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# Stability of a Leslie-Gower type predator-prey model with a strong Allee effect with delay

# Estabilidad de un modelo depredador-presa tipo Leslie Gower con un efecto Allee fuerte con retardo

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**Abstract**

*In this paper, a modified Leslie-Gower type predator-prey model introducing in prey population growth a delayed strong Allee effect is studied. The Leslie-Gower model with Allee effect has none, one or two positive equilibrium points but the incorporation of a time delay in the growth rate destabilizes the system, breaking the stability when the delay cross a critical point. The existence of a Hopf bifurcation is studied in detail and the numerical simulations confirm the theoretical results showing the different scenarios. We present biological interpretations for species prey-predator type.*

**Keywords** . Allee effect, Leslie-Gower predator-prey model, delay parameter, stability, Hopf bifurcation.

**Resumen**

*En este trabajo se estudia un modelo depredador-presa del tipo Leslie-Gower modificado que introduce en el crecimiento de la población de presas un fuerte efecto Allee retardado. El modelo Leslie-Gower con efecto Allee no tiene ninguno, uno o dos puntos de equilibrio positivos, pero la incorporación de un retardo temporal en la tasa de crecimiento desestabiliza el sistema, rompiendo la estabilidad cuando el retardo cruza un punto crítico. Se estudia en detalle la existencia de una bifurcación de Hopf y las simulaciones numéricas confirman los resultados teóricos mostrando los diferentes escenarios. Presentamos interpretaciones biológicas para especies de tipo presa-predador.*

**Palabras clave**. Efecto Allee, modelo depredador-presa tipo Leslie Gower, parametro de retardo, estabilidad, bifurcación de Hopf.

**1. Introduction.** The Leslie-Gower predator-prey model is characterized by the following aspects:

i) the functional response or predator consumption rate is linear [22, 28], and

ii) the equation for predator is a logistic-type growth function [22, 28].

In the last characteristic, the conventional environmental carrying capacity for predators $K_y$ is expressed by a function of the available prey quantity [28], particularly is assumed proportional to prey abundance $x = x(t)$, that is, $K_y = K(x) = nx$. Denoting by $y = y(t)$ the predator population size in the logistic predator model, the quotient $\frac{y}{nx}$ is called the Leslie-Gower term. It measures the loss in the predator population due to rarity (per capita $\frac{y}{x}$) of its favorite food [1]. So, the model is expressed the following

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autonomous bidimensional differential equation system of Kolmogorov type [9]

$$X_\mu : \begin{cases} \dfrac{dx}{dt} & = & \left(r\left(1-\dfrac{x}{K}\right)-qy\right)\,x \\ \dfrac{dy}{dt} & = & s\left(1-\dfrac{y}{nx}\right)\,y \end{cases}, \tag{1.1}$$

where $x = x(t)$ and $y = y(t)$ represent the prey and predators population size, respectively, with $\mu = (r, K, q, s, n) \in \mathbb{R}^5_+$ and the parameters having different biological meanings [12]. It is well known that system (1) is not defined for $x = 0$ and the equilibrium points are $(K, 0)$ and $\left(\frac{Kr}{r+Knq}, \frac{Krn}{r+Knq}\right)$. The unique positive equilibrium point (equilibrium at the interior of the first quadrant) is globally asymptotically stable [22, 28]. In this work, we introduce two modifications to the prey growth function in the Leslie-Gower model considering

i) the prey is affected by the Allee effect phenomenon, and
ii) a time lag appearing in the intraspecific interaction of prey, representing a delayed prey growth effect [13, 20].

Any mechanism leading to a positive relationship between a component of individual fitness and the number or density of conspecifics can be named as a *Allee effect* [26, 27]; it describes a scenario in which populations at low population sizes, are affected by a positive relationship between population growth rate and density, increasing their likelihood of extinction [6, 7]; it has been denominated in different ways in Population Dynamics [15] and *depensation* in Fisheries Sciences [5, 15]. The main characterization of Allee effect is that the per capita growth rate is positive for low population sizes. This is a common phenomenon in some animal populations and populations may exhibit Allee effect dynamics due to a wide range of biological phenomena (Table 1 in [3] or Table 2.1 in [7]). Recent ecological research suggests that two or more Allee effects can lead to these mechanisms acting simultaneously on a single population (Table 2 in [3]); the combined influence of some of these phenomena is known as *multiple (double) Allee effect* [3].

