Journal homepage http://revistas.unitru.edu.pe/index.php/SSMM

SELECCIONES MATEMÁTICAS
Universidad Nacional de Trujillo
ISSN: 2411-1783 (Online)
2022; Vol. 9(2): 243-257.

Relatives Geometries

Geometrías relativas

Armando M. V. Corro and Marcelo Lopes Ferro

Received, May. 30, 2022 *Accepted, Set. 23, 2022*

How to cite this article:
Corro VA, Ferro ML. *Relatives Geometries*. Selecciones Matemáticas. 2022;9(2):243–257. http://dx.doi.org/10.17268/sel.mat.2022.02.03

Abstract
In this paper we consider M a fixed hypersurface in Euclidean space and we introduce two types of spaces relative to M, of type I and type II. We observe that when M is a hyperplane, the two geometries coincides with the isotropic geometry. By applying the theory to a Dupin hypersurface M, we define a relative Dupin hypersurface $\mathcal{M}$ of type I and type II , we provide necessary and sufficient conditions for a relative hypersurface $\mathcal{M}$ to be relative Dupin parametrized by relative lines of curvature, in both spaces. Moreover, we provides a relationship between the Dupin hypersurfaces locally associated to M by a Ribaucour transformation and the type II Dupin hypersurfaces relative $\mathcal{M}$. We provide explicit examples of the Dupin hypersurface relative to a hyperplane, torus, $S^1 \times \mathbb{R}^{n-1}$ and $S^2 \times \mathbb{R}^{n-2}$, in both spaces.
Keywords. Relative hypersurface, Relative Dupin hypersurface , Isotropic geometry, Ribaucour transformations.

Resumen
En este artículo consideramos M una hipersuperficie fija en el espacio euclidiano e introducimos dos tipos de espacios relativos a M de tipo I y tipo II. Observamos que cuando M es un hiperplano, las geometrías coinciden con la geometría isotrópica. Aplicando la teoría a una hipersuperficie de Dupin M, definimos una hipersuperficie de Dupin relativa $\mathcal{M}$ de tipo I y tipo II, proporcionamos condiciones necesarias y suficientes para que una hipersuperficie relativa $\mathcal{M}$ sea Dupin relativo parametrizado por líneas relativas de curvatura, en ambos espacios. Además, proporcionamos una relación entre las hipersuperficies de Dupin asociadas localmente a M mediante una transformación de Ribaucour y las hipersuperficies de Dupin relativas $\mathcal{M}$ de tipo II . Proporcionamos ejemplos explícitos de la hipersuperficie de Dupin relativa a un hiperplano, toroide, $S^1 \times \mathbb{R}^{n-1}$ y $S^2 \times \mathbb{R}^{n-2}$, en ambos espacios.
Palabras clave. Hipersuperficie relativa, Hipersuperficie de Dupin relativa, geometría isotrópica, Transformación de Ribaucour.

1. Introduction. The isotropic geometry introduced by Strubecker in [13], [14] and [15], and developed by several authors, study of the properties invariant by the action of the 6-parameter group G_6 in $\mathbb{R}^3$

$$\begin{aligned} x' &= a + x\cos\phi - y\sin\phi \\ y' &= b + x\sin\phi + y\cos\phi \\ z' &= c + c_1 x + c_2 y + z, \end{aligned}$$

where a, b, c, c_1, c_2, $\phi \in \mathbb{R}$.
In other word, G_6 is the group of rigid motions. Notice that on the xy-plane this geometry looks exactly like the plane Euclidean geometry $\mathbb{R}^2$. The projection of a vector $u = (u_1, u_2, u_3)$ on the xy-plane is the top view of u and we shall denote it by $\widetilde{u} = (u_1, u_2, 0)$. The top view concet plays a fundamental role in

*IME, Universidade Federal de Goiás, Caixa Postal 131, 74001-970, Goiânia, GO, Brazil. (corro@ufg.br).
†IME, Universidade Federal de Goiás, Caixa Postal 131, 74001-970, Goiânia, GO, Brazil. (marceloferro@ufg.br).

the isotropic space $\mathbb{I}^3$, since the z-direction is preserved by the action of G_6. A line with this direction is called an isotropic line and a plane that contained an isotropic line is said to be an isotropic plane. One may introduce a isotropic inner product between two vectors u, v as

$$\langle (u_1\,,\,u_2\,,\,u_3)\,,\,(v_1\,,\,v_2\,,\,v_3)\rangle_Z = u_1v_1 + u_2v_2,$$

from which the isotropic distance is defined as

$$d_K(A\,,\,B) = \sqrt{\langle A-B\,,\,A-B\rangle_Z}.$$

The inner product and distance above are just the plane Euclidean counterparties of the top views $\widetilde{u}$ and $\widetilde{v}$. In addition, since the isotropic metric is degenerate, the distance from a point $A = (a_1, a_2, a_3)$ to $v = (b_1, b_2, b_3)$ is zero if $\widetilde{A} = \widetilde{B}$. In this cases, one may define a codistance by

$$Cd_Z(A\,,\,B) = |b_3 - a_3|,$$

which is then preserved by G_6. To study the geometry of a surface in isotropic space, it is considered a surface as a graph of a function, given by

$$X(u_1\,,\,u_2) = (u_1\,,\,u_2\,,\,h(u_1\,,\,u_2)).$$

The isotropic Gauss map, is given by

$$N_I = (-h_{,1}\,,\,-h_{,2}\,,\,1).$$

The coefficients of the first and second fundamental forms are defined by

$$g_{ij} \;=\; \delta_{ij}, \qquad b_{ij} \;=\; \langle X_{,ij}\,,\,N_I\rangle.$$

In this work motivated by isotropic geometry, we generalize the idea of isotropic distance. Let $\mathbb{R}^{n+1}$ be the Euclidean space with the usual metric $\langle\,,\,\rangle$ and consider M a hypersurface fixed in $\mathbb{R}^{n+1}$, with Gauss map N. We introduce the space relative to M, as being

$$\mathbb{R}^{n+1}_M \;=\; \{p + tN \,|\, p\in M,\, t\in\mathbb{R},\, p_1 + t_1N \neq p_2 + t_2N,\, p_1,\, p_2 \in M,\, p_1\neq p_2,\, t_1,\, t_2\in\mathbb{R}\}.$$

We define the space relative to M of the type I, as being the space $\mathbb{R}^{n+1}_M$, with the distance defined by

$$d_{R_1}\big(p_1 + t_1N(p_1)\,,\,p_2 + t_2N(p_2)\big) \;=\; d\big(p_1\,,\,p_2\big), \tag{1.1}$$

where $d\big(p_1\,,\,p_2\big)$ is the distance between p_1 and p_2, considering M as a metric space.
On the other hand, let $q = p\,+\,tN(p)$ be a point in $\mathbb{R}^{n+1}_M$, we can consider $T_q\mathbb{R}^{n+1}_M = T_p\mathbb{R}^{n+1}$, and we define the space relative to M of the type II, as being the space $\mathbb{R}^{n+1}_M$, with the metric defined by

$$\langle V_q\,,\,W_q\rangle_{R_2} \;=\; \langle V^T_p\,,\,W^T_p\rangle, \tag{1.2}$$

where V^T_p denotes the orthogonal projection of V_q on T_pM.

