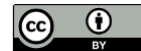# A positively invariant attracting set for a predator-prey model with a non-differentiable functional response

Un conjunto atractor positivamente invariante para un modelo depredador-presa con respuesta funcional no-diferenciable

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Abstract
In this work, the ecologically well-posedness of a predator-prey model is demonstrated by showing that a region of the first quadrant is a positively invariant attracting set for the solutions of that model. The predator-prey model considers logistic-type growth in both populations and a non-differentiable functional response that generalizes previous ones. Due to non-differentiability, there is no uniqueness of solutions, and the standard methodology cannot be applied directly. Topological equivalences, geometrical arguments, and the Poincare-Bendixson theorem are used to obtain our result.

Keywords. Predator-prey model, invariant set, functional response.

Resumen
En el presente trabajo demostramos que un modelo depredador-presa está bien puesto en el sentido ecológico a través de evidenciar que existe un conjunto atractor positivamente invariante para las soluciones del modelo. El modelo depredador-presa en estudio considera crecimiento logístico en ambas poblaciones y una respuesta funcional no-diferenciable. Esta respuesta funcional es una generalización de otras consideradas en anteriores trabajos. Por causa de la no-diferenciabilidad, no podemos garantizar unicidad de soluciones y los métodos clásicos no pueden ser aplicados directamente. Usando equivalencias topológicas, argumentos geométricos y el teorema de Poincaré-Bendixon obtenemos nuestro resultado.

Palabras clave. Modelo depredador-presa, conjunto invariante, respuesta funcional.

1. Introduction. For the well-posedness of a mathematical model in ecology, we must ensure that the size of each population involved in it is always a non-negative number that can be increased to a certain limit. Usually, the non-negativity and boundedness of the functions that represent the population sizes are proved. This work aims to prove that there exists a positively invariant attracting set for the predator-prey model described by the following system of ordinary differential equations

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= r\left(1 - \frac{x}{k}\right)x - q\frac{x^\alpha y^\beta}{x^m + a}, \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= s\left(1 - \frac{y}{nx + c}\right)y
\end{aligned}
\tag{1.1}
$$

where $x(t)$ and $y(t)$ represent respectively the prey and predator population size at the positive time t, and with ecological parameters (see Table 1.1) satisfying

$$0 < r, k, q, s, n, c; \qquad 0 < \alpha, \beta \le 1,$$

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and

$$m = a = 0 \qquad \text{or} \qquad 0 < m, a.$$

More precisely, we prove that any solution of system (1.1) with initial condition in the interior of the first quadrant enters in the region

$$\left\{ (x, y) \in \mathbb{R}^2 \mid 0 \le x \le k,\, c \le v \le nk + c \right\},$$

and remains in it.

System (1.1) considers the following: the predator is generalist with logistic-type growth equation in which the environmental carrying capacity is proportional to the prey population size plus a positive constant that represents an alternative food; the prey natural growth is given by a logistic equation; the action of the predator consuming the prey, called functional response, is represented by the function $h(x) = q \frac{x^\alpha}{x^m + a}$, which is a generalization of the *Rosenzweig* functional response [1] and describes anti-predator behaviors (*prey herd behavior*, *group defense formation*); the depredation rate is given by $h(x) y^\beta$, which is a modification of the *Cobb-Douglas* type function [2, 3] and describes predator mutual interference [4, 5]. We point out that the logistic-type growth of predators is in fact a modified *Leslie-Gower* scheme [6, 7]. Note that for $\alpha = 1$ and $m = 1$, $h(x) = q \frac{x}{x + a}$ is the Holling type II functional response.

System (1.1) with $m = a = 0$ was studied in the past [8, 9]. The dynamics of the general case $0 < m, a$ was not analyze. Therefore, our result is helpful to continue with the research on predator-prey dynamics.

System (1.1) is defined on $\mathbb{R}^2_{+,0}$. But, in general, (1.1) is not differentiable on the coordinate axes. For instance, when we consider α, β, or m less than 1. Therefore the analysis must be careful. We can expect uniqueness of solutions only in the interior of the first quadrant. The equilibrium points of (1.1) are: $(0,0)$, $(k,0)$, and $(0,c)$ over the coordinate axes, and (x_e, y_e) satisfaying the isocline equations

$$r \left(1 - \frac{x}{k} \right) x = q \frac{x^\alpha y^\beta}{x^m + a} \quad \text{and} \quad y = nx + c, \tag{1.2}$$

in the interior of the first quadrant. In fact, the above equations can have several solutions. Moreover, it is clear that

$$\frac{r \left(1 - \frac{x_e}{k} \right) x_e^{1-\alpha} (x_e^m + a)}{(nx_e + c)^\beta} = q > 0,$$

and therefore x_e must belong to $]0, k[$. Consequently, y_e must belong to $]c, nk + c[$.

