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The Hausdorff–Young Inequality for n–dimensional Hermite Expansions

La desigualdad de Hausdorff-Young para las expansiones de Hermite n-dimensionales

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Abstract

We discuss a sharpened Hausdorff–Young inequality for n-dimensional Hermite expansions.

Keywords. Hausdorff–Young inequality, n-dimensional Hermite expansions.

Resumen

Consideramos una desigualdad de Hausdorff–Young refinada para expansiones de Hermite n–dimensionales.

Palabras clave. Desigualdad de Hausdorff–Young, expansiones de Hermite n–dimensionales.

1. Introduction. This note concerns the sharpened Hausdorff–Young inequality in the context of n–dimensional Hermite expansions. The corresponding 1–dimensional result was considered in [4].

The Hermite functions constitute an ONS in $\mathbb{R}$ with respect to the Lebesgue measure there, and are defined as follows [5, 12, 14]. Szegö introduced the Hermite polynomials, $H_m(x)$, in Chapter V of [12]. Earlier, Hille had also considered the Hermite polynomials, and proved some remarkable formulas and estimates [5, 14]. In particular, Hille considered the generating formula

$$\sum_{m=0}^{\infty} H_m(x)\frac{u^m}{m!} = e^{2xu - u^2},$$

and defined the Hermite functions $\mathcal{H}_m(x)$, $m \geq 0$, by

$$\mathcal{H}_m(x) = \frac{1}{(m!)^{1/2}}\frac{1}{2^{m/2}}H_m(x)\, e^{-x^2/2}, \quad x \in \mathbb{R}\,.$$

The n–dimensional Hermite functions are obtained as products of the 1–dimensional Hermite functions [10, 14], and constitute an ONS in $\mathbb{R}^n$ with respect to the Lebesgue measure there. To the point, given $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$ and an n–tuple of nonnegative integers $m = (m_1, \ldots, m_n)$, let the Hermite functions $\mathcal{H}_m(x)$ be given by

$$\mathcal{H}_m(x) = \mathcal{H}_{m_1}(x_1)\cdots\mathcal{H}_{m_n}(x_n)\,.$$

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Now, for a function $f(x)$ defined on $\mathbb{R}^n$, the Hermite expansion of f is given by

$$f(x) \sim \sum_m C_m \mathcal{H}_m(x)\,, \quad x \in \mathbb{R}^n\,,$$

where the Hermite coefficients of $f(x)$, C_m, are defined by

$$C_m = \int_{\mathbb{R}^n} f(x)\mathcal{H}_m(x)\,dx\,. \tag{1.1}$$

In order to verify that the n–dimensional Hermite functions are an ONS, it suffices to show that if all the Hermite coefficients of an $L^2(\mathbb{R}^n)$ function $f(x) \sim \sum_m C_m \mathcal{H}_m(x)$ vanish, then $f(x) = 0$ a.e. with respect to the Lebesgue measure on $\mathbb{R}^n$. For simplicity we assume that $n = 2$, and let $f(x_1, x_2)$ be such that $C_m = 0$ for all $m = (m_1, m_2)$ with $m_1, m_2 = 0, 1, 2, \ldots$

Let then

$$\varphi_{m_2}(x_1) = \int_{\mathbb{R}} f(x_1, x_2)\,\mathcal{H}_{m_2}(x_2)\,dx_2\,, \quad x_1 \in \mathbb{R}\,, m_2 = 0, 1, 2, \ldots,$$

and observe that all the Hermite coefficients C'_{m_1} of φ_{m_2} vanish. Indeed, with $m = (m_1, m_2)$,

$$\begin{aligned} C'_{m_1} &= \int_{\mathbb{R}} \varphi_{m_2}(x_1)\,\mathcal{H}_{m_1}(x_1)\,dx_1 \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x_1, x_2)\,\mathcal{H}_{m_2}(x_2)\,dx_2\,\mathcal{H}_{m_1}(x_1)\,dx_1 = C_m = 0\,. \end{aligned}$$

Hence, by the completeness of the Hermite functions in $\mathbb{R}$, $\varphi_{m_2}(x_1)$ vanishes a.e. with respect to the Lebesgue measure on the line for each $m_2 = 0, 1, 2, \ldots$ Let E_{m_2} be the set of Lebesgue measure 0 in the line outside of which φ_{m_2} vanishes, and let $E = \bigcup_{m_2=0}^{\infty} E_{m_2}$; E is a set of Lebesgue measure 0 in $\mathbb{R}$.