This phenomenon has become crucial for population dynamics since in fact it has a surprising number of ramifications towards different branches of ecology [7] and the knowledge of this effect on simple models is essential to understand more complicated ones, specially when the weak Allee effect is considered. In this work we employ the simplest form to express the growth rate of a population affected by the Allee effect, which is described by the cubic polynomial differential equation [2, 17]. On the other hand, delay can be incorporated to different aspect of the autonomous models [18, 25]. Here, we consider a single discrete delay $\tau$ modifying also the logistic prey growth. In the work by Çelik [4] the prey growth is malthusian and the delay effect is considered in the prey population that is available to be consumed by predators. Similarly Nindjin et al [23] consider an alternative food for predators in a modified Leslie-Gower model, assuming moreover, the logistic growth for prey. Meanwhile, Li and Li [14] consider the same delay affecting the intraspecific interaction of prey and the available to be consumed by predators. In our knowledge, very little Leslie-Gower models with delay and Allee effect have been studied [8].

Time delays are very important for ecology populations dynamics due to they use to model realistic delay in predator prey interactions [10, 19]. In the other hand, time delays produce changes in the number of critical points or their stability (bifurcations) [21, 24]. In a predator-prey relationship, a common delay is taken in account by gestation of newborn predators as consequence of predation, [29].

**2. The delayed mathematical model and main results.** We consider the modified Leslie-Gower predator-prey model with strong Allee effect and a discrete delay appearing on the prey equation:

$$X_\mu : \begin{cases} \dfrac{dx}{dt} & = & \left(r\left(1-\dfrac{x\,(t-\tau)}{K}\right)(x-m)\,-qy\right)\,x \\ \\ \dfrac{dy}{dt} & = & s\left(1-\dfrac{y}{nx}\right)\,y \end{cases}, \tag{2.1}$$

with the initial conditions: $x\,(\theta) = \phi\,(\theta) \geq 0$, $\theta \in [-\tau, 0]$, $\phi \in C\left([-\tau, 0]\,, \mathbb{R}\right)$, $x\,(0) > 0$, $y\,(0) > 0$, where $x = x(t)$ and $y = y(t)$ represent the prey and predators population sizes for $t \geq 0$, respectively, (measured as number of individuals, biomass or density by area or volume unity), and $\mu = (r, K, q, s, n, m) \in \mathbb{R}^5_+ \times ]0, K[$.

The parameters having different biological meanings, [11] $\tau$ indicates the time lag. The model without Allee effect is analized by Ho et al. [13]; we intent to determine the new dynamics that can appear when the Allee effect with a discrete delay is include. The equilibrium points of system (2) are $(m, 0)$, $(K, 0)$ and $(x_e, y_e)$, with $x_e = A > 0$ and $y_e = nA$, under the assumption that there exists a positive root $A$ of the

quadratic equation: $x^2 + bx + c = 0$, where $b = m + K(1 - qn)$ and $c = -km$. The model without delay is analyzed partially in the book's Kot [17] amd the uniqueness of the limit cycle of this model is given in [11, 12].
Using the change of variables

$$x = Ku, \;\; y = \frac{rK}{q}v,$$

the time rescaling given by $t = \frac{u}{rK}T$ and the delay parameter rescaling $\tau = \frac{u}{rK}\Upsilon$; then we construct the function

$$\varphi : \tilde{\Omega} \times \mathrm{R} \longrightarrow \Omega \times \mathrm{R}$$

such that

$$\varphi(u, v, \Upsilon, T) = \left(Ku, \frac{rK}{q}v, \frac{u}{rK}\Upsilon, \frac{u}{rK}T\right) = (x, y, \tau, t),$$

with

$$\det D\varphi(u, v, \tau, t) > 0.$$

Then, $\varphi$ is a diffeomorphism preserving time orientation: it means that we get a topologically equivalent system $Z_n = \varphi^* \chi_\mu$ with

$$Z_n(u, v) = P(u, v)\frac{\partial}{\partial u} + Q(u, v)\frac{\partial}{\partial v},$$

obtaining the four parameter polynomial differential equations system of delayed Kolgomorov-type predator-prey model given by

$$X_\mu : \begin{cases} \dfrac{dx}{dt} = [(1 - x(t - \tau))(x - M) - y]x^2 \\ \\ \dfrac{dy}{dt} = \beta(\alpha x - y)y \end{cases}, \tag{2.2}$$

where $\alpha = \frac{qn}{r}$, $M = \frac{m}{K}$ and $\beta = \frac{s}{Kqn}$ are the dimensionless parameters, with the following conditions: $x(\theta) = \phi(\theta) \geq 0$, $\theta \in [-\tau, 0]$, $\phi \in C([-\tau, 0], \mathbb{R})$, $x(0) > 0$, $y(0) > 0$.
The vector field $X_\mu$ is a continuous extension of system (2.1) and it is defined at $\tilde{\Omega} = \{(x, y) \in \mathrm{R}^2, x \geq 0, y \geq 0\}$.