Let $\mathcal{M}$ be a hypersurface in $\mathbb{R}^{n+1}_M$, locally, M can be parametrized by $X : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}_M$

$$X(u) \;=\; Y(u) + h(u)N(u), \quad u\in U, \tag{1.3}$$

where $h : U \subset \mathbb{R}^n \to R$ is a real function and $Y : U \to M$ is a local parameterization of the M. In this case, the function h is called the height function.

Let $\mathcal{N}$ be the Gauss map of the $\mathcal{M}$. We define the relative Gauss map by

$$N_R(u) \;=\; \frac{\mathcal{N}(u)}{\langle \mathcal{N}(u)\,,\,N(u)\rangle}. \tag{1.4}$$

Considering $\mathcal{M}$ to be a hypersurface in the relative space of the type I, the coefficients of the first and second fundamental forms of the $\mathcal{M}$ relative to M are given by

$$g^1_{ij}(u) \;=\; \langle Y_{,i}\,,\,Y_{,j}\rangle, \qquad b_{ij} \;=\; \langle X_{,ij}(u)\,,\,N_R(u)\rangle. \tag{1.5}$$

Moreover, we define the relative Weingartem matrix W of the $\mathcal{M}$ by $W = -BG_1^{-1}$, where B is the matrix of coefficients of the second fundamental form and G_1 is the matrix of coefficients of the first fundamental

form.
Analogously, considering $\mathcal{M}$ to be a hypersurface in the relative space of the type II, the coefficients of the first and second fundamental forms of the $\mathcal{M}$ relative to M are given by

$$g^2_{ij}(u) = \langle Y,_i + hN,_i \, , \, Y,_j + hN,_j \rangle, \qquad b_{ij} = \langle X,_{ij}(u) \, , \, N_R(u) \rangle. \tag{1.6}$$

We define the relative Weingartem matrix W of the $\mathcal{M}$ by $W = -BG_2^{-1}$, where B and G_2 are, respectively, the matrices of the coefficients of the first and second fundamental forms of the $\mathcal{M}$ relative to M.

We observe that in the two relative spaces, $\mathcal{M}$ has the same relative normal map and the same second fundamental form, what are different are its first fundamental forms.

Ribaucour transformations for hypersurfaces, parametrized by lines of curvature, were classically studied by Bianchi [4]. They can be applied to obtain surfaces of constant Gaussian curvature and surfaces of constant mean curvature, from a given such surface, respectively, with constant Gaussian curvature and constant mean curvature. The first application of this method to minimal and cmc surfaces in $\mathbb{R}^3$ was obtained by Corro, Ferreira, and Tenenblat in [6]-[8].

Dupin's surfaces in Euclidean space are classified. There are several equivalent definitions of Dupin cyclides, for example, in Euclidean space, they can be defined as any inversion of a torus, cylinder or double cone, i.e, Dupin cyclide is invariant under Möbius transformations. Classically the cyclides of Dupin were characterized by the property that both sheets of the focal set are curves. Another equivalent definition says that such surfaces can also be given as surfaces that are the envelope of two families at 1-parameter spheres (including planes as degenerate spheres). For more on Dupin cyclides see [2] and [3].

We consider M a fixed hypersurface in Euclidean space and we introduce two types of spaces relative to M, of type I and type II. We observe that when M is a hyperplane, the two geometries coincides with the isotropic geometry. By applying the theory to a Dupin hypersurface M, we define a relative Dupin hypersurface $\mathcal{M}$ of type I and type II , we provide necessary and sufficient conditions for a relative hypersurface $\mathcal{M}$ to be relative Dupin parametrized by relative lines of curvature, in both spaces. Moreover, we provides a relationship between the Dupin hypersurfaces locally associated to M by a Ribaucour transformation and the type II Dupin hypersurfaces relative $\mathcal{M}$. We provide explicit examples of the Dupin hypersurface relative to a hyperplane, torus, $S^1 \times \mathbb{R}^{n-1}$ and $S^2 \times \mathbb{R}^{n-2}$, in both spaces. This work is organized as follows. In section 1, we provide the basic local theory of the Ribaucour transformation and Dupin hypersurface definition. In section 2, we provide a local characterization of the hypersurfaces relatives, to a fixed hypersurface parametrized by lines of curvature, in both types. Moreover, we provide the relative Weingarten matrix and a necessary and sufficient condition for a relative hypersurface $\mathcal{M}$ to be a relative Dupin parametrized by lines of relative curvature, in both types. In section 3, we highlight the type I relative Dupin hypersurfaces and we generate families of type I Dupin hypersurface relative to a hyperplane, a torus, $\mathbb{S}^1 \times \mathbb{R}^{n-1}$ and $\mathbb{S}^2 \times \mathbb{R}^{n-2}$. In section 4, we highlight the type II relative Dupin hypersurfaces, we provides a relationship between the Dupin hypersurfaces locally associated to M by a Ribaucour transformation and the type II Dupin hypersurfaces relative to M. Moreover, we generate families of type II Dupin hypersurface relative to a hyperplane, a torus, $\mathbb{S}^1 \times \mathbb{R}^{n-1}$ and $\mathbb{S}^2 \times \mathbb{R}^{n-2}$.

2. Preliminaries. This section contains definitions and basic concepts that will be used in later sections.

A sphere congruence is an n-parameter family of spheres whose centers lie on an n-dimensional manifold M_0 contained in $\mathbb{R}^{n+1}$. Locally, we may condider M_0 parametrized by $X_0 : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$. For each $u \in U$, we consider a sphere centered at $X_0(u)$ with radius $r(u)$, where r is a differentiable real funcition. Two hypersurfaces M and $\widetilde{M}$ are said to be associated by a sphere congruence if there is a difeomorphism $\psi : M \to \widetilde{M}$, such that at corresponding points p and $\psi(p)$ the manifolds are tangent to the same sphere of the sphere congruence. A special case occurs when ψ preserves lines of curvature.

Let M and $\widetilde{M}$ be orientable hypersurfaces of $\mathbb{R}^{n+1}$. We denote by N and $\widetilde{N}$ their Gauss map. We say that M and $\widetilde{M}$ are associated by a Ribaucour transformation, if and only if, there exists a differentiable function h defined on M and a diffeomorphism $\psi : M \to \widetilde{M}$

(a) for all $p \in M$, $p + h(p)N(p) = \psi(p) + h(p)\widetilde{N}(\psi(p))$, where $\widetilde{N}$ is the Gauss map of $\widetilde{M}$.
(b) The subset $p + h(p)N(p)$, $p \in M$, is a n-dimensional submanifold.
(c) ψ preserves lines of curvature.