Now, for each $x_1 \in \mathbb{R} \setminus E$, we have

$$\int_{\mathbb{R}} f(x_1, x_2)\,\mathcal{H}_{m_2}(x_2)\,dx_2 = 0\,, \quad m_2 = 0, 1, 2, \ldots,$$

and by the completeness of the Hermite expansion in $\mathbb{R}$, $f(x_1, x_2) = 0$ a.e. x_2 in $\mathbb{R}$ whenever $x_1 \in \mathbb{R} \setminus E$.

Then, by Tonelli's theorem, on account of the above observations it follows that

$$\int_{\mathbb{R}^2} |f(x)|^2 dx = \int_{\mathbb{R}}\int_E |f(x_1, x_2)|^2 dx_1 dx_2 + \int_{\mathbb{R}\setminus E}\int_{\mathbb{R}} |f(x_1, x_2)|^2 dx_2 dx_1 = 0\,,$$

and so, $f(x_1, x_2) = 0$ a.e. on $\mathbb{R}^2$.

Thus, in particular, the n–dimensional Hermite expansions satisfy the Parseval–Plancherel formula in $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \sum_m \big|C_m\big|^2,$$

and, in order to discuss the Hausdorff–Young inequality, we introduce some preliminary material concerning Lebesgue, Lorentz and Orlicz spaces.

2. Preliminaries. Given a function f defined on $\mathbb{R}^n$, with ν the Lebesgue measure on R^n, let $m(f, \lambda)$ denote the *distribution function* of f,

$$m(f, \lambda) = \nu\big(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\big), \quad \lambda > 0\,.$$

$m(f, \lambda)$ is nonincreasing and right continuous, and the *nonincreasing rearrangement* f^* of f defined for $t > 0$ by

$$f^*(t) = \inf\{\lambda : m(f, \lambda) \le t\}\,, \quad \inf \emptyset = 0\,,$$

is informally its inverse (this statement is made precise in [9, p. 43]). f^* is nonincreasing and right continuous and, at its points of continuity t, $f^*(t) = \lambda$ is equivalent to $m(f, \lambda) = t$.

The *Lorentz space* $L^{p,q}(\mathbb{R}^n) = L(p, q)$, $0 < p < \infty$, $0 < q \le \infty$, consists of those measurable functions f with finite quasinorm $\|f\|_{p,q}$ given by

$$\|f\|_{p,q} = \left(\frac{q}{p}\int_0^{\infty} \big(t^{1/p} f^*(t)\big)^q \,\frac{dt}{t}\right)^{1/q}, \quad 0 < q < \infty\,,$$

and,

$$\|f\|_{p,\infty} = \sup_{t>0} \big(t^{1/p} f^*(t)\big) = \sup_{\lambda>0} \lambda\, m(f,\lambda)^{1/p}\,, \quad q=\infty\,.$$

The Lorentz spaces are monotone with respect to the second index, that is, if $0<q<q_1\le\infty$, then $L(p,q)\subset L(p,q_1)$, and

$$\|f\|_{p,q_1} \lesssim \|f\|_{p,q}\,, \tag{2.1}$$

with $L(p,p)$ the Lebesgue space $L^p(\mathbb{R}^n)$, and $L(p,\infty)$ the space weak–$L^p(\mathbb{R}^n)$.

