pi/38, 18*2*pi/38)
plot(Omega, periodograma, type=“l”)
```

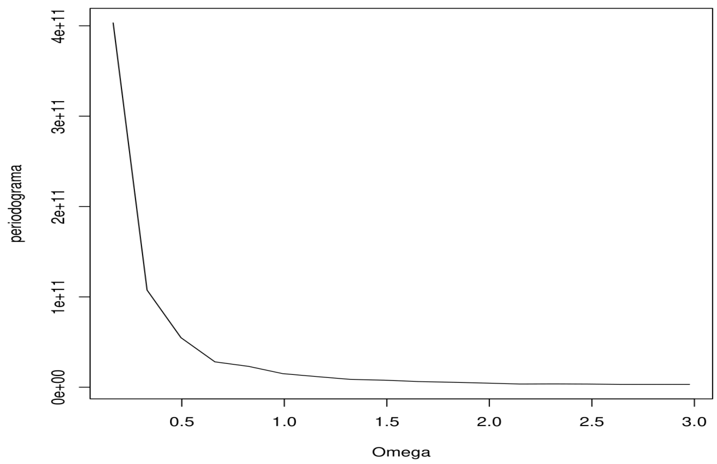

Figure 6.1: Periodogram of the example 1.

7. Calculation of the periodogram through the Discrete Fourier Transform. Taking N samples of a periodic signal $y_k = f(t_k)$ of period T at times separated by regular intervals:

$$t_0 = 1, t_1 = \frac{T}{N}, t_2 = \frac{2T}{N}, ..., t_k = \frac{kT}{T}, ..., t_{N-1} = \frac{(N-1)\,T}{N}.$$

It can be approximated by a combination $g(t)$ of known T-periodic functions that take the same value as f at this points. This procedure is known as trigonometric interpolation.

The T-periodic functions used are the complex harmonics $e^{in\omega t}$ with $\omega = \frac{2\pi}{T}$ and since there are N points, if we want the problem to have a unique solution we must combine a total of N harmonics [26]. The function $g(t)$ used in the approximation then takes the general form

$$g(t) = \frac{1}{N}\left(\beta_0 + \beta_1 e^{i\omega t} + \beta_2 e^{i2\omega t} + ... + \beta_{N-1} e^{i(N-1)\omega t}\right) = \frac{1}{N}\sum_{n=0}^{N-1} \beta_n e^{in\omega t}, \qquad (7.1)$$

such that $y_k = f(t_k)$ for each $k = 0, 1, 2, 3, ..., N-1$. Then:

$$g(t) = \frac{1}{N}\sum_{n=0}^{N-1} \beta_n e^{in\omega t} = \frac{1}{N}\sum_{n=0}^{N-1} \beta_n e^{ink\frac{2\pi}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} \beta_n w_N^{nk}, \ k = 0, 1, ..., N-1, \qquad (7.2)$$

where $w_n = \exp\left(\frac{2\pi i}{N}\right)$ is the primitive N-th root of unity. In matrix form, we have:

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ . \\ . \\ . \\ y_k \\ . \\ . \\ . \\ y_{N-1} \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & ... & 1 & ... & 1 \\ 1 & w & w^2 & ... & w^n & ... & w^{N-1} \\ 1 & w^2 & w^4 & ... & w^{2n} & ... & w^{2(N-1)} \\ . & . & . & ... & . & ... & . \\ . & . & . & ... & . & ... & . \\ . & . & . & ... & . & ... & . \\ 1 & w^k & w^{2k} & ... & w^{nk} & ... & w^{(N-1)k} \\ . & . & . & ... & . & ... & . \\ . & . & . & ... & . & ... & . \\ . & . & . & ... & . & ... & . \\ 1 & w^{N-1} & w^{2(N-1)} & ... & w^{n(N-1)} & ... & w^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ . \\ . \\ . \\ \beta_k \\ . \\ . \\ . \\ \beta_{N-1} \end{bmatrix} \qquad (7.3)$$

where $F_N = \left[w^{nk}\right]_{n,k=0}^{N-1}$ the Fourier matrix of order N. The vector β is called the discrete Fourier transform of the vector y_k and is denoted as: $\beta = DTF(y)$.