**2.1. Existence of equilibrium points.** Without conditions over the parameters, the delayed model (2.2) has a minimum of three equilibrium points

- $P_0 = (0, 0)$,
- $P_1 = (1, 0)$,
- $P_m = (M, 0)$.

Let be $\Delta = \alpha - 1 - M$ and $W^2 = \Delta^2 - 4M$.

**Lemma 2.1.** *Existence of positive equilibrium points*

*1) Suppose $-\Delta > 0$ and $M > 0$ in equation (2.2), then we have*
   *a) If $W^2 > 0$, then it has two singularities in the interior of the first quadrant given by $(x_i^*, \alpha x_i^*)$.*
   *b) If $W^2 = 0$, then the system has a unique positive equilibrium point, given by $(\sqrt{M}, \alpha\sqrt{M})$.*
   *c) If $W^2 < 0$, then the system has no singularities at the interior of the first quadrant.*

*2) If $-\Delta < 0$, the system has no singularities at the interior of the first quadrant.*

*3) When $M = 0$ a particular case of the weak Allee effect, the system has a unique positive equilibrium point.*

*Proof:* The coexisting (positive equilibrium) point(s) is given by
$P_i^* = (x_i^*, y_i^*)$, $i = 1, 2$,
with

$x_i^* = \dfrac{-\Delta \pm \sqrt{\Delta^2 - 4M}}{2}$ and $y_i^* = \alpha x_i^*$.
The number of coexisting points depend on the linear coefficient $\Delta = \alpha - 1 - M$ and $W^2 = \Delta^2 - 4M$.□

In consequence, the delayed model has a minimum of three equilibrium points and a maximum of five.

**Lemma 2.2.** *Every solution of system with the indicated initial conditions exist in the interval* $[0, \infty[$ *and is positive for all* $t \geq 0$.
*Proof:* Let be

$$F_1(t, x, y, \tau) = [(1 - x(t-\tau))(x - M) - y]x^2,$$

$$F_2(t, x, y, \tau) = \beta(\alpha x - y)y,$$

and

$$F_1(t, 0, y, \tau) = 0,$$

$$F_2(t, x, 0, \tau) = 0.$$

By Theorem 3.4 [25], we have that solution $(x(t), y(t))$ of the system satisfy $x(t) > 0$ and $y(t) > 0$ for al $t > s$ where it is defined. □

**3. Local stability.** In this section, we analyze the local stability of the predator-free equilibrium point.
**Theorem 3.1.** *For the equilibrium point* $P_1 = (1, 0)$.
*1) if* $\tau = 0$, $P_1$ *is*
*a) an unstable focus if* $\beta + 1 - M > 0$,
*b) a local stable focus if* $\beta + 1 - M < 0$.
*2) if* $\tau \neq 0$ *and* $2\alpha + 1 - M$, $P_2$ *is a locally unstable focus.*
*Proof:* The characteristic equation of system (2.2) at $P_1 = (1, 0)$ is

$$P(\lambda, \tau) = \lambda^2 + B\lambda + D + (A\lambda + C)e^{-\lambda\tau} = 0,$$

where

$$\begin{cases} A &= 1 - M \\ B &= \beta \\ C &= \beta(1 - M) \\ D &= 2\beta\alpha \end{cases} . \tag{3.1}$$

1) If $\tau = 0$
the polynomial

$$\lambda^2 + (B + A)\lambda + D + C = 0,$$

then
- If $\beta + 1 - M < 0$, $P_1$ is a locally stable focus,
- If $\beta + 1 - M > 0$, $P_1$ is a locally unstable focus,

2) If $\tau \neq 0$
the characteristic equation

$$P(\lambda, \tau) = \lambda^2 + B\lambda + D + (A\lambda + C)e^{-\lambda\tau},$$

as

$$P(0, \tau) = D + C = \beta(2\alpha + 1 - M) < 0,$$

and $P(\lambda, \tau) \to +\infty$ when $\lambda \to +\infty$, then for every $\tau > 0$ there exists at least a $\lambda = \lambda_0 \in \mathrm{R}$ such that $P(\lambda_0, \tau) = 0$.
In consequence, the equilibrium point $P_1 = (1, 0)$ remains a locally unstable focus. □