The paper is organized as follows. In Section 2, we begin by proving the non-negativity and boundedness of solutions. Next, a differentiable system, topologically equivalent to (1.1), is obtained. The nature of the boundary equilibrium points is determined in Section 3. In Section 4, our result follows from the application of the *Poincare-Bendixson* theorem and some geometrical arguments over the topologically equivalent system.

Parameter	Meaning
r	Intrinsic prey growth rate
k	Prey environmental carrying capacity
q	Consuming rate per capita of the predators
s	Intrinsic predator growth rate
n	Measure of the quality of food
c	Amount of alternative food available for the generalist predator
α	Anti-predator behavior constant
β	Mutual interference constant
a	Attack efficiency of predator to prey population
m	Describes the shape of the functional response

Table 1.1: The parameters of the system (1.1) and their ecological meanings.

2. Boundedness and topological equivalences. System (1.1) describes the interaction between prey and predators. hence, the solutions of this system must be bounded and non-negative. These properties are guaranteed by the following proposition.

Proposition 2.1. *The solutions of system (1.1) are non-negative and uniformly bounded. More precisely, for all (x_0, y_0) in $\mathbb{R}^2_{+,0}$, $\phi(t, (x_0, y_0))$ belongs to $\mathbb{R}^2_{+,0}$ and*

$$\|\phi(t, (x_0, y_0))\|_\infty \leq \max\{x_0, y_0, k, nx_0 + c, nk + c\},$$

for all $t \geq 0$, where $\phi(\cdot, (x_0, y_0))$ is the trajectory passing through the initial condition (x_0, y_0).

Proof: The non-negativity of solutions follows from the fact that $dx/dt = 0$ on the positive y-axis and $dy/dt = 0$ on the positive x-axis. Therefore, every solution with initial condition in the first quadrant $\mathbb{R}^2_{+,0}$ remains in it. Next we prove that the solutions of (1.1) are uniformly bounded. Let $(x(t), y(t)) = \phi(t, (x_0, y_0))$ for any positive time t, with $(x(0), y(0)) = (x_0, y_0)$. Since the solutions are non-negative, the term $\frac{q(x(t))^\alpha (y(t))^\beta}{(x(t))^m + a}$ in the first equation of (1.1) is non-negative. Consequently,

$$\frac{dx}{dt}(t) \leq r\left(1 - \frac{x(t)}{k}\right) x(t) \quad \text{for all } t \geq 0.$$

Integrating this inequality over $[0, t]$ we obtain

$$x(t) \leq \frac{kx(0)\exp(rt)}{k + x(0)(\exp(rt) - 1)} \leq \max\{x(0), k\}.$$

Let $x_{\max} = \max\{x(0), K\}$. Since $0 < nx(t) + c \leq nx_{\max} + c$, from the second equation of (1.1) it follows that

$$\frac{dy}{dt}(t) \leq s\left(1 - \frac{y(t)}{nx_{\max} + c}\right) y(t) \quad \text{for all } t \geq 0.$$

Integrating this inequality over $[0, t]$ we obtain

$$y(t) \leq \frac{(nx_{\max} + c) y(0)\exp(st)}{(nx_{\max} + c) + y(0)(\exp(st) - 1)} \leq \max\{y(0), nx_{\max} + c\}.$$

Therefore, we conclude that

$$\|(x(t), y(t))\|_\infty \leq \max\{x(0), y(0), k, nx(0) + c, nk + c\},$$

$\square$

In order to simplify the calculations, we reduce system (1.1) to a polynomial form making a change of variable and a time rescaling. We show this in our next lemma.

Lemma 2.1. *System (1.1) is topologically equivalent on the interior of the first quadrant to the system*

$$\begin{aligned}
\frac{du}{d\tau} &= \left((1 - u)(u^m + A)u - Qu^\alpha v^\beta\right)(u + C), \\
\frac{dv}{d\tau} &= S(u + C - v)(u^m + A)v,
\end{aligned} \tag{2.1}$$

with $Q = qn^\beta / \left(rk^{1+m-\alpha-\beta}\right)$, $C = c/(nk)$, $S = s/r$, and $A = a/k^m$.