One way to obtain the DFT [26] is through the FFT (Fast Fourier Transform) [27, 28] algorithm, developed by designed by J.W. Cooley and John Tukey [29, 30]. If the function we interpolate is a real function of period T, $g(t_k) = y_k$, where $k = 0, 1, 2, .., N-1$, which uses the general form:

$$g(t) = \sum_{n=1}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right],$$

with $\omega = 2\pi/T$, supposing that $N = 2M$, if $\beta = DTF(y)$, then

$$a_0 = \frac{\beta_0}{N},\ a_n = \frac{2Re(\beta_0)}{N},\ b_n = -\frac{2Im(\beta_0)}{N},\ \ (n = 1, 2, ..., M-1),\ a_M = \frac{\beta_M}{N}, \qquad (7.4)$$

and the trigonometric polynomial

$$g(t) = a_0 + \sum_{n=1}^{M-1} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right] + a_M \cos(M\omega t).$$

Example 2. Utilizando los datos del Ejemplo 1

```
# Diferencias de medias
Y ¡- PIB-mean(PIB)
# transformadas
y ¡- fft(Y)
# transformada inversa
PIB ¡- fft(y,inverse=TRUE)/38
```

```
# periodogramas con la transformada rápida de Fourier
CF = abs(fft(Y)/sqrt(38))^2
P = CF[2:20]/(2*pi)
f = (0:18)/38
plot(f, P, type= "l")
```

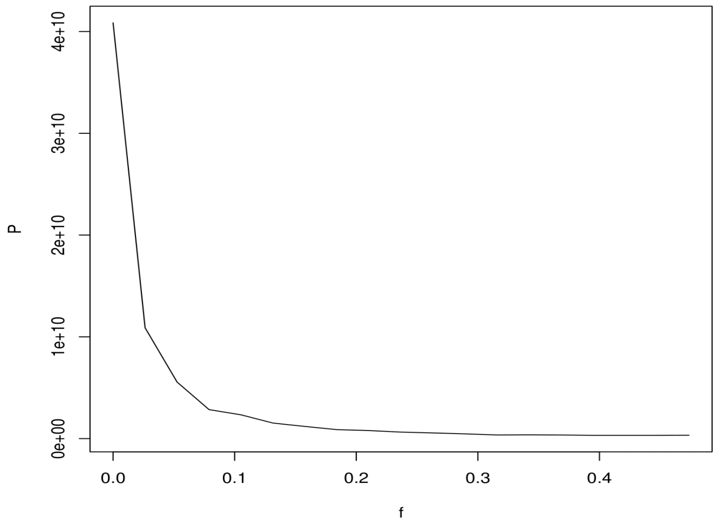

Figure 7.1: Periodogram with Fast Fourier Transform.

8. Test on the periodogram. A test to study the serial dependence (Durbin, 1969) [31, 32, 33, 34] in series of stationary observations $y_1, ..., y_T$ is carried out on the graph of the accumulated periodogram:

$$s_j = \frac{\sum_{r=1}^{j} p_r}{\sum_{r=1}^{m} p_r}, \tag{8.1}$$

where $r = 1, 2, ..., m$ is the ordinary periodogram:

$$p_r = \frac{2}{T} \left| \sum_{t=1}^{T} y_t e^{2\pi i r t} \right|^2. \tag{8.2}$$

The periodogram p_j computed for series $y_1, ..., y_T$ of independent variables $N(\mu, \sigma^2)$, is calculated:

$$a_j = \sqrt{\tfrac{2}{T} \sum_{t=1}^{T} y_t \cos\left(\tfrac{2\pi j t}{T}\right)}, \quad b_j = \sqrt{\tfrac{2}{T} \sum_{t=1}^{T} y_t \sin\left(\tfrac{2\pi j t}{T}\right)}, \quad p_j = a_j^2 + b_j^2, \quad \left(j = 1, 2, ..., \left[\tfrac{1}{2}T\right]\right), \tag{8.3}$$

And its plot of p_j against j presents a high appearance of irregularity on visual inspection. Therefore, a better way to present the information of the $p_j's$ is to do it through the graph of the accumulated periodogram, s_j.

It is assumed that when $y_1, ..., y_T$ is independently and normally distributed, $s_1, ..., s_{m-1}$ is distributed equal to the statistical order of $m - 1$ independent samples from the uniform distribution $(0, 1)$. Bartlett's [35] suggests to test for serial independence, to test for the maximum discrepancy between s_j and its expectation, ie. j/m. For a test for an excess of low relative frequencies against high frequencies, which would be equivalent to the expectation of the presence of positive serial correlation, this approach leads to the statistic:

$$c^+ = \max_j \left(s_j - \frac{j}{m} \right). \tag{8.4}$$

On the contrary, a test against excesses of high frequency variations, the appropriate statistic is:

$$c^{+} = \max_{j}\left(\frac{j}{m} - s_j\right). \tag{8.5}$$

The statistic corresponding to the two parts of the test would be:

$$c = \max_{j}\left|s_j - \frac{j}{m}\right| = \max\left(c^{+}, c^{-}\right). \tag{8.6}$$

This statistic is closely related to the Kolmogoroiv-Smirnov statistic D_n^{+}, D_n^{-}, D_n and its modified form C_n^{+}, C_n^{-}, C_n considered by Pyke [36] and Stephens [37]. For example,

$$D^{-} = \max_{j}\left\{s_j - \left(\frac{j-1}{m-1}\right)\right\}$$

and $C^{-} = c^{+}$.