**4. Stability and Hopf Bifurcation.** The characteristic equation of system (2.2) is

$$\lambda^2 - (a_{11} + a_{22})\lambda - c_{11}\lambda e^{-\lambda\tau} + (a_{11}a_{22} - a_{21}a_{12}) + a_{21}c_{11}e^{-\lambda\tau} = 0, \tag{4.1}$$

where

$$\begin{cases} a_{11} &= x^*[(1-x^*)(3x^* - 2M) - 2\alpha x^*] \\ a_{12} &= -x^{*2} \\ a_{21} &= \beta\alpha x^* \\ a_{22} &= -\beta\alpha x^* \\ c_{11} &= -(x^* - M)x^{*2} \end{cases} . \tag{4.2}$$

**4.1. Stability with absence of delay.** The characteristic equation of system (2.2), with absence of delay is

$$\lambda^2 + (A+B)\lambda + C + D = 0,$$

where $A = -(a_{11} + a_{22})$, $B = -c_{11}$, $C = a_{11}a_{22} - a_{21}a_{12}$, $D = a_{21}c_{11}$.
If $A + B > 0$ and

$$C + D = -\beta(x^*)^2[(1-x^*)(3x^* - 2M) + \alpha x^*(x^* - M - 3)],$$

we get the following results.

**Lemma 4.1.** *If $x^* < \min\{1, \frac{2m}{3}\}$ and $A + B > 0$ then, the roots of the characteristic equation must have negative real parts; then we know that the positive equilibrium $P(x^*, y^*)$ of system (2.1) is locally stable in the absence of delay.* and

**Lemma 4.2.** *If $\max\{1, \frac{2m}{3}\} < x^* < M + 3$ and $A + B > 0$ then, the roots of the characteristic equation must have negative real parts; in consequence the positive equilibrium $P(x^*, y^*)$ of system (2.1) is locally stable in the absence of delay.*

**4.2. Stability in the Model with delay.** For $\tau > 0$, let $\lambda_k(\tau) = \alpha_k + i\omega_k$ be the root of the characteristic equation near the $\tau = \tau_k$ satisfying $\alpha_k = 0$, $\omega_k = \omega_0$, $k = 1, 2, 3, \ldots$.

Let $A_1 = -(a_{11} + a_{22})$, $B_1 = a_{21}c_{11}$, $C_1 = -c_{11}$ and $D_1 = a_{11}a_{22} - a_{21}a_{12}$,
Multiplying by $e^{\lambda\tau}$ on both sides of (4.1), we obtain

$$(\lambda^2 + A_1\lambda + D_1)e^{\lambda\tau} + (B_1 + C_1\lambda) = 0. \tag{4.3}$$

Let denote

$$\begin{cases} \Lambda_1 &= (D_1\omega_0 - \omega_0^3)\cos(\omega_0\tau_0) - (A_1\omega_0^2)\sin(\omega_0\tau_0) \\ \Lambda_2 &= (D_1\omega_0 - \omega_0^3)\sin(\omega_0\tau_0) + (A_1\omega_0^2)\cos(\omega_0\tau_0) \\ \Lambda_3 &= -(2\omega_0\cos(\omega_0\tau_0) + A_1\sin(\omega_0\tau_0) \\ \Lambda_4 &= C_1 + A_1\cos(\omega_0\tau_0) \end{cases} . \tag{4.4}$$

**Theorem 4.1.** *Under the assumption*

$$\Lambda_3\Lambda_1 + \Lambda_2\Lambda_4 \neq 0,$$

*we have*

$$\frac{dRe\lambda_k(\tau)}{d\tau} > 0, k = 0, 1, 2, \ldots$$

*and the transversality condition holds.*

*Thus system (2.1) undergoes Hopf bifurcations at positive equilibrium* $P^* = (x^*, y^*)$ *for* $\tau = \tau_k, k = 0, 1, 2, \ldots$.

*Proof:* Let us assume a purely imaginary solution of (4.3) in the form $\lambda = i\omega$, $\omega > 0$,then we have

$$\begin{cases} ((i\omega)^2 + A_1(i\omega) + D_1)e^{i\omega\tau} + (B_1 + C_1(i\omega)) & = & 0, \\ (-\omega^2 + (A_1\omega)i + D_1)(\cos(\omega\tau) + i\sin(\omega\tau)) + (B_1 + (C_1\omega)i) & = & 0. \end{cases}$$