Proof: System (2.1) is obtained by applying to (1.1) the change of coordinates and the time rescaling given by the diffeomorphism

$$\varphi : \mathbb{R}^2_+ \times \mathbb{R}_{+,0} \to \mathbb{R}^2_+ \times \mathbb{R}_{+,0}$$

$$\varphi(u, v, \tau) = \left(ku, nkv, \frac{1}{r}\left(u + \frac{c}{nk}\right)\left(u^m + \frac{a}{k^m}\right)\tau\right).$$

Indeed, by the change of coordinates $x = ku$, $y = nkv$ we obtain

$$\begin{aligned}
\frac{du}{dt} &= r(1 - u)u - \left(\frac{qn^\beta}{k^{1+m-\alpha-\beta}}\right)\frac{u^\alpha v^\beta}{u^m + a/k^m} \\
\frac{dv}{dt} &= s\left(1 - \frac{v}{u + c/nk}\right)v
\end{aligned},$$

and applying the time rescaling $t = \frac{1}{r}\left(u + c/nk\right)\left(u^m + a/k^m\right)\tau$ it follows that

$$\frac{du}{d\tau} = \left(r\left(1-u\right)u\left(u^m + \frac{a}{k^m}\right)\tau - \left(\frac{qn^\beta}{rk^{1+m-\alpha-\beta}}\right)u^\alpha v^\beta\right)\left(u + \frac{c}{nk}\right).$$

$$\frac{dv}{d\tau} = \frac{s}{r}\left(u + \frac{c}{nk} - v\right)v\left(u^m + a/k^m\right)$$

From this we obtain (2.1) with the new parameters Q, C, S, and A. The Jacobian matrix of φ is

$$D\varphi\left(u,v,\tau\right) = \begin{pmatrix} k & 0 & 0 \\ 0 & nk & 0 \\ \frac{\tau}{r} & 0 & \frac{1}{r}\left(u + \frac{c}{nk}\right)\left(u^m + \frac{a}{k^m}\right) \end{pmatrix}.$$

Since $\det\left(D\varphi\left(u,v,\tau\right)\right) = nk^2\left(u + c/\left(nk\right)\right)\left(u^m + a/k^m\right)/r$ is positive, the diffeomorphism φ preserves the time orientation. Therefore, (1.1) and (2.1) are topologically equivalent on the interior of the first quadrant. $\qquad\square$

Although system (1.1) is not differentiable, it is possible to obtain an equivalent differentiable system. The equivalence is established inside the first quadrant.

Proposition 2.2. *System (1.1) is topologically equivalent on the interior of the first quadrant to a differentiable system. More precisely, we distinguish two cases. Let $\lambda = \lambda\left(m\right)$ be the exponent function defined as*

$$\lambda\left(m\right) = \begin{cases} 0 & if \quad m = 0 \text{ or } m \geq 1 \\ 1 & if \quad 0 < m < 1 \end{cases}.$$

1. *If $\alpha = 1$, then the system (1.1) is topologically equivalent on $\mathbb{R}_+^2$ to the system*

$$\frac{dz}{d\kappa} = \left(\left(1-z\right)\left(z^m + A\right)z - Qzw\right)\left(z + C\right)z^\lambda$$
$$\frac{dw}{d\kappa} = \beta S\left(z + C - w^{\frac{1}{\beta}}\right)\left(z^m + A\right)wz^\lambda \tag{2.2}$$

   which is differentiable on $\mathbb{R}^2$ and leaves the coordinate axes invariant.

2. *If $\alpha < 1$, then the system (1.1) is topologically equivalent on $\mathbb{R}_+^2$ to the system*

$$\frac{dz}{d\kappa} = \frac{1-\alpha}{1+\lambda}\left(\left(1 - z^{\frac{1+\lambda}{1-\alpha}}\right)\left(z^{m\frac{1+\lambda}{1-\alpha}} + A\right)z^{1+\lambda} - Qw\right)\left(z^{\frac{1+\lambda}{1-\alpha}} + C\right)$$
$$\frac{dw}{d\kappa} = \beta S\left(z^{\frac{1+\lambda}{1-\alpha}} + C - w^{\frac{1}{\beta}}\right)\left(z^{m\frac{1+\lambda}{1-\alpha}} + A\right)wz^\lambda \tag{2.3}$$

   which is differentiable on $\mathbb{R}^2$ and orthogonal to the w-axis.