Example 3. Y ¡- c(11244, 11237, 11777,
12116, 12655, 13672,
14202, 15241, 16205,
17279, 17759, 18916,
19834, 20827, 22052,
22548, 22817)
logaritmos
y ¡- (Y)
# periodogramas
FFy = abs(fft(y)/sqrt(17))ˆ2
P = FFy[2:9]/(2*pi)
P ¡- cumsum (P)/sum(P)
test periodograma acumulado
fs ¡- 0.3148+(1:8)/8
0.3148 es el valor de la tabla no1 para un α=0.01 y n=17
fi ¡- (1:8)/8-0.3148
z ¡- ts(matrix(c(P,fs,fi),8,3)) plot(z,plot.type="single")

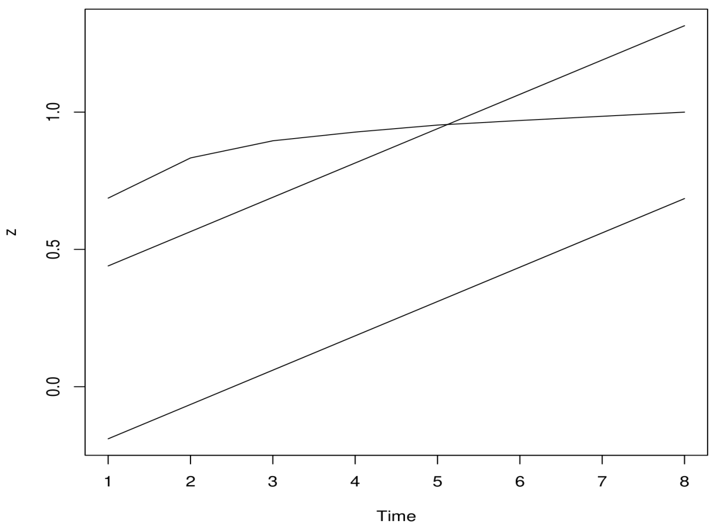

Figure 8.1: Cumulative Periodigram Test.

9. Linear filters . A linear filter [38] is defined as

$$a(L) = \sum_{j=-\infty}^{\infty} a_j L^j, \tag{9.1}$$

where the weights are real numbers, i. and. $a_j \in \mathbb{R}$; they do not depend on time and satisfy $\sum_{j=-\infty}^{\infty} a_j^2 < \infty$. Applying the linear filter $a(L)$ to a stationary stochastic process, x_t, results in a new stochastic process

$$y_t = a(L) x_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j}, \tag{9.2}$$

where the properties of x_t are passed to y_t by means of the linear filter $a(L)$. To examine the effect of a linear filter, it must be analyzed in the frequency domain.

Using the Fourier transform, the spectrum of the linear filter applied to x_t:

$$a\left(e^{-i\omega}\right) = G(\omega) e^{-iF(\omega)}, \tag{9.3}$$

where

$$G(\omega) = \left|a\left(e^{-i\omega}\right)\right|, \tag{9.4}$$

and

$$F(\omega) = \tan^{-1}\left[\frac{-\sum_{j=-\infty}^{\infty} a_j \sin(j\omega)}{\sum_{j=-\infty}^{\infty} a_j \cos(j\omega)}\right], \tag{9.5}$$

are respectively the modulus and the argument of the frequency response.

In this context the modulus, $G(\omega)$, is known as the filter gain; which determines the extent to which the amplitude of motions observed at a certain frequency in the spectrum of x_t are transferred to the spectrum of y_t. For example a gain of zero around the frequency $\omega_t \in [0, \pi]$ means that the filtered process will show no motion around that frequency.

For its part, the argument, $F(\omega)$, is known as the phase shift of the filter, which is associated with shifts of the series in the time domain. It is important to note that when $a_j = -a_j$ for all j, that is, when it is a symmetric filter; the phase shift of the filter is equal to zero[2], which implies that $F(\omega) = 0$.

10. Types of Filters. The most used filters [39] in time series analysis are variation rates and moving averages.

Rates of change are linear time-invariant but nonlinear operators. Since the elementary theory of filters refers to invariant linear operators, we have to approximate the rates to difference operators. Thus the first difference of a logarithm is a good approximation of a monthly variation rate. A simple moving average is the arithmetic mean of the previous n data points.

Centered moving averages are characterized in that the number of observations that enter into their calculation is odd, assigning each moving average to the central observation. Thus, a moving average centered on t of length $2n+1$ is given by the following expression:

$$MM(2n+1) = \frac{1}{2n+1} \sum_{t=-n}^{n} \frac{x_{t-n} + x_{t-n+1} + ... + x_t + ... + x_{t+n-1} + x_{t+n}}{2n+1}. \tag{10.1}$$

As can be seen, the subscript assigned to the moving average, t, is the same as that of the central observation, Y_t. Note also that, by construction, moving averages for the first n and last n observations cannot be computed. In asymmetric moving averages, each moving average is assigned to the period corresponding to the most advanced observation

[2]To understand this property of linear filters, the following trigonometric functions $\sin(-\omega) + \sin\omega = 0 \Rightarrow \sin\omega = 0$ are used. This implies that when $h_j = -h_j$, the product at $\sum_{j=-\infty}^{\infty} h_j \sin(j\omega)$ is equal to zero, which in turn implies that $F(\omega) = 0$ since $\tan^{-1}(0) = 0$.