We say that M and $\widetilde{M}$ are locally associated by a Ribaucour transformation if, for all $p \in M$, there exists a neighborhood of p in M which is associated by a Ribaucour transformation to an open subset of $\widetilde{M}$. Similarly, one may consider the notion of parametrized hypersurfaces locally associated by a Ribaucour transformation.

A hypersurface $M \subset \mathbb{R}^{n+1}$ is a Dupin submanifold if its principal curvatures are constant along the corresponding lines of curvature. Whenever, the principal curvatures are constant, M is a called an isoparametric submanifold.

Let M be an orientable hypersurface in $\mathbb{R}^{n+1}$, N its Gauss map and suppose that M has an orthogonal parameterization by lines of curvature $Y : U \subset \mathbb{R}^n \to M$, with principal curvatures $-\lambda_i$, $1 \le i \le n$. Then,

$$\langle Y,_i , Y,_j \rangle = \delta_{ij} L_{ij}, \qquad 1 \le i, j \le n, \tag{2.1}$$

$$N,_i = \lambda_i Y,_i \,, \qquad 1 \le i \le n, \tag{2.2}$$

$$\langle N, N \rangle = 1, \tag{2.3}$$

where $\langle , \rangle$ denotes the Euclidean metric in $\mathbb{R}^{n+1}$.

Moreover for $1 \le i \ne j \le n$, we have

$$Y,_{ij} - \Gamma^i_{ij} Y,_i - \Gamma^j_{ij} Y,_j = 0, \tag{2.4}$$

$$\big(\lambda_j - \lambda_i\big)\Gamma^i_{ij} = \lambda_{i,j}, \tag{2.5}$$

where Γ^i_{ij} are the Christoffel symbols.

The Christoffel symbols in terms of the metric (2.1) are given by

$$\Gamma^i_{ii} = \frac{L_{ii,i}}{2L_{ii}}, \quad \Gamma^j_{ii} = -\Gamma^i_{ij}\frac{L_{ii}}{L_{jj}}, \quad \Gamma^i_{ij} = \frac{L_{ii,j}}{2L_{ii}}, \tag{2.6}$$

where $1 \le i, j \le 2$ are distinct.

From (2.1)-(2.3) and (2.6), we get

$$Y,_{ii} = \Gamma^i_{ii} Y,_i - \sum_{j \ne i}^{n} \Gamma^i_{ij} \frac{L_{ii}}{L_{jj}} Y,_j - \lambda_i L_{ii} N. \tag{2.7}$$

3. Relative Weingarten matrix. In this section, we start with a local characterization of a relative hypersurface of the type I and (or) type II to a fixed hypersurface in $\mathbb{R}^{n+1}$. We provide the relative Weingarten matrix and a necessary and sufficient condition for a relative hypersurface $\mathcal{M}$ has a parameterization by lines of relative curvature.

Theorem 3.1. *Let M be an orientable hypersurface in $\mathbb{R}^{n+1}$, N its Gauss map and suppose that M has an orthogonal parameterization by lines of curvature $Y : U \subset \mathbb{R}^n \to M$, with principal curvatures $-\lambda_i$, $1 \le i \le n$. Let $\mathcal{M}$ be a hypersurface in $\mathbb{R}^{n+1}_M$ of the type I or type II. Then $\mathcal{M}$ can be parabeterized by*

$$X = Y + hN, \tag{3.1}$$

with the relative normal N_R, given by

$$N_R = -\sum_{r=1}^{n} \frac{h,_r \, Y,_r}{\big(1 + \lambda_r h\big) L_{rr}} + N, \tag{3.2}$$

where $L_{rr} = \langle Y,_r \, , \, Y,_r \rangle$. Moreover, the type I (or type II) relative Weingarten matrix of X, $V = \big(V_{ij}\big)$ is given by

$$V_{ij} = \frac{1}{g_{jj}}\left[h,_{ij} - A_i\left[\sum_{r=1}^{n} \frac{\Gamma^r_{ij} h,_r}{A_r} + \delta_{ij} L_{ii}\lambda_i\right] - h,_i \, h,_j \left[\frac{\lambda_j}{A_j} + \frac{\lambda_i}{A_i}\right] - \frac{h\lambda_{i,j} h,_i}{A_i}\right], \tag{3.3}$$

where $A_i = 1 + h\lambda_i$, Γ^r_{ij} $1 \le i, j, r \le n$ are given by (2.6) and g_{jj} are the coefficients of the first fundamental form of X given by (1.5), if type I and by (1.6), if type II.

Proof: Let $\mathcal{M}$ be a hypersurface in $\mathbb{R}^{n+1}_M$. Since that $Y : U \subset \mathbb{R}^n \to M$ is a parameterization by lines of curvature for M, we have that $\mathcal{M}$ can be parametrized by $X = Y + h\,N$, where N is a vector field normal to Y. Differentiating X with respect to u_i and u_j $1 \le i \, , \, j \le n$, we get

$$X,_i = \big(1 + \lambda_i h\big) Y,_i + h,_i \, N, \tag{3.4}$$

$$X,_{ij} = h,_{ij} \, N + (1 + \lambda_i h)\left[\sum_{r=1}^{n} \Gamma^r_{ij} Y,_r - \delta_{ij} L_{ii} \lambda_i N\right] + \lambda_{i,j} h Y,_i + \lambda_i h,_j \, Y,_i + \lambda_j h,_i \, Y,_j \,, \tag{3.5}$$

where λ_i, $1 \le i \le n$ are the principal curvatures of the M.

In order, we will consider $\mathcal{N}$ the unit vector field normal to $\mathcal{M}$ given by

$$\mathcal{N} = \sum_{r=1}^{n} b^r Y,_r + b^{n+1} N, \tag{3.6}$$

where

$$\sum_{r=1}^{n} \left(b^r\right)^2 L_{rr} + \left(b^{n+1}\right)^2 = 1.$$

Since $\langle X,_i , \mathcal{N}\rangle = 0$, for all $1 \leq i \leq n$, using (3.4) we get

$$b^i(1+\lambda_i h)L_{ii} + b^{n+1}h,_i = 0.$$

Substituting in (3.6), we have

$$\mathcal{N} = b^{n+1}\Big(-\sum_{r=1}^{n} \frac{h,_r}{(1+\lambda_r h)L_{rr}} Y,_r + N\Big).$$

Therefore the relative normal

$$N_R = \frac{\mathcal{N}}{\langle \mathcal{N} , N\rangle},$$

is given by (3.2).

Finally, let $V = \left(V_{ij}\right)$ be the type I (or type II) relative Weingarten matrix of X. Thus

$$V_{ij} = \frac{-\langle N_R , X,_{ij}\rangle}{g_{jj}}$$

where g_{jj} are the coefficients of the first fundamental form of X given by (1.5), if type I and by (1.6), if type II. Using (3.2) and (3.5), we have (3.3).