Proof: Systems (2.2) and (2.3) are obtained by applying to system (2.1) (which is topologically equivalent on $\mathbb{R}_+^2$ to (1.1) by Lemma 2.1) the change of coordinates and the time rescaling given by the diffeomorphism

$$\varphi : \mathbb{R}_+^2 \times \mathbb{R}_{+,0} \to \mathbb{R}_+^2 \times \mathbb{R}_{+,0}$$
$$\varphi\left(z,w,\kappa\right) = \begin{cases} \left(z, w^{\frac{1}{\beta}}, z^\lambda\kappa\right) & \text{if } \alpha = 1 \\ \left(z^{\frac{1+\lambda}{1-\alpha}}, w^{\frac{1}{\beta}}, z^\lambda\kappa\right) & \text{if } \alpha < 1 \end{cases}.$$

If $\alpha = 1$ by the change of coordinates $\left(u,v\right) = \left(z, w^{1/\beta}\right)$ we obtain

$$\frac{dz}{d\tau} = \left(\left(1-z\right)\left(z^m + A\right)z - Qzw\right)\left(z + C\right)$$
$$\frac{dw}{d\tau} = \frac{\beta}{w^{\frac{1}{\beta}-1}}\frac{dv}{d\tau} = \beta S\left(z + C - w^{\frac{1}{\beta}}\right)\left(z^m + A\right)w,$$

and applying the time rescaling $\tau = z^\lambda\kappa$ it follows that

$$\frac{dz}{d\kappa} = \left(\left(1-z\right)\left(z^m + A\right)z - Qzw\right)\left(z + C\right)z^\lambda$$
$$\frac{dw}{d\kappa} = \beta S\left(z + C - w^{\frac{1}{\beta}}\right)\left(z^m + A\right)wz^\lambda.$$

If $\alpha < 1$ by the change of coordinates $(u, v) = \left(z^{(1+\lambda)/(1-\alpha)}, w^{1/\beta} \right)$ we obtain

$$
\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}\tau} &= \frac{(1-\alpha)}{(1+\lambda) z^{\frac{1+\lambda}{1-\alpha}-1}} \frac{\mathrm{d}u}{\mathrm{d}\tau} \\
&= \frac{(1-\alpha)}{(1+\lambda)} \left(\left(1 - z^{\frac{1+\lambda}{1-\alpha}} \right) \left(z^{m\frac{1+\lambda}{1-\alpha}} + A \right) z - Q z^{-\lambda} w \right) \left(z^{\frac{1+\lambda}{1-\alpha}} + C \right), \\
\frac{\mathrm{d}w}{\mathrm{d}\tau} &= \frac{\beta}{w^{\frac{1}{\beta}-1}} \frac{\mathrm{d}v}{\mathrm{d}\tau} = \beta S \left(z^{\frac{1+\lambda}{1-\alpha}} + C - w^{\frac{1}{\beta}} \right) \left(z^{m\frac{1+\lambda}{1-\alpha}} + A \right) w
\end{aligned}
$$

and applying the time rescaling $\tau = z^{\lambda} \kappa$ gives

$$
\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}\kappa} &= \frac{(1-\alpha)}{(1+\lambda)} \left(\left(1 - z^{\frac{1+\lambda}{1-\alpha}} \right) \left(z^{m\frac{1+\lambda}{1-\alpha}} + A \right) z^{1+\lambda} - Qw \right) \left(z^{\frac{1+\lambda}{1-\alpha}} + C \right) \\
\frac{\mathrm{d}w}{\mathrm{d}\kappa} &= \beta S \left(z^{\frac{1+\lambda}{1-\alpha}} + C - w^{\frac{1}{\beta}} \right) \left(z^{m\frac{1+\lambda}{1-\alpha}} + A \right) w z^{\lambda}
\end{aligned}
$$.

The Jacobian matrix of the diffeomorphism φ is

$$
D\varphi(z, w, \kappa) = \begin{pmatrix} D_{11} & 0 & 0 \\ 0 & \frac{1}{\beta} w^{\frac{1}{\beta}-1} & 0 \\ \kappa \lambda z^{\lambda-1} & 0 & z^{\lambda} \end{pmatrix},
$$

where $D_{11} = 1$ when $\alpha = 1$ and $D_{11} = \frac{1+\lambda}{1-\alpha} z^{\frac{1+\lambda}{1-\alpha}-1}$ when $\alpha < 1$. Since $\det\left(D\varphi(z, w, \kappa) \right) = D_{11} w^{1/\beta-1} z^{\lambda}/\beta$ is positive, the diffeomorphism φ preserves the time orientation. Therefore, the initial system (1.1) is topologically equivalent on the interior of the first quadrant to (2.2) (resp. (2.3)) when $\alpha = 1$ (resp. $\alpha < 1$). Note that the exponent function $\lambda = \lambda(m)$ ensures that all exponents in the expanded equations of (2.2) and (2.3) are greater than or equal to 1, and therefore (2.2) and (2.3) are differentiable on $\mathbb{R}^2_{+,0}$. $\square$