of all those involved in its calculation. Thus, the asymmetric moving average of n points associated with observation t will have the following expression:

$$MMA\,(n) = \frac{1}{n}\sum_{t=-n}^{n} Y_{t+i} = \frac{x_{t-n+1} + x_{t-n+2} + ... + x_{t-1} + x_t}{n}. \tag{10.2}$$

Linear filters associated with moving averages are denoted as follows:

$$a\,(L)\,x_t = \frac{1}{n}\sum_{j=0}^{n} x_j L^j. \tag{10.3}$$

11. The filter as a product of convolution. Let y and z be two vectors of dimension N. Their convolution product $y \star z$ is defined as the vector:

$$y \star z = \begin{bmatrix} z_0y_0 + z_1y_{N-1} + z_2y_{N-2} + ... + z_{N-2}y_2 + z_{N-1}y_t \\ z_0y_1 + z_1y_0 + z_2y_{N-1} + ... + z_{N-2}y_3 + z_{N-1}y_2 \\ z_0y_2 + z_1y_1 + z_2y_0 + ... + z_{N-2}y_4 + z_{N-1}y_3 \\ . \\ . \\ . \\ z_0y_{N-2} + z_1y_{N-3} + z_2y_{N-4} + ... + z_{N-2}y_0 + z_{N-1}y_{N-1} \\ z_0y_{N-1} + z_1y_{N-2} + z_2y_{N-3} + ... + z_{N-2}y_1 + z_{N-1}y_0 \end{bmatrix}. \tag{11.1}$$

The convolution product can be expressed in matrix form:

$$y \star z = \begin{bmatrix} y_0 & y_{N-1} & y_{N-2} & . & . & . & y_2 & y_1 \\ y_1 & y_0 & y_{N-1} & . & . & . & y_3 & y_2 \\ y_2 & y_1 & y_0 & . & . & . & y_4 & y_3 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ y_{N-2} & y_{N-1} & y_{N-4} & . & . & . & y_0 & y_{N-1} \\ y_{N-1} & y_{N-2} & y_{N-3} & . & . & . & y_1 & y_0 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ . \\ . \\ . \\ z_{N-2} \\ z_{N-1} \end{bmatrix}. \tag{11.2}$$

The square matrix of the seizure product is called the circulating matrix since the elements of the first column rotate their position in the successive columns.

The discrete Fourier transform of the convolution product of $y \star z$ is the Hadamard product of the corresponding y and z transforms: $DFT\,(y \star z) = DFT\,(Y) \cdot DFT\,(Z)$. One way to calculate $y \star z$ is through coordinate-by-coordinate multiplication of the transforms of y and z, obtaining the inverse transform of this vector $\overline{DFT}\,(y \star z)$.

Filtering a series can be thought of as the product of a convolution; thus, for example, when using the linear filter $(1 - L)$ the following convolution product would be carried out:

$$y \star z = \begin{bmatrix} -1 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & -1 & 1 & . & . & . & 0 & 0 \\ 0 & 0 & -1 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & -1 & 1 \\ 1 & 0 & 0 & . & . & . & 0 & -1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ . \\ . \\ . \\ z_{N-2} \\ z_{N-1} \end{bmatrix}, \tag{11.3}$$

where the vector y would been

$$y = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

A three-term centered moving average would be expressed by the following convolution product:

$$y \star z = \begin{bmatrix} 1/3 & 1/3 & 0 & . & . & . & 0 & 0 \\ 0 & 1/3 & 1/3 & . & . & . & 0 & 0 \\ 0 & 0 & 1/3 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1/3 & 0 & 0 & . & . & . & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & . & . & . & 0 & 1/3 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ . \\ . \\ . \\ z_{N-2} \\ z_{N-1} \end{bmatrix}, \tag{11.4}$$

where the vector z would be

$$y = \begin{bmatrix} 1/3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

Example 4. Using R we are going to filter the series

```
Y ¡- PIB
# transformadas
y ¡- fft(Y)
#filtro de diferencia
F ¡- c(-1, rep(0, 36), 1)
f ¡- fft(F)
D_Y ¡- fft(fft(F)*y,inverse=TRUE)/38
plot.ts (D_Y[1 :37], type="l")
```

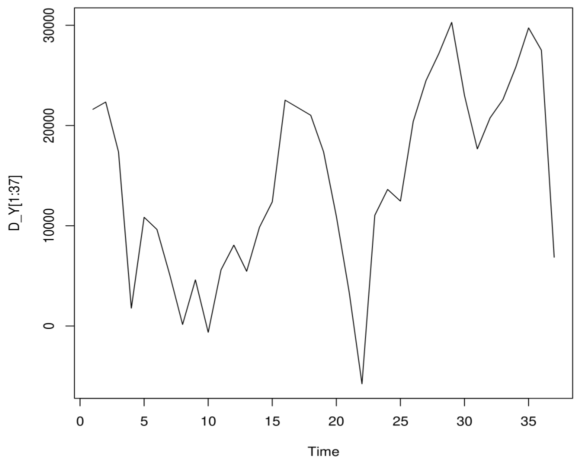

Figure 11.1: Transforms and difference filter.