As for the Lorentz sequence spaces, given n–tuples of non–negative integers m, and a sequence $c=\{c_m\}$, let $\{c_k^*\}$ denote the sequence obtained by ordering $\{|c_m|\}$ in a nonincreasing fashion. The *Lorentz sequence space* $\ell(p,q)$, $1\le p<\infty$, $1\le q\le\infty$, consists of those sequences $c=\{c_m\}$ with finite quasinorm $\|c\|_{\ell^{p,q}}$ given by

$$\|c\|_{\ell^{p,q}} = \Big(\sum_{k=1}^{\infty} \big(k^{1/p} c_k^*\big)^q \,\frac{1}{k}\Big)^{1/q}, \quad 1\le q<\infty\,,$$

and, with μ the atomic measure concentrated on the lattice of n-tuples of nonnegative integer atoms m taking the value $\mu(m)=1$ on each such atom,

$$\|c\|_{\ell^{p,\infty}} = \sup_{k\ge1} k^{1/p} c_k^* = \sup_{\lambda>0} \lambda \mu(\{m : |c_m|>\lambda\})^{1/p}\,, \quad q=\infty\,. \tag{2.2}$$

As for the Orlicz spaces, the letters A, B are reserved for *Young's functions*, i.e., for functions $A(t)$ defined for $t\ge0$ that are zero at zero, increasing, and convex, or, more generally, $A(t)/t$ increasing to ∞ as $t\to\infty$. The *Orlicz space* $L^A(\mathbb{R}^n)$ consists of those measurable functions f (modulo equality a.e.) such that $\int_{\mathbb{R}^n} A\big(|f(x)|/M\big)\,dx<\infty$ for some M, normed by

$$\|f\|_A = \inf\Big\{\lambda>0 : \int_{\mathbb{R}^n} A\Big(\frac{|f(x)|}{\lambda}\Big)\,dx\le 1\Big\}.$$

The *Orlicz sequence space* ℓ^A consists of those sequence $c=\{c_m\}$ such that for some M,

$$\sum_m A\big(|c_m|/M\big) < \infty\,,$$

normed by

$$\|c\|_{\ell^A} = \inf\Big\{\lambda>0 : \sum_m A\Big(\frac{|c_m|}{\lambda}\Big)\le 1\Big\}.$$

Finally, an operator T of a class of functions f on $\mathbb{R}^n$ into a linear class of functions is said to be *linear* provided that, if T is defined for f_0, f_1, and $\lambda\in\mathbb{R}$, then T is defined for $f_0+\lambda\, f_1$, and $T(f_0+\lambda\, f_1)(x) = T(f_0)(x)+\lambda\, T(f_1)(x)$.

A linear operator T defined for $f\in L^A(\mathbb{R}^n)$ and taking values $T(f)=\{c_m\}$ in ℓ^B is said to be *bounded* if there is a constant $K>0$ such that

$$\sum_m B\Big(\frac{|c_m|}{K}\Big)\le 1$$

, whenever

$$\int_{\mathbb{R}^n} A\big(|f(x)|\big)\,dx\le 1.$$

A bounded operator T from ℓ^A to $L^B(\mathbb{R}^n)$ is defined similarly. In either case, the smallest K above is called the *norm* of T, is denoted by $\|T\|$, and the operator is said to be of *type* (A,B). These operators satisfy $\|T(f)\|_{\ell^B}\lesssim\|f\|_A$, and $\|T(\{c_m\})\|_B\lesssim\|\{c_m\}\|_{\ell^A}$, respectively. When $A(t)=t^p$ and $B(t)=t^q$, we say that T is of *type* (p,q). If the mapping T is from an $L^p(\mathbb{R}^n)$ space into an $L^q(\mathbb{R}^n)$, or a sequence space $\ell(q,\infty)$, the mapping is said to be of *weak–type* (p,q). Similarly for mappings from ℓ^p into weak–$L^q(\mathbb{R}^n)$ spaces.

For further consideration of the Lorentz and Orlicz spaces the reader may consult [1, 6, 8, 9].

3. The Hausdorff–Young Inequality. The sharpened Hausdorff–Young inequality for $n = 1$ proved in [4, Theorem 4.1] rests on a remarkable estimate for the Hermite functions established by Hille [5, p. 436], [12, p. 240], to wit,

$$\big|\,\mathcal{H}_m(x)\,\big| \lesssim m^{-1/12}. \tag{3.1}$$

Hille notes that (3.1) is the best possible estimate, but that in applications he will only use the weaker formula $|\mathcal{H}_m(x)| \lesssim 1$. On the other hand, as in [4], we will use (3.1) to obtain a sharperned Hausdorff–Young inequality for Hermite expansions on $\mathbb{R}^n$. We refer to these estimates as sharpened because they are of type (p, q) with $q < p'$.