□

Let M be an orientable hypersurface in $\mathbb{R}^{n+1}$ and consider $\mathcal{M}$ a hypersurface in $\mathbb{R}^{n+1}_M$ of the type I (or type II). For each $p \in \mathcal{M}$ there exists a type I (or type II) relative orthonormal basis $\{e_1^R , e_2^R , \dots , e_n^R\}$ of $T_p\mathcal{M}$ such that $dN_R(e_i^R) = \lambda_i^R e_i^R$, $1 \leq i \leq n$. The functions $-\lambda_i^R$ are called the type I (or type II) relative principal curvatures at p and the corresponding directions, that is, e_i^R are called type I (or type II) relative principal directions at p. We say that a hypersurface $\mathcal{M}$ in $\mathbb{R}^{n+1}_M$ is parametrized by lines of relative curvature X of the type I (or type II), if for each $p \in \mathcal{M}$, $e_i^R = \dfrac{X,_i(p)}{\sqrt{g_{ii}(p)}}$, are type I (or type II) relative principal directions.

Theorem 3.2. *Let M be an orientable hypersurface in $\mathbb{R}^{n+1}$, N its Gauss map and suppose that M has an orthogonal parameterization by lines of curvature $Y : U \subset \mathbb{R}^n \to M$, with principal curvatures $-\lambda_i$, $1 \leq i \leq n$. Then (3.1) is an orthogonal parameterization by lines of relative curvature of the type I (or type II) for a hypersurface $\mathcal{M}$ relative to M of the type I (or type II), if and only if, there exists nonvanishing functions Ω, Ω^i and W, where $h = \frac{\Omega}{W}$, such that*

$$\begin{aligned} \Omega^i,_j &= \Omega^j \frac{a_{j,i}}{a_i}, \qquad for \;\; i \neq j, \\ \Omega,_i &= a_i \Omega^i, \\ W,_i &= -a_i \Omega^i \lambda_i, \end{aligned} \tag{3.7}$$

with $a_i = \sqrt{\langle Y,_i \;, Y,_i\rangle}$, and $W(W + \lambda_i \Omega) \neq 0$. Moreover X given by (3.1) becomes

$$X = Y + \frac{\Omega}{W} N. \tag{3.8}$$

Proof: From Theorem 1, the type I (or type II) relative Weingarten matrix of X, $V = \left(V_{ij}\right)$, is given by (3.3). Therefore, if V is diagonal, then X is a parameterization by lines of relative curvature of the type I (or type II). Thus, for $V_{ij} = 0$, $i \neq j$, we get

$$h,_{ij} - \frac{1+\lambda_i h}{1+\lambda_j h}\Gamma^j_{ij} h,_j - \frac{1+\lambda_j h}{1+\lambda_i h}\Gamma^i_{ij} h,_i - \left(\frac{\lambda_j}{1+\lambda_j h} + \frac{\lambda_i}{1+\lambda_i h}\right) h,_i h,_j = 0, \qquad 1 \leq i \neq j \leq n. \tag{3.9}$$

From Proposition 2.3 of [6], h is a solution of (3.9), if and only if, there exists nonvanishing functions Ω, Ω^i and W, where $h = \frac{\Omega}{W}$, which satisfy

$$\begin{aligned} \Omega^i,_j &= \Omega^j \frac{a_{j,i}}{a_i}, \qquad for \;\; i \neq j, \\ \Omega,_i &= a_i \Omega^i, \\ W,_i &= -a_i \Omega^i \lambda_i, \end{aligned} \tag{3.10}$$

with $a_i = \sqrt{\langle Y_{,i}\ ,\ Y_{,i}\rangle}$, $\Gamma^i_{ij} = \dfrac{a_{i,j}}{a_i}$ and $W(W+\lambda_i\Omega)\neq 0$.

□

Remark 3.1. Let $\mathcal{M}$ be a hypersurface in $\mathbb{R}^{n+1}_M$ of the type I (or type II), parametrized by lines of relative curvature of the type I (or type II), as in Theorem 2. Then the type I (or type II) relative Weingarten matrix, $V = \big(V_{ij}\big)$ given by in the Theorem 1, can be rewritten as follows $V_{ij} = 0$, $1\leq i\neq j\leq n$ and

$$V_{ii} = \frac{W+\lambda_i\Omega}{W^2 g_{ii}}\left[\Omega_{,ii} - \sum_{r=1}^{n}\Gamma^r_{ii}\Omega_{,r} - L_{ii}\lambda_i W\right]. \tag{3.11}$$

where W and Ω satisfies (3.10).

Remark 3.2. Let $\mathcal{M}$ be a hypersurface in $\mathbb{R}^{n+1}_M$ of the type I (or type II), parametrized by lines of relative curvature of the type I (or type II). Then, the relative principal curvatures of $\mathcal{M}$ of the type I (or type II) λ_i^R, are given by $\lambda_i^R = V_{ii}$, $1\leq i\leq n$.

Definition 3.1. A hypersurface $\mathcal{M}\subset\mathbb{R}^{n+1}_M$ is a relative Dupin submanifold of the type I (or type II) if its relative principal curvatures of type I (or type II) are constant along the corresponding relative lines of curvature of type I (or type II). Whenever, the relative principal curvatures of type I (or type II) are constant, $\mathcal{M}$ is a called a relative isoparametric submanifold of type I (or type II).

Using Remark 2 and Definition 1, we immediately get the corollary.

Corollary 3.1. *Let M be a Dupin hypersurface in $\mathbb{R}^{n+1}$ and suppose that M has an orthogonal parameterization by lines of curvature $Y : U\subset\mathbb{R}^n\to M$. Let $\mathcal{M}$ be a hypersurface in $\mathbb{R}^{n+1}_M$ of the type I (or type II), and consider the relative Weingarten matrix of the type I (or type II), $V = \big(V_{ij}\big)$ given by (3.11). Then $\mathcal{M}$ is a Dupin hypersurface in $\mathbb{R}^{n+1}_M$ of the type I (or type II) if, and only if, $V_{ii,i} = 0$.*

Remark 3.3. If M is the hyperplane $\mathbb{R}^n$, then the hypersurface $\mathcal{M}$ relative to M of the type I and type II is an isotropic hypersurface.

4. Relative geometry of the type I. In this section, we highlight the relative Dupin hypersurfaces type I. We start by providing a relationship between the Dupin hypersurfaces locally associated to $\mathbb{R}^n$ by a Ribaucour transformation and the type I Dupin hypersurfaces relative to $\mathbb{R}^n$. We will generate families of type I Dupin hypersurfaces relative to a hyperplane, a torus, $S^1\times\mathbb{R}^{n-1}$ and $\mathbb{S}^2\times\mathbb{R}^{n-2}$.