Example 5. # periodograma de filtro
CF = abs(fft(F)/sqrt(38))ˆ2
P = CF[1:20]/(2*pi)
f = (0:19)/38
plot(f, P, type="l")

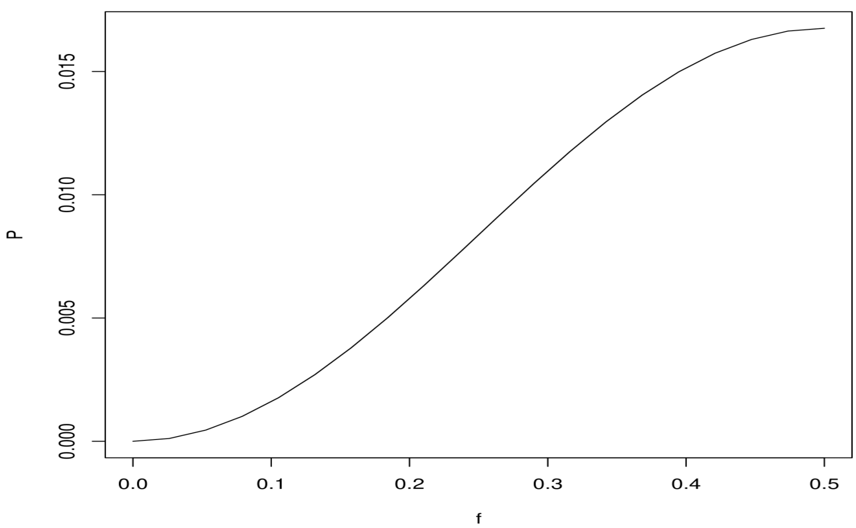

Figure 11.2: Periodograma de filtro.

The first difference filter periodogram filters out low frequencies, and amplifies high frequencies or short-term movements.

Example 6. #filtro de media movil
F ¡- c(1/4, rep(0, 34), rep(1/4,3))
f ¡- fft(F)
M_Y ¡- fft(fft(F)*y,inverse=TRUE)/38
plot.ts (M_Y[1:34], type="l")

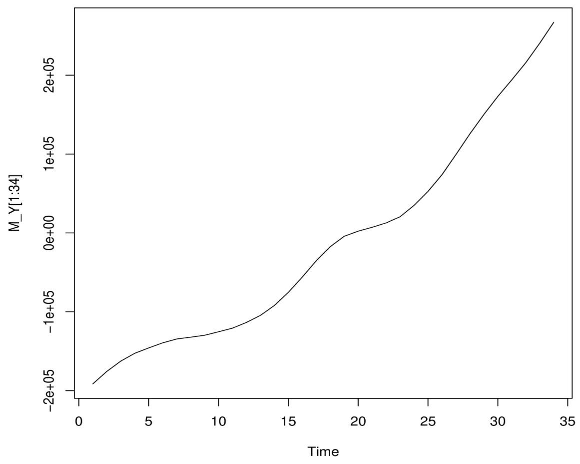

Figure 11.3: Moving average filter.

Example 7. # periodograma de filtro
CF = abs(fft(F)/sqrt(38))^2
P = CF[1:20]/(2*pi)
f = (0:19)/38
plot(f, P, type="l")

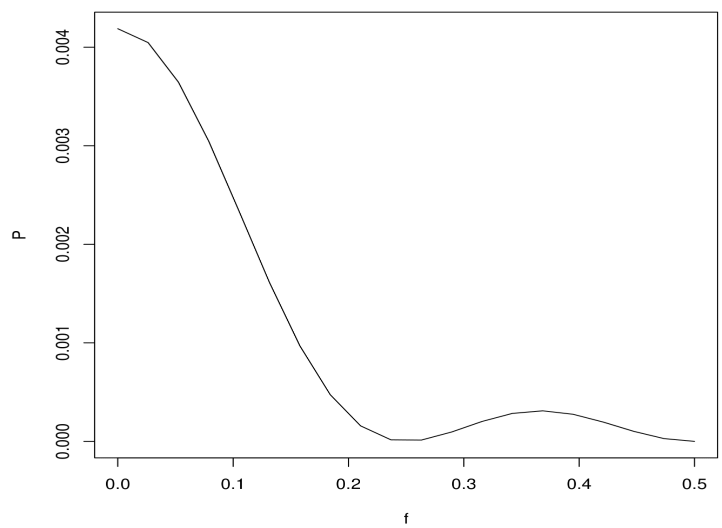

Figure 11.4: Filter Periodogram.

The four-term moving average filter periodogram filters out low frequencies, and amplifies high frequencies or long-term movements.