We then have,

Theorem 3.1. *Let $f(x) \sim \sum_m C_m\mathcal{H}_m(x)$ denote the expansion of a function f defined on $\mathbb{R}^n$ in a Hermite series, and let T be the mapping that assigns to f its sequence of Hermite coefficients $\{C_m\}$. Then, T maps the Lorentz space $L(p, s)$ continuously into the Lorentz sequence space $\ell(q, s)$, $1 \le s \le \infty$, provided that p, q verify*

$$1 < p < 2, \quad \textit{and,} \quad \Big(1 - \frac{1}{6n}\Big)\frac{1}{p} + \frac{1}{q} = 1 - \frac{1}{12n}\,. \tag{3.2}$$

In particular,

$$\|T(f)\|_{\ell^q} = \|\{C_m\}\|_{\ell^q} \lesssim_p \|f\|_p, \tag{3.3}$$

and T is of type (p, q) whenever (3.2) holds.

Moreover, if A, B are Young's functions such that $\int_0^t B(s)/s^{12n}\, ds/s \lesssim B(t)/t^{12n}$ and $B(t)/t^2$ increases, T is of type (A, B), provided that A, B verify

$$B^{-1}(t) = t^{\left((12n-1)/12\,n\right)} A^{-1}\Big(t^{\left((1-6n)/6\,n\right)}\Big), \quad t > 0\,. \tag{3.4}$$

Proof: For simplicity, since no new ideas are required for general n, we will carry out the proof for $n = 2$. Let $f(x) \sim \sum_m C_m\mathcal{H}_m(x)$ denote the Hermite expansion of f in a Hermite series, where the $\{C_m\}$ are defined as in (1.1) above.

Note that since $\mathcal{H}_0(t) = 1$, it follows that if $m_0 = (0, 0)$, then $|C_{m_0}| \le \|f\|_1$, and also that

$$\big|C_m\big| \lesssim \|f\|_1 {m_1}^{-1/12}, \quad m = (m_1, 0)\,, m_1 \ge 1, \tag{3.5}$$

and,

$$\big|C_m\big| \lesssim \|f\|_1 {m_2}^{-1/12}, \quad m = (0, m_2)\,, m_2 \ge 1\,. \tag{3.6}$$

Also, for $m = (m_1, m_2)$ with $m_1 \cdot m_2 \ne 0$, we have

$$\big|C_m\big| \lesssim \|f\|_1 (m_1 m_2)^{-1/12}, \quad m = (m_1, m_2)\,, m_1 \cdot m_2 \ne 0\,. \tag{3.7}$$

Let μ denote the atomic measure concentrated on the lattice of 2-tuples of integer atoms $m = (m_1, m_2)$ with $m_1, m_2 = 0, 1, 2, \ldots$, taking the value $\mu(m) = 1$ on each such atom.

Given $\lambda > 0$, let $\mathcal{I}_\lambda = \{m : |C_m| > \lambda\}$. Now, if $m = (m_1, m_2)$ is in $\mathcal{I}_\lambda$ and $m_1 \cdot m_2 \ne 0$, by (3.7) we have

$$\lambda < |C_m| \lesssim \|f\|_1\,(m_1 m_2)^{-\frac{1}{12}},$$

and, consequently,

$$m_1 m_2 \le \Big(\|f\|_1/\lambda\Big)^{12},$$

which, since $m_1, m_2 \ge 1$ implies that

$$m_1 \lesssim \Big(\|f\|_1/\lambda\Big)^{12}, \quad m_2 \lesssim \Big(\|f\|_1/\lambda\Big)^{12}.$$