3. Equlibrium of boundary points. Here we analyze the nature of equilibrium points over the coordinate axes.

Proposition 3.1. *The equilibrium point $(0, 0)$ is an unstable node. The equilibrium point $(k, 0)$ is a saddle point and its unstable manifold $W^u(k, 0)$ is transversal to the x-axis.*

Proof: Let $F(x, y)$ and $G(x, y)$ be the functions on the right side of (1.1), that is, $\mathrm{d}x/\mathrm{d}t = F(x, y)$ and $\mathrm{d}y/\mathrm{d}t = G(x, y)$. Since

$$
\begin{aligned}
F_x(0, 0) &= \lim_{h \to 0} \frac{F(h, 0) - F(0, 0)}{h} = \lim_{h \to 0} r(1 - h/k) = r, \\
F_y(0, 0) &= \lim_{h \to 0} \frac{F(0, h) - F(0, 0)}{h} = \lim_{h \to 0} 0/h = 0, \\
G_x(0, 0) &= \lim_{h \to 0} \frac{G(h, 0) - G(0, 0)}{h} = \lim_{h \to 0} 0/h = 0, \quad \text{and} \\
G_y(0, 0) &= \lim_{h \to 0} \frac{G(0, h) - G(0, 0)}{h} = \lim_{h \to 0} s(1 - h/c) = s,
\end{aligned}
$$

the Jacobian matrix of (1.1) at $(0, 0)$ is

$$
J_{(0,0)} = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix},
$$

and therefore the equilibrium point $(0, 0)$ is an unstable node. On the other hand, note that if $\beta < 1$ then the partial derivative of F at the equilibrium point $(k, 0)$

$$
F_y(k, 0) = \lim_{h \to 0} \frac{F(k, h) - F(k, 0)}{h} = \lim_{h \to 0} -q \frac{k^{\alpha} h^{\beta-1}}{k^m + a},
$$

diverges. So, we consider the change of variables $(u, v) = \left(u, w^{1/\beta} \right)$ in the topologically equivalent system (2.1). Thus, system (2.1) becomes

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}\tau} &= \left((1 - u)(u^m + A)u - Qu^{\alpha} w \right)(u + C), \\
\frac{\mathrm{d}w}{\mathrm{d}\tau} &= \frac{\beta}{w^{\frac{1}{\beta}-1}} \frac{\mathrm{d}v}{\mathrm{d}\tau} = \beta S \left(u + C - w^{1/\beta} \right)(u^m + A)w,
\end{aligned}
$$

and the equilibrium point $(k, 0)$ in the new coordinates is $(1, 0)$. If we denote by $M(u, w)$ and $N(u, w)$ the functions on the right side of the above equations, that is, $du/d\tau = M(u, w)$ and $dw/d\tau = N(u, w)$, then we have

$$M_u(1, 0) = \lim_{h \to 0} \frac{M(1+h, 0) - M(1, 0)}{h} = -(1 + A)(1 + C),$$

$$M_w(1, 0) = \lim_{h \to 0} \frac{M(1, h) - M(1, 0)}{h} = -Q(1 + C),$$

$$N_u(1, 0) = \lim_{h \to 0} \frac{N(1+h, 0) - N(1, 0)}{h} = 0, \quad \text{and}$$

$$N_w(1, 0) = \lim_{h \to 0} \frac{N(1, h) - N(1, 0)}{h} = \beta S(1 + C)(1 + A).$$

It follows that the Jacobian matrix at $(1, 0)$ is

$$J_{(1,0)} = \begin{pmatrix} -(1 + A)(1 + C) & -Q(1 + C) \\ 0 & \beta S(1 + A)(1 + C) \end{pmatrix}.$$

Since $\det J_{(1,0)} = -\beta S(1 + C)^2(1 + A)^2$ is negative, the equilibrium point $(1, 0)$ is a saddle point. Moreover, we have

$$J_{(1,0)} \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mp (1 + A)(1 + C) \\ 0 \end{pmatrix},$$

and $\frac{dw}{d\tau}(u, 0) = 0$ for all u. Therefore the stable manifold $W^s(1, 0)$ belongs to the z-axis and the unstable manifold $W^u(1, 0)$ must be transversal to the z-axis. $\square$

4. The positively invariant attracting set.
Our main result is the following.

Theorem 4.1. *Let $\mathcal{A}$ be the set defined by*

$$\mathcal{A} = \left\{ (x, y) \in \mathbb{R}^2_{+,0} \mid 0 \le x \le k \text{ and } c \le v \le nk + c \right\}.$$

Then, $\mathcal{A}$ is positively invariant and every solution of system (1.1) with initial condition in the interior of the first quadrant converges to $\mathcal{A}$.