12. Trend and empirical cycle of a series. Starting then from a representation of the trend or relevant movement of a time series obtained, for example from a quadratic trend, a specification of a harmonic model for a series would be as follows:

$$y_t = a + b\frac{2t}{t+1} + c\frac{6t}{(T+1)(T+2)} + \sum_{p}^{k} \left[a_p \cos(p\omega_0 t) + b_p \sin(p\omega_0 t)\right] + v_t, \quad (12.1)$$

where k is the number of harmonics corresponding to the cycles that explain a certain percentage of the variance of the series in the periodogram. The empirical cycle of the series would then be [40]:

$$CE_t^y = \sum_{p}^{k} \left[a_p \cos\left(p\omega_0 t \right) + b_k \sin\left(p\omega_0 t \right) \right]. \tag{12.2}$$

If the empirical cycle series presents in its periodogram a few cycles that explain a significant percentage of its variance, we can obtain an estimator of the cycle of said time series from the $|\omega_k|$ and of the harmonics corresponding to the cycles.

Example 8. Plot.ts (PIB)

```
t ¡- c(0:37)
t_2 ¡- tˆ2
X ¡- matrix(c(t,t_2),nrow=38)
Y ¡- matrix(c(PIB),nrow=38)
LS ¡- lsfit(X,Y)
ciclo = LS$residuals
tend= PIB-ciclo
#estimaciones armonicos
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t1),cos(t1)),nrow=38)
LS1 ¡- lsfit(X,Y)
Armo1 = ciclo-LS1$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t2),cos(t2)),nrow=38)
LS2 ¡- lsfit(X,Y)
Armo2 = ciclo-LS2$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t3),cos(t3)),nrow=38)
LS3 ¡- lsfit(X,Y)
Armo3 = ciclo-LS3$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t4),cos(t4)),nrow=38)
LS4 ¡- lsfit(X,Y)
Armo4 = ciclo-LS4$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t5),cos(t5)),nrow=38)
LS5 ¡- lsfit(X,Y)
Armo5 = ciclo-LS5$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t6),cos(t6)),nrow=38)
LS6 ¡- lsfit(X,Y)
Armo6 = ciclo-LS6$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t7),cos(t7)),nrow=38)
LS7 ¡- lsfit(X,Y)
Armo7 = ciclo-LS7$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t8),cos(t8)),nrow=38)
LS8 ¡- lsfit(X,Y)
Armo8 = ciclo-LS8$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t9),cos(t9)),nrow=38)
LS9 ¡- lsfit(X,Y)
Armo9 = ciclo-LS9$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t10),cos(t10)),nrow=38)
LS10 ¡- lsfit(X,Y)
Armo10 = ciclo-LS10$residuals
```

```
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t11),cos(t11)),nrow=38)
LS11 ¡- lsfit(X,Y)
Armo11 = ciclo-LS11$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t12),cos(t12)),nrow=38)
LS12 ¡- lsfit(X,Y)
Armo12 = ciclo-LS12$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t13),cos(t13)),nrow=38)
LS13 ¡- lsfit(X,Y)
Armo13 = ciclo-LS13$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t14),cos(t14)),nrow=38)
LS14 ¡- lsfit(X,Y)
Armo14 = ciclo-LS14$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t15),cos(t15)),nrow=38)
LS15 ¡- lsfit(X,Y)
Armo15 = ciclo-LS15$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t16),cos(t16)),nrow=38)
LS16 ¡- lsfit(X,Y)
Armo16 = ciclo-LS16$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t17),cos(t17)),nrow=38)
LS17 ¡- lsfit(X,Y)
Armo17 = ciclo-LS17$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t18),cos(t18)),nrow=38)
LS18 ¡- lsfit(X,Y)
Armo18 = ciclo-LS18$residuals
Y¡-matrix(c(ciclo),nrow=38)
X¡-matrix(c(sin(t19),cos(t19)),nrow=38)
LS19 ¡- lsfit(X,Y)
Armo19 = ciclo-LS19$residuals
Armo ¡- (Armo1+
Armo2+Armo3+
Armo4+ Armo5+
Armo6+ Armo7+
Armo8+ Armo9+
Armo10+ Armo11+
Armo12+ Armo13+
Armo14+ Armo15+
Armo16+ Armo17+
Armo18+ Armo19)
plot.ts (Armo)
```

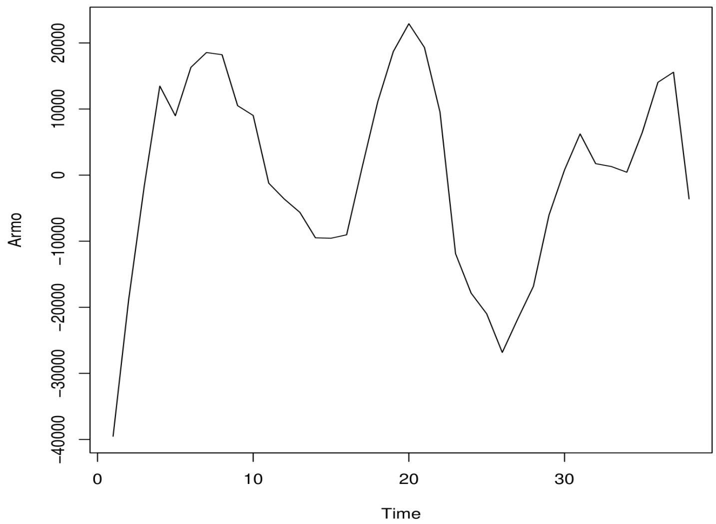

Figure 12.1: Harmonic estimates.