Hence,

$$\mu(\{m = (m_1, m_2) \in \mathcal{I}_\lambda : m_1 \cdot m_2 \ne 0\}) \lesssim \Big(\|f\|_1/\lambda\Big)^{12}\Big(\|f\|_1/\lambda\Big)^{12} = \Big(\|f\|_1/\lambda\Big)^{24}. \tag{3.8}$$

Also, since from (3.5) and (3.6) above

$$|C_m| \lesssim \|f\|_1 m_1^{-1/12} \lesssim \|f\|_1 m_1^{-1/24}\,, \quad m=(m_1,0)\,,$$

and

$$|C_m| \lesssim \|f\|_1 m_2^{-1/12} \lesssim \|f\|_1 m_2^{-1/24}\,, \quad m=0,(m_2)\,,$$

it follows that

$$\mu(\{m=(m_1,m_2)\in\mathcal{I}_\lambda : m_1=0 \text{ or } m_2=0\}) \lesssim \Big(\|f\|_1/\lambda\Big)^{24}\,,$$

which combined with (3.8) above yields

$$\lambda^{24}\,\mu\big(\{m=(m_1,m_2),(m_1,m_2)\neq(0,0) : |C_m|>\lambda\}\big) \lesssim \|f\|_1^{24}\,. \tag{3.9}$$

Now, if $m_0=(0,0)\in\mathcal{I}_\lambda$, since as observed above $|C_{m_0}|\le\|f\|_1$, it follows that $\lambda<|C_{m_0}|\le\|f\|_1$, and so

$$\lambda^{24}\,\mu(m_0)=\lambda^{24}\le\|f\|_1^{24}\,,$$

which combined with (3.9) above gives that

$$\lambda^{24}\,\mu(\,\mathcal{I}_\lambda\,) \lesssim \|f\|_1^{24}\,.$$

Therefore, by (2.2), it follows that

$$\|\{C_m\}\|_{\ell^{24,\infty}} = \sup_{\lambda>0}\lambda\,\mu\big(\{m : |C_m|>\lambda\}\big)^{1/24} \lesssim \|f\|_1\,, \tag{3.10}$$

and T is continuous from $L^1(\mathbb{R}^2)$ into the sequence space $\ell(24,\infty)$.

Also, T is of type (2.2) as established by the Parseval–Plancherel formula, and, in particular,

$$\|\{C_m\}\|_{\ell^2} = \Big(\sum_m |C_m|^2\Big)^{1/2} \lesssim \|f\|_2\,. \tag{3.11}$$

We are, therefore, in the appropriate framework to interpolate for the Orlicz spaces. We remind the reader the underlying principle to obtain these interpolation results [13]. If a linear mapping T is of type, or weak–type, or mixed types, (p_0,q_0) and (p_1,q_1), with $p_0\neq p_1$, and the equation of the straight line passing through the points $(1/p_0,1/q_0)$, $(1/p_1,1/q_1)$ is given by $y=\varepsilon x+\gamma$, then, under appropriate growth conditions on the Young's functions A,B, the mapping T is of type (A,B) provided that

$$B^{-1}(t)=t^\gamma\,A^{-1}(t^\varepsilon)\,.$$

In our case T is of weak–type $(1,24)$ and of type (2,2), and the equation of the line passing through the points $(1,1/24)$ and $(1/2,1/2)$ is given by

$$y=-\frac{11}{12}\,x+\frac{23}{24}\,.$$

Hence, by [12, Theorem 2.8, p. 184], T is of type (A,B) provided that

$$B^{-1}(t)=t^{23/24}\,A^{-1}(t^{-11/12})\,,\quad t>0\,, \tag{3.12}$$

which is precisely (3.4) for $n=2$.