Let M be an orientable hypersurface in $\mathbb{R}^{n+1}$, N its Gauss map and suppose that M has an orthogonal parameterization by lines of curvature $Y : U\subset\mathbb{R}^n\to M$, with principal curvatures $-\lambda_i$, $1\leq i\leq n$. Let $\mathcal{M}$ be a hypersurface in $\mathbb{R}^{n+1}_M$ of the type I. Then $\mathcal{M}$ can be parametrized by

$$X = Y + hN,$$

where h is a differentiable real function defined on $\mathcal{M}$. Moreover, for (1.5), the coefficients of the first and second fundamental forms of X are given by

$$g^1_{ij} = \langle Y_{,i}\ ,\ Y_{,j}\rangle = \delta_{ij}L_{ii},\quad b^1_{ij} = \langle N_R\ ,\ X_{,ij}\rangle,$$

where the normal relative N_R is given by (3.2).

The first theorem provides a relationship between the Dupin hypersurfaces locally associated to $\mathbb{R}^n$ by a Ribaucour transformation and the type I Dupin hypersurfaces relative to $\mathbb{R}^n$.

Theorem 4.1. *Let $\mathbb{R}^n$ be a hyperplane parametrized by $Y(u_1,\ldots,u_n) = (u_1,\ldots,u_n,0)$. Consider $\widetilde{M}$ the hypersurface locally associated to $\mathbb{R}^n$ by a Ribaucour transformation. Let $\mathcal{M}$ be a type I hypersurface relative to $\mathbb{R}^n$, then $\mathcal{M}$ is a type I Dupin hypersurface relative to $\mathbb{R}^n$, if and only if, $\widetilde{M}$ is a Dupin hypersurface.*

Proof: From Corollary 1, $\mathcal{M}$ is a type I Dupin hypersurface relative to $\mathbb{R}^n$ if, and only if, $V_{ii,i} = 0$, where

$$V_{ii} = \frac{W+\lambda_i\Omega}{W^2 L_{ii}}\left[\Omega_{,ii} - \sum_{r=1}^{n}\Gamma^r_{ii}\Omega_{,r} - L_{ii}\lambda_i W\right],$$

with functions W and Ω satisfying (3.10).

On the other hand, from [9], $\widetilde{M}$ locally associated to $\mathbb{R}^n$ by a Ribaucour transformation, is a Dupin hypersurface, if and only if, $T_{i,i} = 0$, where

$$T_i = \frac{2}{L_{ii}}\left[\Omega_{,ii} - \sum_{r=1}^{n}\Gamma^r_{ii}\Omega_{,r} - L_{ii}\lambda_i W\right],$$

with functions W and Ω satisfying (3.10).

Since the principal curvatures of Y are $\lambda_i = 0$ and the metric $L_{ij} = \delta_{ij}$, for $1 \leq i, j \leq n$, it follows from equation (3.10) that

$$\Omega = \sum_{i=1}^{n} f_i(u_i), \qquad W = c \neq 0,$$

where $f_i(u_i)$ are differentiable functions. Therefore, $V_{ii,i} = 0$, if and only if, $T_{i,i} = 0$.

□

Remark 4.1. When M is the hyperplane $\mathbb{R}^n$, the geometry of $\mathbb{R}^{n+1}_M$ coincides with the isotropic geometry. Then in the theorem 3, we show that the hypersurface $\widetilde{M}$ locally associated to $\mathbb{R}^n$ by a Ribaucour transformation is a Dupin hypersurface, if and only if, the hypersurface $\mathcal{M}$ is an isotropic Dupin hypersurface. Moreover, $\mathcal{M}$ is the hypersurface of center of the Ribaucour transformation.

In the next results, we provide families of type I Dupin hypersurfaces relative to a hyperplane, a torus, $\mathbb{S}^1 \times \mathbb{R}^{n-1}$ and $\mathbb{S}^2 \times \mathbb{R}^{n-2}$.

Proposition 4.1. *Consider the hyperplane in the Euclidean space $\mathbb{R}^{n+1}$, parametrized by $Y(u_1, ..., u_n) = (u_1, ..., u_n, 0)$. Then $\mathcal{M}$ is a type I Dupin hypersurface relative to $\mathbb{R}^{n+1}$, if and only if, $\mathcal{M}$ can be parametrized by*

$$X(u_1, ..., u_n) = \left(u_1, ..., u_n, \frac{\sum_{i=1}^{n} f_i(u_i)}{c}\right), \tag{4.1}$$

where $f_i(u_i) = c_{i2}u_i^2 + c_{i1}u_i + c_{i0}$, $1 \leq i \leq n$, and $c \neq 0$, $c_{i2},\ c_{i1},\ c_{i0} \in \mathbb{R}$.

Proof: Since the principal curvatures of Y are $\lambda_i = 0$ and the metric $L_{ij} = \delta_{ij}$, for $1 \leq i, j \leq n$, it follows from equation (3.10) that

$$\Omega = \sum_{i=1}^{n} f_i(u_i), \qquad W = c \neq 0,$$

where $f_i(u_i)$ are differentiable functions. In order, to obtain type I Dupin hypersurface relative to $\mathbb{R}^n$, we consider V_{ii} given by (3.11),

$$V_{ii} = \frac{f_i''}{c}.$$

From Corollary 1, $\mathcal{M}$ parametrized by $X = Y + \dfrac{\Omega}{W} e_{n+1}$, where $e_{n+1} = (0, 0, ..., 0, 1)$ is a unit vector field normal to $\mathbb{R}^n$, is a type I Dupin hypersurface relative to $\mathbb{R}^n$, if and only if, $V_{ii,i} = 0$. Therefore, $f_i(u_i) = c_{i2}u_i^2 + c_{i1}u_i + c_{i0}$, with $c_{i2},\ c_{i1},\ c_{i0} \in \mathbb{R}$ and from (3.8), X is given by (5.2).

□

Proposition 4.2. *Consider the torus in $\mathbb{R}^3$, parametrized by*

$$Y(u_1, u_2) = \big((a + r\cos u_2)\cos u_1\, ,\ (a + r\cos u_2)\sin u_1\, ,\ r\sin u_2\big).$$

Then $\mathcal{M}$ is a Type I Dupin hypersurface relative to torus, if and only if, $\mathcal{M}$ can be parametrized by

$$X = \begin{bmatrix} \dfrac{-\cos u_1\big(-aB_2\sin u_2 + (-aB_1 + rB + A)\cos u_2\big)}{\cos u_2(A_2\sin u_1 + A_1\cos u_1) + B_2\sin u_2}, \\ \dfrac{-\sin u_1\big(-aB_2\sin u_2 + (-aB_1 + rB + A)\cos u_2\big)}{\cos u_2(A_2\sin u_1 + A_1\cos u_1) + B_2\sin u_2}, \\ \dfrac{-\sin u_2\big(aA_2\sin u_1 + aA_1\cos u_1 - aB_1 + rB + A\big)}{\cos u_2(A_2\sin u_1 + A_1\cos u_1) + B_2\sin u_2} \end{bmatrix}. \tag{4.2}$$

where $B_i,\ A_i,\ A$ and B are real constants.