Proof: First, we claim that the set $\mathcal{A}$ is positively invariant. Indeed, since

$$\frac{dy}{dt}(0 \le x \le k, nk + c) = s\left(1 - \frac{nk + c}{nx + c}\right)(nk + c) \le 0,$$

on the upper boundary of $\mathcal{A}$,

$$\frac{dy}{dt}(0 \le x \le k, c) = s\left(1 - \frac{c}{nx + c}\right)c \ge 0,$$

on the lower boundary of $\mathcal{A}$,

$$\frac{dx}{dt}(k, c \le y \le nk + c) = -q\frac{k^\alpha y^\beta}{k^m + a} < 0$$

on the right boundary of $\mathcal{A}$, and

$$\frac{dx}{dt}(0, y) = 0,$$

on the left boundary of $\mathcal{A}$, we conclude that the trajectories that enter in $\mathcal{A}$ remain in it. Moreover, note that $\mathcal{A}$ contains the the equilibrium points $(0, c)$ and (x_e, y_e) (interior equilibrium point).

Let (x_0, y_0) be an arbitrary initial condition in $\mathbb{R}^2_+$. By Proposition (2.1), there exists a positive constant R such that $k, nk + c \le R$ and $\|\phi(t, (x_0, y_0))\|_\infty \le R$ for all $t \ge 0$. It follows that the closed square $\mathcal{S} = [0, R] \times [0, R]$ contains the trajectory $\phi(\cdot, (x_0, y_0))$ $(t \ge 0)$ and the invariant set $\mathcal{A}$. Note that all equilibrium points belong to the square $\mathcal{S}$: $(0, c)$ and (x_e, y_e) are in $\mathcal{A}$; $(0, 0)$ and $(0, k)$ are in $\mathcal{S} \backslash \mathcal{A}$.

If (x_0, y_0) belongs to $\mathcal{A}$, then $\phi(t, (x_0, y_0)) \in \mathcal{A}$ for all $t \ge 0$, because $\mathcal{A}$ is a positively invariant set.

When (x_0, y_0) belongs to $\mathcal{S} \backslash \mathcal{A}$, we consider separately the following two cases:
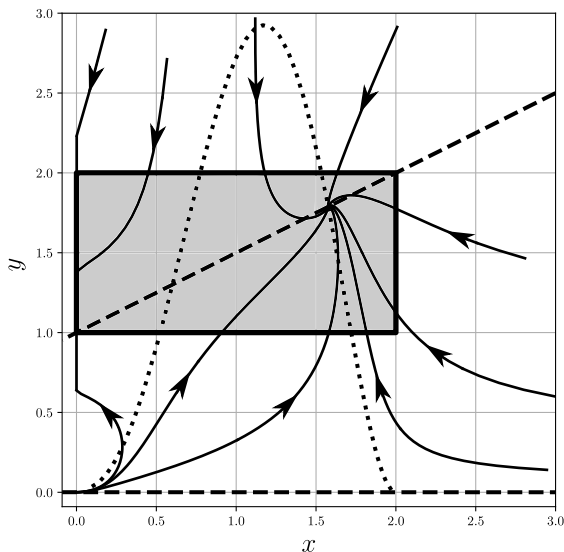
Figure 4.1: Numerical solutions of system (1.1) with parameters $r = 1.5$, $q = 0.5$, $k = 2$, $s = 1$, $n = 0.5$, $c = 1.$, $\alpha = 0.5$, $\beta = 0.5$, $m = 1$, and $a = 0.1$. The set $\mathcal{A}$ is given by the gray rectangle. The isoclines are in dotted and dashed lines.

1. Case $\alpha = 1$. By Proposition (2.2), the initial system (1.1) is topologically equivalent in the interior of the first quadrant to the differentiable system (2.2). In the coordinates of system (2.2) (z-w), we denote by (z_0, w_0) the corresponding initial condition. Let $\psi(\cdot, (z_0, w_0))$ be the trajectory of (2.2) passing through (z_0, w_0). The sets $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{A}}$ denote the close square $\mathcal{S}$ and the invariant set $\mathcal{A}$ in this new system of coordinates. We recall the following:

   - System (2.2) leaves the coordinate axes invariant by Proposition (2.2). Therefore, the trajectory $\psi(\cdot, (z_0, w_0))$ remains on the interior of $\tilde{\mathcal{S}}$.