Example 9. PIBfit ¡- tend+Armo
plot.ts (PIBfit)

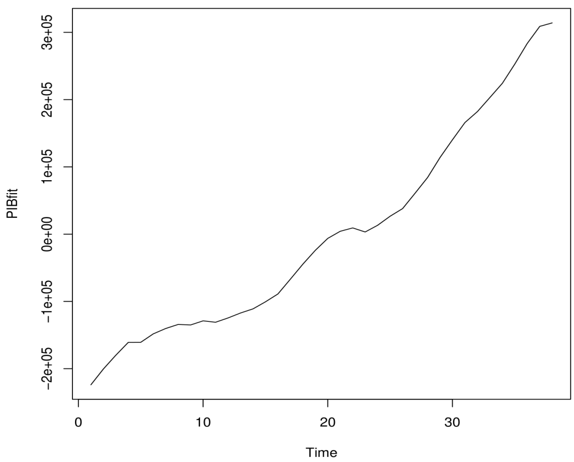

Figure 12.2: Harmonics+trend.

Example 10. # Periodograma de ciclo
y ¡- fft(ciclo)
CF = abs(fft(Y)/sqrt(38))ˆ2
P = CF[2:20]/(2*pi)
f = (0:18)/38
plot(f, P, type="l")

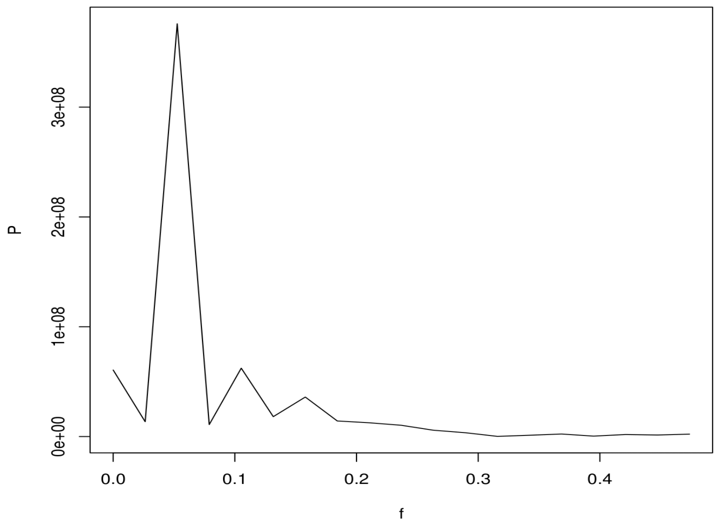

Figure 12.3: Cycle periodogram.

The most relevant harmonics in the empirical cycle periodogram are 1, 3, 5 and 7.

Example 11. Armo ¡- (Armo1+ Armo3+ Armo5+ Armo7)
plot.ts (Armo)

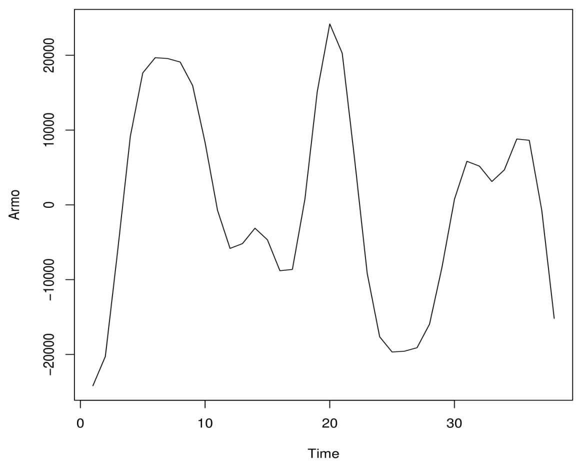

Figure 12.4: Harmonics.

Example 12. PIBfit ¡- tend+Armo
plot.ts (PIBfit)

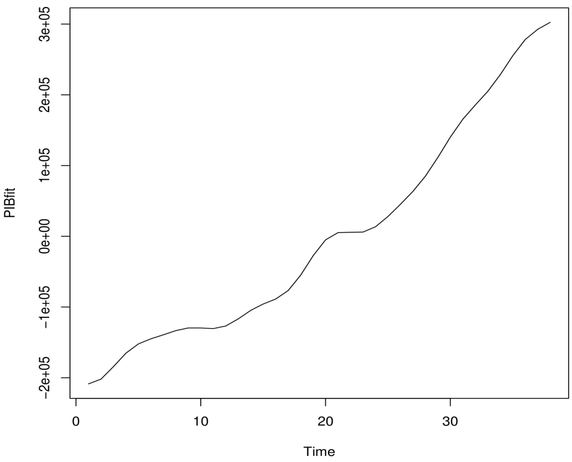

Figure 12.5: Harmonic trend.