Proof: The principal curvatures of the torus and coefficients of the metric of the torus are

$$\lambda_1 = \frac{\cos u_2}{a - r\cos u_2}, \qquad \lambda_2 = \frac{1}{r}, \qquad L_{11} = \big(a + r\cos u_2\big)^2, \qquad L_{22} = r^2.$$

Using (3.10), we obtain

$$\Omega = \big(a + r\cos u_2\big)f_1 + rf_2 + A, \quad W = -\cos u_2\, f_1 - f_2 + B,$$

where A, B are constants and f_1, f_2 are differentiable functions of u_1 and u_2, respectively.

Consider V_{ii} given by (3.11). Thus

$$V_{11} = \frac{W + \Omega\lambda_1}{(a + r\cos u_2)W^2}\Big[f_1'' + f_1 - \sin u_2 f_2' - \cos u_2(B - f_2)\Big], \tag{4.3}$$

$$V_{22} = \frac{W + \Omega\lambda_2}{rW^2}\Big[f_2'' + f_2 - B\Big]. \tag{4.4}$$

From Corollary 1, $\mathcal{M}$ parametrized by (3.8) is a type I Dupin hypersurface relative to torus, if and only if, $V_{ii,i} = 0$ for all $1 \le i \le 2$.
Since $\big(W + \lambda_i\Omega\big)_{,i} = 0$ and $\big(a + r\cos u_2\big)_{,1} = 0$, we conclude that $V_{ii,i} = 0$, if and only if,

$$\left[\frac{1}{W^2}\Big(f_1'' + f_1 - \sin u_2 f_2' - \cos u_2(B - f_2)\Big)\right]_{,1} = 0, \tag{4.5}$$

$$\left[\frac{1}{W^2}\Big(f_2'' + f_2 - B\Big)\right]_{,2} = 0. \tag{4.6}$$

If $f_1' = 0$, then we have $V_{11,1} = 0$. Then suppose $f_1' \neq 0$.
Since (4.6), we get

$$\frac{-2W_{,2}}{W}\big(f_2'' + f_2 - B\big) + f_2''' + f_2' = 0.$$

This last equation can be rewritten as

$$2W_{,2}\,\big(f_2'' + f_2 - B\big) = W\big(f_2''' + f_2'\big). \tag{4.7}$$

Differentiating with respect to u_1, we get

$$2W_{,12}\,\big(f_2'' + f_2 - B\big) = W_{,1}\,\big(f_2''' + f_2'\big).$$

As $W_{,12} = \sin(u_2)f_1'$ and $W_{,1} = -\cos(u_2)f_1'$, then if $f_2'' + f_2 - B \neq 0$, we get $f_2'' + f_2 - B = A_1\cos^2 u_2$. Substituting in (4.7) and using that $W_{,2} = \sin u_2 f_1 - f_2'$, we obtain a contradiction. Therefore, we have $f_2'' + f_2 - B = 0$. Thus

$$f_2 = B_1\cos u_2 + B_2\sin u_2 + B. \tag{4.8}$$

Substituting (4.8) in (4.5), we obtain

$$\left[\frac{1}{W^2}\Big(f_1'' + f_1 + B_1\Big)\right]_{,1} = 0, \tag{4.9}$$

Thus

$$\frac{-2W_{,1}}{W}\big(f_1'' + f_1 + B_1\big) + f_1''' + f_1' = 0.$$

This last equation can be rewritten as

$$2W_{,1}\,\big(f_1'' + f_1 + B_1\big) = W\big(f_1''' + f_1'\big). \tag{4.10}$$

Differentiating with respect to u_2, we get

$$2W_{,12}\,\big(f_1'' + f_1 + B_1\big) = W_{,2}\,\big(f_1''' + f_1'\big).$$

As $W_{,12} = \sin u_2 f_1'$ and $W_{,2} = \sin u_2 f_1 - f_2'$, then if $f_1''' + f_1' \neq 0$, we get $f_1'' + f_1 + B_1 \neq 0$ and

$$\frac{2f_1'\big(f_1'' + f_1 + B\big)}{f_1''' + f_1'} - f_1 = \frac{-f_2'}{\sin u_2}.$$

Hence $\dfrac{2f_1'\big(f_1'' + f_1 + B_1\big)}{f_1''' + f_1'} - f_1 = B_1$, since (4.8). Thus $f_1'' + f_1 + B_1 = c(f_1 + B_1)^2$, which is a contradiction, since (4.10), $W_{,1} = -\cos(u_2)f_1'$ and $W = -\cos u_2 f_1 - f_2 + B$. Therefore, we have $f_1''' + f_1' = 0$. Substituting in (4.10), we get $f_1'' + f_1 + B_1 = 0$, since $W_{,1} \neq 0$. Thus

$$f_1(u_1) = A_1\cos u_1 + A_2\sin u_1 - B_1. \tag{4.11}$$

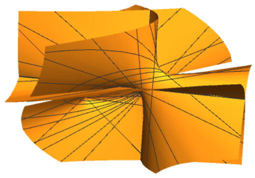

Figure 4.1: On the surface above we have a type I Dupin surface relative to torus, with $a = 4,\ r = 1,\ A = 3,\ B = -2,\ A_2 = B_2 = 1,\ B_1 = -1$ and $A_1 = B = -2$.

Finally, considering the unit vector field normal to Y

$$N = \Big(\cos\, u_1 \cos\, u_2\,,\, \sin\, u_1 \cos\, u_2\,,\, \sin\, u_2\Big),$$

and substituting f_1, f_2, Ω and W in $X = Y + \frac{\Omega}{W}N$ we obtain (5.3).

□

Proposition 4.3. *Consider the submanifold $\mathbb{S}^2 \times \mathbb{R}^{n-2}$ in $\mathbb{R}^{n+1}$, parametrized by*

$$Y(u_1, ..., u_n) = \big(\sin\, u_1 \cos\, u_2\,,\, \sin\, u_1 \sin\, u_2\,,\, \cos\, u_1\,,\, u_3\,,\, u_4\,,\, ...\,,\, u_n\big).$$

Then $\mathcal{M}$ is a type I Dupin hypersurface relative to Y, if and only if, $\mathcal{M}$ can be parametrized by

$$X = Y + \frac{\Omega}{W}N,$$

where

$$N(u_1, ..., u_n) = \big(\sin\, u_1 \cos\, u_2\,,\, \sin\, u_1 \sin\, u_2\,,\, \cos\, u_1\,,\, 0\,,\, ...\,,\, 0\big),$$

$$\Omega = \sin\, u_1 f_2 + \sum_{i\neq 2}^{n} f_i, \qquad W = -\sin\, u_1 f_2 - f_1 + C, \qquad f_1(u_1) = A_1 \cos\, u_1 + C,$$

$$f_2(u_2) = B_1 \cos\, u_2 + B_2 \sin\, u_2 \qquad f_j(u_j) = C_{j2}u_j^2 + C_{j1}u_j + C_{j0},\ \ 3 \leq j \leq n,$$

with C, A_i, B_i, C_{j2}, C_{j1} and C_{j0} are real constants.