Separating real and imaginary parts, we get

$$\begin{cases} -\omega^2\cos(\omega\tau) - A_1\omega\sin(\omega\tau) + D_1\cos(\omega\tau) + B_1 & = & 0, \\ A_1\omega\cos(\omega\tau) - \omega^2\sin(\omega\tau) + D_1\sin(\omega\tau) + C_1\omega & = & 0. \end{cases}$$

$$\begin{cases} (D_1 - \omega^2)\cos(\omega\tau) - A_1\omega\sin(\omega\tau) & = & -B_1, \\ (D_1 - \omega^2)\sin(\omega\tau) + A_1\omega\cos(\omega\tau) & = & -C_1\omega. \end{cases} \tag{4.5}$$

Solving the above equations, we get

$$\begin{cases} \sin(\omega\tau) & = & \dfrac{-(D_1 - \omega^2)C_1\omega + A_1B_1\omega}{(D_1 - \omega^2)^2 + A_1^2\omega^2}, \\ \\ \cos(\omega\tau) & = & \dfrac{(\omega^2 - D_1)B_1 - A_1C_1\omega^2}{(D_1 - \omega^2)^2 + A_1^2\omega^2}. \end{cases}$$

Eliminating trigonometric functions in (4.5), we get the following eight degree equation

$$\omega^8 + e_3\omega^6 + e_2\omega^4 + e_1\omega^2 + e_0 = 0, \tag{4.6}$$

where

$$\begin{cases} e_3 & = & -C_1^2 + 2A_1^2 - 4D_1, \\ \\ e_2 & = & -B_1^2 + 2D_1(C_1^2 - 2A_1^2) - {A_1}^2{C_1}^2 + {A_1}^4 + 6{D_1}^2, \\ \\ e_1 & = & 2D_1{B_1}^2 - A_12{B_1}^2 - D_12({C_1}^2 - 2{A_1}^2) - 4{D_1}^3, \\ \\ e_0 & = & -{D_1}^2{B_1}^2 + {D_1}^4. \end{cases}$$

Let $\nu = \omega^2$, then (4.6) becomes

$$\nu^4 + e_3\nu^3 + e_2\nu^2 + e_1\nu + e_0 = 0. \tag{4.7}$$

The equation (4.7) has at least one positive real root $\nu$ if $2A_1^2 < C_1^2 + 4D_1$ polyroots. Without loss of generality, we assume that (4.7) has four real positive roots, which are called by $\nu_1, \nu_2, \nu_3, \nu_4$ respectively. Then (4.6)has four positive roots $\omega_k = \sqrt{\nu_k}$, $k = 1, 2, 3, 4$.
Therefore,

$$\tau_k^{(j)} = \frac{1}{\omega_k}\arccos\left(\frac{(B_1 - A_1C_1)\omega_k^2 - D_1B_1}{(D_1 - \omega_k^2)^2 + A_1^2\omega_k^2}\right) + \frac{2j\pi}{\omega_k}, \tag{4.8}$$

where $k = 1, 2, 3, 4$ and $j = 0, 1, 2, \ldots$. Then $\pm i\omega_k$ are a pair of imaginary roots of (4.8) with $\tau = \tau_k^{(j)}$.

Define

$$\tau_0 = \tau_k^{(0)} = \min\{\tau_k^{(j)}\}, \tag{4.9}$$

where $\omega_0 = \omega_{k0}$ and $k = 1, 2, 3, 4$. Then, we get

$$\tau_j = \frac{1}{\omega_0} \arccos\left(\frac{(B_1 - A_1C_1)\omega_0^2 - D_1B_1}{(D_1 - \omega_0^2)^2 + A_1^2\omega_0^2}\right) + \frac{2j\pi}{\omega_0}. \tag{4.10}$$

Differentiating the two sides of the characteristic equation (4.3) with respect to $\tau$, we get

$$(2\lambda + A_1)e^{\lambda\tau}\frac{d\lambda}{d\tau} + (\lambda^2 + A_1\lambda + D_1)e^{\lambda\tau}\left(\frac{d\lambda}{d\tau} + \lambda\right) + C_1\frac{d\lambda}{d\tau} = 0,$$

then

$$\frac{d\lambda}{d\tau} = \frac{-\lambda(\lambda^2 + A_1\lambda + D_1)e^{\lambda\tau}}{(2\lambda + A_1)e^{\lambda\tau} + \tau(\lambda^2 + A_1\lambda + D_1)e^{\lambda\tau}},$$

and

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = -\frac{(2\lambda + A_1)e^{\lambda\tau} + C_1}{\lambda(\lambda^2 + A_1\lambda + D_1)e^{\lambda\tau}} - \frac{\tau}{\lambda}.$$