   - In the new coordinate system, we denote by $\left(\tilde{k}, 0\right)$ and $(0, \tilde{c})$ the boundary equilibrium points, and by (z_e, w_e) the interior equilibrium point. Thus, we have $(0, \tilde{c})$, (z_e, w_e) in $\tilde{\mathcal{A}}$ and $(0, 0)$, $\left(\tilde{k}, 0\right)$ in $\tilde{\mathcal{S}} \backslash \tilde{\mathcal{A}}$.

   - By the topological equivalence and Proposition (3.1), for system (2.2), the equilibrium point $(0, 0)$ is an unstable node and the equilibrium point $\left(\tilde{k}, 0\right)$ is a saddle point, with its unstable manifold $W^{\mathrm{u}}\left(\tilde{k}, 0\right)$ transversal to the z-axis.

   So, applying the Poincaré-Bendixson theorem to the set $\tilde{\mathcal{S}}$, it follows that the ω-limit set of the trajectory $\psi(\cdot, (z_0, w_0))$ is contained in $\tilde{\mathcal{A}}$. Therefore, in the initial coordinate system x-y, the ω-limit set of the trajectory $\phi(\cdot, (x_0, y_0))$ is contained in $\mathcal{A}$ by the topological equivalence.

2. Case $\alpha < 1$. By Proposition (2.2), the initial system (1.1) is topologically equivalent in the interior of the first quadrant to system (2.3), which is differentiable. We use the same notation as in the previous case. Consider the closed set $\mathcal{L} = \tilde{\mathcal{S}} \backslash \left(\operatorname{int} \tilde{\mathcal{A}} \cup \left\{(0, w) \in \tilde{\mathcal{A}}\right\}\right)$ and its boundary $\partial \mathcal{L}$. Since system (2.3) is ortogonal to the w-axis, the point $(0, \tilde{c})$ is not a equilibrium point for it. Hence, the equilibrium points contained in $\mathcal{L}$ are $(0, 0)$ and $\left(\tilde{k}, 0\right)$, and we deduce that the trajectory $\psi(\cdot, (z_0, w_0))$ escapes from $\mathcal{L}$ by the nature of these equilibrium points. That is, there exists a finite time $\kappa^\star > 0$ such that $\psi(\kappa^\star, (z_0, w_0))$ belongs to $\partial \mathcal{L}$.

   - Let $\partial \mathcal{L}_1$ be the part of $\partial \mathcal{L}$ contained in the z-axis. Since

   $$\frac{\mathrm{d}w}{\mathrm{d}\kappa}(z, 0) = \beta S \left(z^{\frac{1+\lambda}{1-\alpha}} + C - 0\right)\left(z^{m\frac{1+\lambda}{1-\alpha}} + A\right)(0) z^\lambda = 0,$$

   the z-axis is an invariant set. Therefore, the trajectory $\psi(\cdot, (z_0, w_0))$ does not cross $\partial \mathcal{L}_1$ by the uniqueness of solutions of the continuously differentiable system (2.3).

- Let $\partial\mathcal{L}_2$ be the part of $\partial\mathcal{L}$ contained in the interior of the first quadrant $\mathbb{R}_+^2$ which coincides with the boundary of $\tilde{\mathcal{S}}$. Since $\mathcal{S}$ contains the trajectory $\phi\left(\cdot,(x_0,y_0)\right)$ for all $t \geq 0$, and systems (1.1) and (2.3) are topologically equivalents in $\mathbb{R}_+^2$, the trajectory $\psi\left(\cdot,(z_0,w_0)\right)$ cannot escape by $\partial\mathcal{L}_2$.

- Let $\partial\mathcal{L}_3$ be the part of $\partial\mathcal{L}$ contained in the interior of the first quadrant $\mathbb{R}_+^2$ which coincides with the boundary of $\tilde{\mathcal{A}}$. If there exist a finite time $\kappa^\star > 0$ such that $\psi\left(\kappa^\star,(z_0,w_0)\right)$ belongs to $\partial\mathcal{L}_3$, since $\mathcal{A}$ is an invariant set, the topological equivalence in $\mathbb{R}_+^2$ implies

$$\psi\left(\kappa^\star,(z_0,w_0)\right) \in \tilde{\mathcal{A}} \quad \text{for all } \kappa \geq \kappa^\star.$$

Consequenlty, the ω-limit set of the trajectory $\phi\left(\cdot,(x_0,y_0)\right)$ is contained in $\mathcal{A}$.