Furthermore, since the Lorentz norms are monotone with respect to the second index, from (2.1) it follows that

$$\|\{C_n\}\|_{\ell^{2,\infty}} \lesssim \|\{C_n\}\|_{\ell^2} \lesssim \|f\|_2 \lesssim \|f\|_{2,1}\,,$$

and, thus, together with (3.10) we are in the right framework to interpolate for the Lorentz spaces, and so, by [3, Corollary to Theorem 10, p. 293] it follows that T maps the Lorentz space $L(p,s)$ continuously into the Lorentz sequence space $\ell(q,s)$, $1\le s\le\infty$, where, for $0<\theta<1$,

$$\frac{1}{p}=\theta+\frac{1-\theta}{2},\qquad \frac{1}{q}=\frac{\theta}{24}+\frac{1-\theta}{2}\,.$$

Now, replacing θ above by its value,

$$\theta = 2\Big(1-\frac{1}{p}\Big)\,,$$

gives

$$\frac{11}{12}\frac{1}{p}+\frac{1}{q}=\frac{23}{24}\,,$$

which is (3.2) for $n=2$. And so, for these values of p,q we have,

$$\|\{C_n\}\|_{\ell^{q,s}} \lesssim_{p,s} \|f\|_{p,s}\,,\quad 1\le s\le\infty\,. \tag{3.13}$$

Moreover, on account of the monotonicity of the Lorentz norms with respect to the second index, since for p,q verifying (3.2) we have $p<2<q$, setting $s=q$ in (3.13 3.14), it follows that

$$\|\{C_n\}\|_{\ell^q} \lesssim \|\{C_n\}\|_{\ell^{q,q}} \lesssim_p \|f\|_{p,q} \lesssim_p \|f\|_{p.p} \lesssim_p \|f\|_p\,,$$

and T is of type (p,q). This conclusion also follows letting $A(t)=t^p$ in (3.12) above. This proves (3.3), and we have finished.

A companion result to the Hausdorff–Young inequality addresses under what conditions $\{c_m\}$ is the sequence of Fourier coefficients of a function f in the Hausdorff–Young range [2], [15, Vol.2, Theorem 2.3, p, 101]. For the Hermite expansions in $\mathbb{R}$, this is done in [4, Theorem 4.2].

In our context, for the Hermite expansions in n dimensions we have,

Theorem 3.2. *Suppose that p,q verify,*

$$\frac{12n}{12n-1}<p<2,\quad \textit{and},\quad \frac{1}{p}+\Big(1-\frac{1}{6n}\Big)\frac{1}{q}=1-\frac{1}{12n}\,. \tag{3.14}$$

Then, given $\{c_m\}$ in the Lorentz sequence space $\ell(p,s)$, there is f in the Lorentz space $L(q,s)$, $1\le s\le\infty$, such that $f(x)\sim\sum_m c_m\,\mathcal{H}_m(x)$, and

$$\|f\|_{q,s}\lesssim_{p,s}\|\{c_n\}\|_{\ell^{p,s}}\,. \tag{3.15}$$

In particular, if τ denotes the mapping that assigns f to the sequence $\{c_m\}$, τ is of type (p,q) whenever (3.14) holds.

Moreover, if A,B are Young's functions such that $B(t)/t^2$ increases, and for some $r>2$, $B(t)/t^r$ decreases and $\int_t^\infty B(s)/s^r\,ds/s\lesssim B(t)/t^r$, then τ is of type (A,B), provided that A,B verify

$$B^{-1}(t)=t^{1/2\big((1-12n)/(1-6n)\big)}A^{-1}\big(t^{-\big(6n/(1-6n)\big)}\big)\,,\quad t>0\,. \tag{3.16}$$

Proof: For simplicity we argue the case $n=2$ as no new ideas are required for general n. Let $b(x)=\{\mathcal{H}_m(x)\}$. Then, as it was shown in the argument leading to (3.10), $b(x)$ is in the Lorentz sequence space $\ell(24,\infty)$, uniformly in x. Therefore, for a sequence $\{c_m\}$ in its conjugate Lorentz sequence space, $\ell(24/23,1)$, it follows that

$$\Big|\sum_m c_m\,\mathcal{H}_m(x)\Big|\lesssim\|\{c_m\}\|_{\ell^{24/23,1}}\,,\quad\text{uniformly in } x\in\mathbb{R}^2\,.$$