13. Flexible Fourier Form (FFF). Gallant [41][42] introduced a functional form with capacities very different from those proposed up to now, whose flexibility properties were in all cases local. The Fourier form used by Gallant has the property of global flexibility, that is, it allows arbitrarily close approximation of both the function and its derivatives over their entire domain of definition. The idea behind this type of approximation (which could be called semi-non-parametric) is to expand the order of the expanding base, when the size of the sample increases, until the asymptotic convergence of the approximating function to the true generating function of the data and its derivatives is achieved.

As it is a Sobolev-flexible form, it is capable of consistently estimating the elasticities over the entire data space (ElBadawi, Gallant and Souza, [43][44]); moreover, unbiased statistical[45] tests can be achieved asymptotically [41][42]. Finally, Gallant and Souza [46] have shown the asymptotic normality of the estimates derived from the Fourier form. On the negative side, the Fourier model can achieve global regularity, but the parametric restrictions that this implies are excessively strong (Gallant, 1981); however, there are weaker conditions (which do not destroy either the flexibility or the consistency of the estimators) with which theoretical regularity can be achieved at least over a finite set of points (Gallant and Golub [47][48], although the implementation of such restrictions is complex . In any case, the Monte Carlo simulations carried out by Fleissig, Kastens, and Terrell [49] and Chalfant and Gallant [50] have shown that the region of regularity of the unrestricted free Fourier form is much larger than the corresponding to the Leontief-Generalized or Translog forms.

Example 13. #logaritmos

```
X ¡- log(K)
Y ¡- log(PIB)
# transformaciones
z ¡- 2*pi*X/max(X)
#estimaciones FFF
X ¡- matrix(c(z,z**2, sin(z),cos(z),
sin(2*z),cos(2*z),
sin(3*z),cos(3*z)), nrow=38)
Y ¡- matrix(c(Y),nrow=38)
LS ¡- lsfit(X,Y)
res = LS$residuals
PIBfit ¡- matrix(c(Y-res),nrow=38)
plot.ts (PIBfit)
```

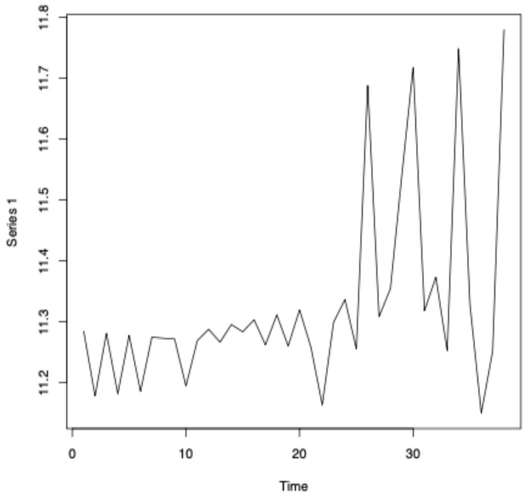

Figure 13.1: FFF estimates.

14. Conclusions . Periodic phenomena involving waves, rotating machines (harmonic motion), or other repetitive driving forces are described by periodic functions. Fourier series are a basic tool for solving problems of ordinary differential equations (ODEs), partial differential equations (PDEs) with periodic boundary conditions, applied mathematics and mathematical physics. Fourier integrals for nonperiodic phenomena are developed and applied with R in the present article. This common name for the field is Fourier analysis. The Fourier theory of trigonometric series is of great practical importance because certain types of discontinuous functions which cannot be expanded in power series can be expanded in Fourier series.

Some of the applications of Fourier Analysis are:

- Generation of electrical current or voltage waveforms by superimposing sinusoids generated by variable amplitude electrical oscillators. The frequencies are already determined.
- Analysis of the harmonic behavior of a signal.
- Signal reinforcement.
- Study of the response in time of an electrical circuit variable where the input signal is not sinusoidal or cosinusoidal, through the use of Laplace transforms and/or solution in permanent sinusoidal regime in the frequency domain.
- The resolution of some differential equations in partial derivatives admit particular solutions in the form of easily computable Fourier series, and that obtain practical solutions, in the theory of heat transmission, etc.

We conclud that the periodogram collects the contribution that each harmonic has when it comes to explaining the variance of each series, and each harmonic is characterized by the frequency in which the cycles take place. Cycles that have a high period (from when one peak occurs to the next peak) will have a low frequency and viceversa. In this work, we study periodograms, linear filters, trend of a series and the flexible Fourier transform.

This article has been developed by MV.

Conflicts of interest. The author declare no conflict of interest.

Abbreviations. The following abbreviations are used in this manuscript:

SelMat Selecciones Matematicas

DOAJ Directory of open access journals.

ORCID and License
Mississippi Valenzuela https://orcid.org/0000-0002-8532-229X

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