Proof: The principal curvatures and coefficients of the metric of the of the $\mathbb{S}^2 \times \mathbb{R}^{n-2}$ are

$$\lambda_1 = \lambda_2 = 1, \quad \lambda_j = 0,\ \ 3 \leq j \leq n, \quad L_{ii} = 1, \quad 1 \leq i \neq 2 \leq n, \quad L_{22} = \sin^2 u_1.$$

Using (3.10), we obtain

$$\Omega\ =\ \sin\, u_1 f_2 + \sum_{i\neq 2}^{n} f_i, \quad W\ =\ -\sin\, u_1 f_2 - f_1 + C,$$

where C is constant and f_i are differentiable functions of u_i, $1 \leq i \leq n$.

Consider V_{ii} given by (3.11). Thus

$$V_{11} = \frac{W+\Omega}{W^2}\Big[f_1'' + f_1 - C\Big], \tag{4.12}$$

$$V_{22} = \frac{W+\Omega}{\sin\, u_1 W^2}\Big[f_2'' + f_2 + \cos\, u_1 f_1' + \sin\, u_1 f_1 - C \sin\, u_1\Big], \tag{4.13}$$

$$V_{jj} = \frac{f_j''}{W}, \quad 3 \leq j \leq n. \tag{4.14}$$

From Corollary 1, $\mathcal{M}$ parametrized by (3.8) is a type I Dupin hypersurface relative to $\mathbb{S}^2 \times \mathbb{R}^{n-2}$, if and only if, $V_{ii,i} = 0$ for all $1 \leq i \leq n$.

Proceeding similarly to the proof of Proposition 2, we obtain that $V_{ii,i} = 0$, $1 \leq i \leq 2$, if and only if, f_1 and f_2 are given by

$$f_1(u_1) = A_1 \cos\, u_1 + A_2 \sin\, u_1 + C, \quad f_2(u_2) = B_1 \cos\, u_2 + B_2 \sin\, u_2 - A_2. \tag{4.15}$$

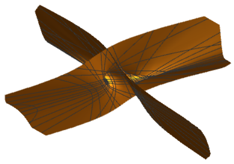

Figure 4.2: On the surface above we have a type I Dupin surface relative to $\mathbb{S}^2$, with $B_2 = 0,\ B_1 = 2,\ A_1 = -1$ and $C = 1$.

Without loss of generality, can be considered $A_2 = 0$. In fact, substituting f_1 and f_2 given above into the expressions of W and Ω, we have that W and Ω do not depend on A_2.
Moreover, since that (4.14) and $W,_j = 0$, we conclude that f_j, $3 \le j \le n$ are given by

$$f_j(u_j) = C_{j2}u_j^2 + C_{j1}u_j + C_{j0}. \tag{4.16}$$

□

Proposition 4.4. *Consider the submanifold $\mathbb{S}^1 \times \mathbb{R}^{n-1}$ in $\mathbb{R}^{n+1}$, parametrized by*

$$Y(u_1, ..., u_n) = \big(\cos\ u_1 \,,\, \sin\ u_1 \,,\, u_2 \,,\, u_3 \,,\, ... \,,\, u_n\big).$$

Then $\mathcal{M}$ is an isotropic Dupin hypersurface relative to Y, if and only if, $\mathcal{M}$ can be parametrized by

$$X = Y + \frac{\Omega}{W}N,$$

where

$$N(u_1, ..., u_n) = \big(\cos\ u_1 \,,\, \sin\ u_1 \,,\, 0 \,,\, ... \,,\, 0\big),$$

$$\Omega = \sum_{i=1}^{n} f_i, \qquad W = -f_1 + C, \qquad f_j(u_j) = C_{j2}u_j^2 + C_{j1}u_j + C_{j0},\ \ 2 \le j \le n,$$

and f_1 satisfies $f_1'' + f_1 - C - C_1(f_1 - C)^2 = 0$, with C, C_1, C_{j2}, C_{j1} and C_{j0} are real constants.
Proof: The principal curvatures and coefficients of the metric of the of the $\mathbb{S}^1 \times \mathbb{R}^{n-1}$ are

$$\lambda_1 = 1, \quad \lambda_j = 0,\ \ 2 \le j \le n, \quad L_{ii} = 1, \quad 1 \le i \le n.$$

Using (3.10), we obtain

$$\Omega \ = \sum_{i \ne 1}^{n} f_i, \quad W \ = \ -f_1 + C,$$

where C is constant and f_i are differentiable functions of u_i, $1 \le i \le n$.
Consider V_{ii} given by (3.11). Thus

$$V_{11} = \frac{W + \Omega}{W^2}\left[f_1'' + f_1 - C\right], \tag{4.17}$$

$$V_{jj} = \frac{f_j''}{W}, \quad 2 \le j \le n. \tag{4.18}$$

From Corollary 1, $\mathcal{M}$ parametrized by (3.8) is a type I Dupin hypersurface relative to $\mathbb{S}^1 \times \mathbb{R}^{n-1}$, if and only if, $V_{ii,i} = 0$ for all $1 \le i \le n$.
Since $\big(W + \Omega\big),_1 = 0$ and $W = C - f_1$, we conclude from $V_{ii,i} = 0$ that the functions f_j are given by

$$f_j(u_j) = C_{j2}u_j^2 + C_{j1}u_j + C_{j0} \quad 2 \le j \le n,$$

and f_1 satisfies $f_1'' + f_1 - C - C_1(f_1 - C)^2 = 0$. where C_1, C_{j2}, C_{j1} and C_{j0} are real constants.

□

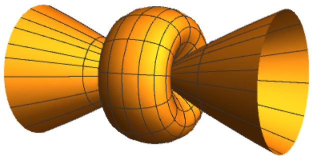

Figure 4.3: On the surface above we have a type I Dupin surface relative to cylinder $\mathbb{S}^1 \times \mathbb{R}$, with $C_{21} = 0$, $C_{22} = -1$, $C_{20} = 1$, $C = C_1 = 1$ and $f_1 = 2$.

5. Relative geometry of the type II. In this section, we highlight the relative Dupin hypersurfaces of the type II. We start by providing a relationship between the Dupin hypersurfaces locally associated to a fixed Dupin hypersurface M by a Ribaucour transformation and the type II Dupin hypersurfaces relative $\mathcal{M}$. We will generate families of type II Dupin hypersurfaces relative to a hyperplane, a torus, $\mathbb{S}^1 \times \mathbb{R}^{n-1}$ and $\mathbb{S}^2 \times \mathbb{R}^{n-2}$.