Thus

$$Re\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{\Lambda_3\Lambda_1 + \Lambda_2\Lambda_4}{\Lambda_1^2\Lambda_2^2},$$

where

$$\begin{cases} \Lambda_1 &= (D_1\omega_0 - \omega_0^3)\cos(\omega_0\tau_0) - (A_1\omega_0^2)\sin(\omega_0\tau_0), \\ \Lambda_2 &= (D_1\omega_0 - \omega_0^3)\sin(\omega_0\tau_0) + (A_1\omega_0^2)\cos(\omega_0\tau_0), \\ \Lambda_3 &= -(2\omega_0\cos(\omega_0\tau_0) + A_1\sin(\omega_0\tau_0), \\ \Lambda_4 &= C_1 + A_1\cos(\omega_0\tau_0). \end{cases}$$

Noting that

$$sign\, Re\left[\frac{d\lambda(\tau_0)}{d\tau}\right] = sign\left[\frac{dRe\,\lambda(\tau_0)}{d\tau}\right]^{-1}.$$

By the hypothesis

$$\Lambda_3\Lambda_1 + \Lambda_2\Lambda_4 \neq 0$$

and the corollary (2.1), we conclude that the transversality condition is satisfied, in consequence the system (2.1) undergoes a Hopf bifurcation around $P^* = (x^*, y^*)$ for $\tau = \tau_0$ □

We can suummarize the above results in the following theorem on stability and Hopf bifurcation.

**Theorem 4.2.** *For system (2.1)*

*(i) If $\tau \in [0, \tau_0)$, then the equilibrium $P^*$ is assymptotically stable.*
*(ii) If $\tau > \tau_0$, then the equilibrium $P^*$ is unstable.*
*(iii) It has a branch of periodic solution bifurcation from zero solution near $\tau = \tau_0$. It means that the system undergoes a Hopf bifurcation around $P^*$ for $\tau = \tau_0$.*

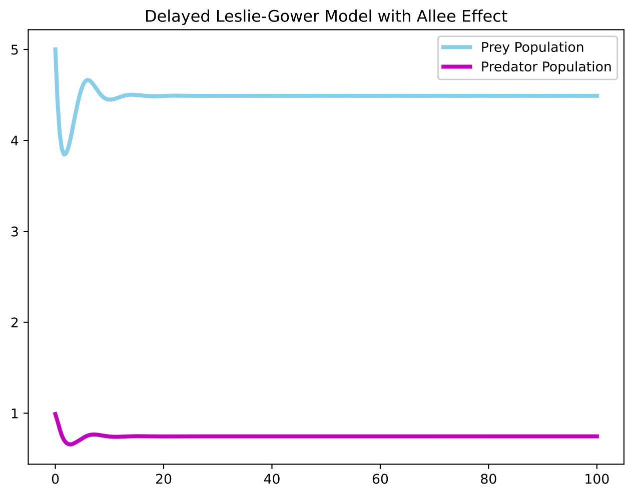


Figure 5.1: Behavior of the system (2.2) without delay i.e. $\tau = 0$.

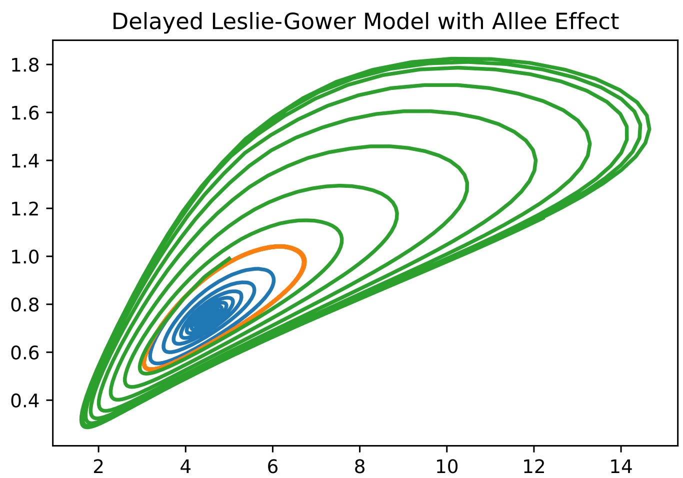


Figure 5.2: Behavior of the system (2.2) with delay $\tau = 1.5$ (green), $\tau = 1.3070$ (red), $\tau = 1.1$ (blue).