- Let $\partial\mathcal{L}_4$ be the part of $\partial\mathcal{L}$ contained in the positive w-axis. In fact, $\partial\mathcal{L}_4$ is a disjoint set. We have

$$\partial\mathcal{L}_4 = \partial\mathcal{L}_4^{\mathrm{a}} \cup \partial\mathcal{L}_4^{\mathrm{b}},$$

where $\partial\mathcal{L}_4^{\mathrm{a}}$ is the subset of $\partial\mathcal{L}_4$ on the top of $\tilde{\mathcal{A}}$ and $\partial\mathcal{L}_4^{\mathrm{b}}$ is the subset of $\partial\mathcal{L}_4$ at the bottom of $\tilde{\mathcal{A}}$. In the original coordinate system, $\partial\mathcal{L}_4^{\mathrm{a}}$ and $\partial\mathcal{L}_4^{\mathrm{b}}$ are the segments $[(0,0),(0,c)]$ and $[(0,nk+c),(0,R)]$ respectively. On the other hand, we know that (2.3) is orthogonal to the w-axis. More precisely, $\frac{\mathrm{d}z}{\mathrm{d}\kappa}(0,w) = -\frac{1-\alpha}{1+\lambda}QCw$. Now suppose that there exists a time $\kappa^\star > 0$ such that $\psi\left(\kappa^\star,(z_0,w_0)\right)$ belongs to $\partial\mathcal{L}_4$. Then there exists $w^\star \in \partial\mathcal{L}_4$ such that $(0,w^\star) = \psi\left(\kappa^\star,(z_0,w_0)\right)$. In the initial coordinate system, we have a positive value $y^\star$ and positive time $t^\star$ corresponding to $w^\star$ and $\kappa^\star$, and the trajectory $\{\phi\left(t,(x_0,y_0)\right) : 0 \leq t < t^\star\}$ which is topologically equivalent to $\{\psi\left(\kappa,(z_0,w_0)\right) : 0 \leq \kappa < \kappa^\star\}$. So, since

$$\lim_{\kappa\to\kappa^\star} \psi\left(\kappa,(z_0,w_0)\right) = (0,w^\star),$$

it follows that

$$\lim_{t\to t^\star} \phi\left(t,(x_0,y_0)\right) = (0,y^\star)$$

by the topological equivalence. But, by the continuity of solutions of (1.1) (due to Peano's theorem) we have $\phi\left(t,(x_0,y_0)\right) \to \phi\left(t^\star,(x_0,y_0)\right)$ when $t \to t^\star$. Hence $\phi\left(t^\star,(x_0,y_0)\right) = (0,y^\star)$. In other words, the solution passing through the initial condition (x_0,y_0) hits the y-axis in finite time $t^\star$. Moreover, $\phi\left(t,(x_0,y_0)\right)$ belongs to the x-axis for all $t \geq t^\star$ since the x-axis is an invariant set for (1.1). Moreover, since $\frac{\mathrm{d}x}{\mathrm{d}t}(0,y) = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}t}(0,y) = s\left(1 - \frac{y}{c}\right)y$, we can obtain a explicit expression of $\phi\left(\cdot,(x_0,y_0)\right)$ for $t \geq t^\star$, namely

$$\phi\left(t,(x_0,y_0)\right) = \left(0, \frac{cy^\star \exp\left(s\left(t-t^\star\right)\right)}{c + y^\star\left(\exp\left(s\left(t-t^\star\right)\right) - 1\right)}\right) \quad t \geq t^\star.$$

Therefore, $\phi\left(t,(x_0,y_0)\right) \to (0,c)$ when $t \to \infty^+$. That is, the ω-limit set of the trajectory $\phi\left(\cdot,(x_0,y_0)\right)$ is contained in $\mathcal{A}$.

$\square$

5. Conclusions. In this work, we have obtained a positively invariant attracting set for a predator-prey model that considers a non-differentiable functional response, which is a generalization of the *Rosenzweig* functional response [1].

In order to simplify the calculations, a reparameterization and a time rescaling were made, obtaining the topologically equivalent system (2.1). Using this, we proved that our original system is topologically equivalent (on the interior of the first quadrant) to a new differentiable system ((2.2) for $\alpha = 1$ and (2.3) for $\alpha < 1$). Also, the nature of boundary equilibrium points was explored. Finally, the Poincaré-Bendixon theorem and geometrical arguments were applied to the differentiable system (2.2)-(2.3) to obtain our main result. Two cases were examined separately, $\alpha = 1$ and $\alpha < 1$.

With our result, we have demonstrated the ecologically well-posedness of system (1.1).

Although (1.1) was studied for particular parameter values [8, 9], the dynamics of the general case remains unexplored. Therefore, our result represents a first step in the analysis of predator-prey system (1.1).

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