Hence, if $f(x)\sim\sum_m c_m\mathcal{H}_m(x)$, then $f\in L^\infty(\mathbb{R}^2)$, and

$$\|f\|_{\infty,\infty}=\|f\|_\infty\lesssim\|\{c_m\}\|_{\ell^{14/23,1}}\,.$$

And, since by the Parseval–Plancherel formula τ is of type (2,2) and we have $\|f\|_{2,\infty}\lesssim\|f\|_2\lesssim\|\{c_m\}\|_2\lesssim\|\{c_m\}\|_{\ell^{2,1}}$, interpolating, by [3, Corollary to Theorem 10, p. 293] it follows that τ maps the Lorentz sequence space $\ell(p,s)$ continuously into the Lorentz space $L(q,s)$, $1\le s\le\infty$, where, $24/23<p<2$, and for $0<\theta<1$,

$$\frac{1}{p}=\frac{23}{24}\,\theta+\frac{1-\theta}{2},\quad\frac{1}{q}=\frac{1-\theta}{2}\,.$$

Now, eliminating θ in the above relations gives (3.14) for $n = 2$, and, provided that (3.14) holds, we get that

$$\|f\|_{p,s} \lesssim_{p,s} \|\{c_n\}\|_{\ell^{q,s}}\,, \quad 1 \le s \le \infty\,.$$

And, since $p < q$, setting $s = q$ in (3.15) gives that τ is of type (p, q), provided that (3.14) holds.

The result for the Orlicz spaces follows now by interpolation. In our case the equation of the line that passes through $(23/24, 0)$ and $(1/2, 1/2)$ is given by

$$y = -\frac{12}{11}\,x + \frac{23}{22}\,,$$

and, consequently, by [12, Theorem 2.8, p. 184], T is of type (A, B) provided that

$$B^{-1}(t) = t^{23/22}\,A^{-1}(t^{-12/11})\,, \quad t > 0\,,$$

which is (3.16) for $n = 2$, and the proof is finished.

The reader will observe that as $n \to \infty$, the expressions (3.2) and (3.4) above relating p, q become $1/p + 1/q = 1$, which is precisely the Haussdorf–Young range in the case of Fourier expansions. And, naturally, the expressions (3.4) and (3.15) above approach the formula $B^{-1}(t) = t\,A^{-1}(1/t)$, which is the condition for the Hausdorff–Young inequality to hold in the case of the Fourier transform [7].

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References

[1] Bennett C and Sharpley R. Interpolation of Operators, Academic Press, Orlando, Florida, 1988.
[2] Butzer PL. The Hausdorff–Young theorems of Fourier analysis and their impact, J. Fourier Anal. Appl. 1994; (1):113–130.
[3] Calderón AP. Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz, Studia Math. XXVI. 1966; 273–299.
[4] Calderón CP and Torchinsky A. Maximal integral inequalities and Hausdorff–Young, preprint.
[5] Hille E. A class of reciprocal functions, Ann. of Math. 1926; (2): 27(4):427-464.
[6] Hunt RA. On L(p, q) spaces, Enseign. Math. 1966; 12(2):249–276.
[7] Jodeit M Jr. and Torchinsky A. Inequalities for Fourier transforms, Studia Math. XXXVII. 1971; 245–276.
[8] Krasnosels'kii MA and Rutickii Ya. B. Convex functions and Orlicz spaces, Nordhoff, Groningen, 1961.
[9] Oklander ET. Interpolación, Espacios de Lorentz, y el Teorema de Marcinkiewicz, Cursos y Seminarios 20, U. de Buenos Aires, 1965.
[10] Pinsky MA and Prather C. Pointwise convergence of n–dimensional Hermite expansions, Jour. Math. Anal. and Appl. 1996; 199:620-628.
[11] Stein EM and Weiss G. Introduction to Fourier Analysis on Euclidean Spaces, Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1971.
[12] Szegö G. Orthogonal polynomials, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc, Providence, R. I., 1959.
[13] Torchinsky A. Interpolation of operations and Orlicz classes, Studia Math. LIX. 1976/77; 2:177–207.
[14] Urbina–Romero W. Gaussian Harmonic Analysis, Springer International Publishing, 2019.
[15] Zygmund A. Trigonometric Series. Vol. I, II, 3rd edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2002.