**5. Numerical simulations.** In order to reinforced our results, here we show some simulations.

The following parameters values are taken for all the graphics:

$$r = .2; K = 1; m = 0.1; q = 0.3; s = 1; n = 0.166.$$

The positive equilibrium point is always stable in the absence of delay, see Figure 5.1. The delay $\tau$, intended as a bifurcation parameter keeping fixed the values of the other parameters and initial conditions. The positive equilibrium point is asymptotically stable when delay is less than critical value, i.e. when $\tau < 1,3$, see Figure 5.3. When delay cross the critical value, i.e. when $\tau \leq 1,3$, the positive equilibrium point losses stability and exhibits complex dynamics in the form of a Hopf bifurcation ($\tau = 1.3070$, $\tau = 1.5$), see Figure 5.4 and Figure 5.5.

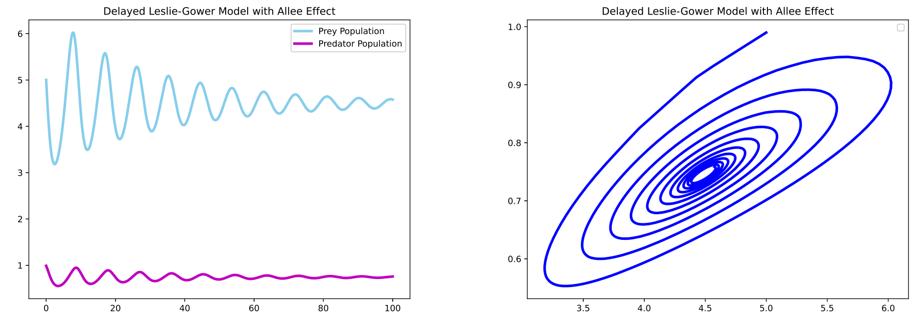


Figure 5.3: Behavior of the system (2.2) with delay $\tau = 1.1$

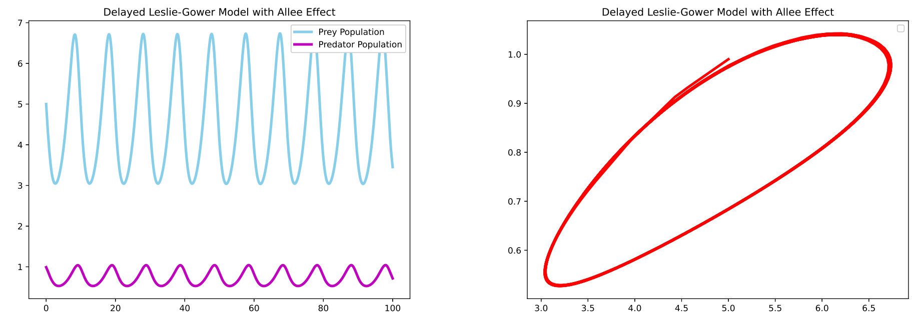


Figure 5.4: Behavior of the system (2.2) with delay $\tau = 1.3$

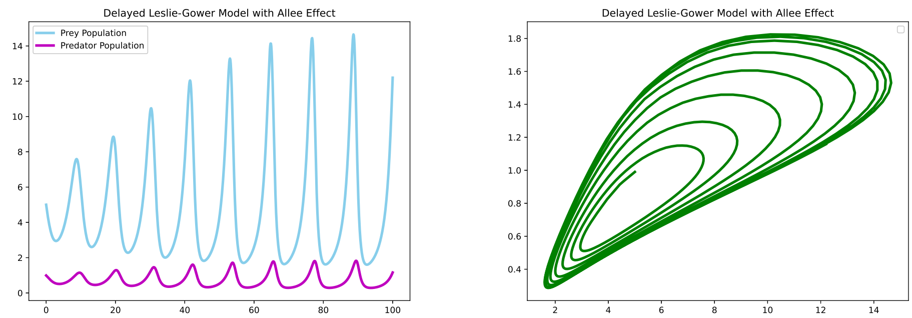


Figure 5.5: Behavior of the system (2.2) with delay $\tau = 1.5$

**6. Conclusions.** We have studied a simple Leslie-Gower model introducing a strong Allee effect in the prey population and a delayed prey growth effect. Our main results show the delay parameter as a bifurcation parameter. The main biological approaches due to our model is that the time delays the finding mates action at low populations in prey population influence on the stability of the predator-prey populations, such as stability switches, bifurcation and so on.

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# References


[1] Aziz-Alaoui MA, Daher Okiye M. Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes, Applied Mathematics Letters 2003; 16:1069-1075.

[2] Bazykin AD. Nonlinear Dynamics of interacting populations, World Scientific 1998.

[3] Berec L, Angulo E, Courchamp F. Multiple Allee effects and population management, Trends in Ecology and Evolution. 2007; 22:185-191.

[4] Çelik C. Hopf bifurcation of a ratio-dependent predator-prey system with time delay, Chaos, Solitons and Fractals. 2009; 42:1474-1484.

[5] Clark CW. Mathematical Bioeconomics: The Mathematics of Management. (2nd ed), John Wiley and Sons Inc. 1990.

[6] Courchamp F, Clutton-Brock T, Grenfell B. Inverse dependence and the Allee effect, Trends in Ecology and Evolution. 1999; 14:405-410.

[7] Courchamp F, Berec L, Gascoigne J. Allee effects in Ecology and Conservation, Oxford University Press 2008.

[8] Das K, Chakraborty M, Chakraborty K, Kar TK. Modelling and analysis of a multiple delayed exploited ecosystem towards coexistence perspective, Nonlinear Dynamics 2014; 78(1):505-523. DOI 10.1007/s11071-014-1457-3.

[9] Freedman HI.Deterministic Mathematical Model in Population Ecology. Marcel Dekker; 1980.

[10] Freedman HI, Rao VSH. The tradeoff between mutual interference and time lag in predator prey models., Bull. Math. Biol. 1983; 45:991-1004.

[11] González-Olivares E, Mena-Lorca J, Rojas-Palma A, Flores JD. Dynamical complexities in the Leslie-Gower predator-prey model as consequences of the Allee effect on prey, Applied Mathematical Modelling. 2011; 35:366-381.

[12] González-Olivares E, Mena-Lorca J, Rojas-Palma A, Flores JD. Erratum to "Dynamical complexities in the Leslie-Gower predator-prey model as consequences of the Allee effect on prey". [Appl. Math. Modell. (2011) 366-381], Applied Mathematical Modelling. 2012; 36:860-862.

[13] Ho C-P, Lin C-H, Huang H-N. On the Lyapunov functional of the Leslie-Gower predator-prey models with time-delay and Hollings's functional response, Electronic Journal of Qualitative Theory of Differential Equations. 2012; 35:1-27.

[14] Li Y, Li C. Stability and Hopf bifurcation analysis on a delayed Leslie-Gower predator-prey system incorporating a prey refuge, Appl. Math. and Comp. 2013; 219:4576-4589.

[15] Liermann M, Hilborn R. Depensation: evidence, models and implications, Fish and Fisheries. 2001; 2:33-58.

[16] Hale JK, Lunel SMV. Introduction to functional differential equations (Vol. 99). Springer Science and Business Media. 2013.

[17] Kot M. Elements of Mathematical Ecology, Cambridge University Press 2001.

[18] Kuang Y. Delay differential equations with applications in Populations Dynamics, Academic Press, Inc. 1993.

[19] Kuang Y, Beretta E. Global qualitative analysis of a ratio-dependent predator-prey system, J. Mat. Biol. 1998; 36:389-406.

[20] Kumar P, Raj S. Modelling and analysis of prey-predator model involving predation of mature prey using delay differential equations. Numerical Algebra, Control and Optimization. 2021.

[21] Liu J. Bifurcation analysis in a stage-structured predator-prey model with maturation delay, Int.J. Mat. Biol. 2014, Vol 7(4):(1-17)

[22] May RM. Stability and complexity in model ecosystems (2nd edition), Princeton University Press 2001.

[23] Nindjin AF, Aziz-Alaoui MA, Cadivel M. Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay, Nonlinear Analysis: RealWorld Applications. 2006; 7:1104-1118.

[24] Pal PJ, Saha T, Sen M, Banerjee M. A delayed predator-prey model with strong Allee effect in prey population growth, Nonlinear Dynamics 2011; 68(1):23-42.

[25] Smith H. An Introduction to Delay Differential Equations with applications to the Life Sciences, Springer 2011.

[26] Stephens PA, Sutherland WJ. Consequences of the Allee effect for behaviour, ecology and conservation. Trends in Ecology and Evolution. 1999; 14:401-405.

[27] Stephens PA, Sutherland WJ, Freckleton RP. What is the Allee effect?, Oikos. 1999; 87:185-190.

[28] Turchin P. Complex population dynamics. A theoretical/empirical synthesis, Monographs in Population Biology 35 Princeton University Press, 2003.

[29] Xua R, Gan Q, Ma Z. Stability and bifurcation analysis on a ratio-dependent predator-prey model with time delay, J. Comput. Appl. Math. 2009; 230:187-203.

[30] Yue Y, Lui H, Wei Y, Ma M, Zhang K. Dynamic study of a Predator Prey Model with Weak Allee Effect and Delay, A. Math. Physics. 2019; 2019:1-15.