Let M be an orientable hypersurface in $\mathbb{R}^{n+1}$, N its Gauss map and suppose that M has an orthogonal parameterization by lines of curvature $Y : U \subset \mathbb{R}^n \to M$, with principal curvatures $-\lambda_i$, $1 \leq i \leq n$. Let $\mathcal{M}$ be a hypersurface in $\mathbb{R}^{n+1}_M$ of the type II. Then $\mathcal{M}$ can be parametrized by

$$X = Y + hN,$$

where h is a differentiable real function defined on $\mathcal{M}$. From (1.6), the coefficients of the first and second fundamental forms of X are given by

$$g^2_{ij} = \left(1 + h\lambda_i\right)^2 \delta_{ij} L_{ii}, \quad b^2_{ij} = \langle N_R\,, X_{,ij}\,\rangle,$$

where $\delta_{ij} L_{ii} = \langle Y_{,i}\,, Y_{,j}\,\rangle$ and the normal relative N_R is given by (3.2). Moreover, since that X is a parameterization by lines of relative curvature, then the relative Weingarten matrix of X, $V = \left(V_{ij}\right)$ is given by $V_{ij} = 0$, $1 \leq i \neq j \leq i$, and

$$V_{ii} = \frac{1}{(W + \lambda_i \Omega) L_{ii}} \left[\Omega_{,ii} - \sum_{r=1}^{n} \Gamma^r_{ii} \Omega_{,r} - L_{ii} \lambda_i W \right] \tag{5.1}$$

where Ω and W satisfies (3.10).

Theorem 5.1. *Let M be a Dupin hypersurface and suppose that it has a parameterization by lines of curvature $Y : U \subset \mathbb{R}^n \to M$, with principal curvatures $-\lambda_i$, $1 \leq i \leq n$. Consider $\widetilde{M}$ the hypersurface locally associated to M by a Ribaucour transformation. Let $\mathcal{M}$ be a type II hypersurface relative to M, then $\mathcal{M}$ is a type II Dupin hypersurface relative to M, if and only if, $\widetilde{M}$ is a Dupin hypersurface.*

Proof: From Corollary 1, $\mathcal{M}$ is a type II Dupin hypersurface relative to M if, and only if, $V_{ii,i} = 0$, where

$$V_{ii} = \frac{1}{(W + \lambda_i \Omega) L_{ii}} \left[\Omega_{,ii} - \sum_{r=1}^{n} \Gamma^r_{ii} \Omega_{,r} - L_{ii} \lambda_i W \right],$$

with functions W and Ω satisfying (3.10).

On the other hand, from [9], $\widetilde{M}$ locally associated to M by a Ribaucour transformation, is a Dupin hypersurface, if and only if, $T_{i,i} = 0$, where

$$T_i = \frac{2}{L_{ii}} \left[\Omega_{,ii} - \sum_{r=1}^{n} \Gamma^r_{ii} \Omega_{,r} - L_{ii} \lambda_i W \right],$$

with functions W and Ω satisfying (3.10).

Since M is a Dupin hypersurface, we have $(W + \lambda_i \Omega)_{,i} = 0$. Therefore, $V_{ii,i} = 0$, if and only if, $T_{i,i} = 0$.

□

In the next results, we provides families of type II Dupin hypersurfaces relative to a hyperplane, a torus, $\mathbb{S}^1 \times \mathbb{R}^{n-1}$ and $\mathbb{S}^2 \times \mathbb{R}^{n-2}$.

Proposition 5.1. *Consider the hyperplane in the Euclidean space $\mathbb{R}^{n+1}$, parametrized by $Y(u_1, ..., u_n) = (u_1, ..., u_n, 0)$. Then $\mathcal{M}$ is a type II Dupin hypersurface relative to $\mathbb{R}^{n+1}$, if and only if, $\mathcal{M}$ can be parametrized by*

$$X(u_1, ..., u_n) = \left(u_1, ..., u_n, \frac{\sum_{i=1}^{n} f_i(u_i)}{c} \right), \tag{5.2}$$

where $f_i(u_i) = c_{i2}u_i^2 + c_{i1}u_i + c_{i0}$, $1 \leq i \leq n$, *and* $c \neq 0$, $c_{i2},\ c_{i1},\ c_{i0} \in \mathbb{R}$.

Proof: Since the principal curvatures of Y are $\lambda_i = 0$ and the metric $L_{ij} = \delta_{ij}$, for $1 \leq i, j \leq n$, it follows from equation (3.10) that

$$\Omega = \sum_{i=1}^{n} f_i(u_i), \qquad W = c \neq 0,$$

where $f_i(u_i)$ are differentiable functions. In order, to obtain type II Dupin hypersurface relative to $\mathbb{R}^{n+1}$, we consider V_{ii} given by (5.1),

$$V_{ii} = \frac{f_i''}{c}.$$

From Corollary 1, $\mathcal{M}$ parametrized by $X = Y + \dfrac{\Omega}{W}e_{n+1}$, where $e_{n+1} = (0, 0, ..., 0, 1)$ is a unit vector field normal to $\mathbb{R}^n$, is a type II Dupin hypersurface relative to $\mathbb{R}^n$, if and only if, $V_{ii,i} = 0$. Therefore, $f_i(u_i) = c_{i2}u_i^2 + c_{i1}u_i + c_{i0}$, with $c_{i2},\ c_{i1},\ c_{i0} \in \mathbb{R}$ and from (3.8), X is given by (5.2).

□

Remark 5.1. In Proposition 5, one observes that the type II Dupin hypersurface X relative to $\mathbb{R}^n$ is an isotropic Dupin hypersurface.

Proposition 5.2. *Consider the torus in* $\mathbb{R}^3$*, parametrized by*

$$Y(u_1, u_2) = \big((a + r\cos u_2)\cos u_1\,,\ (a + r\cos u_2)\sin u_1\,,\ r\sin u_2\big).$$

Then $\mathcal{M}$ *is a type II Dupin hypersurface relative to* Y*, if and only if,* $\mathcal{M}$ *can be parametrized by*

$$X = \begin{bmatrix} \frac{-\cos u_1\left(-aB_2\sin u_2 + (-aB_1 + rB + A)\cos u_2 + aB - aB_3\right)}{\cos u_2(A_1\cos u_1 + A_2\sin u_1 + A_3 + B_1) + B_2\sin u_2 + B_3 - B}, \\ \frac{-\sin u_1\left(-aB_2\sin u_2 + (-aB_1 + rB + A)\cos u_2 + aB - aB_3\right)}{\cos u_2(A_1\cos u_1 + A_2\sin u_1 + A_3 + B_1) + B_2\sin u_2 + B_3 - B}, \\ \frac{-\sin u_2\left(aA_1\cos u_1 + aA_2\sin u_1 + aA_3 + rB + A\right)}{\cos u_2(A_1\cos u_1 + A_2\sin u_1 + A_3 + B_1) + B_2\sin u_2 + B_3 - B} \end{bmatrix}. \qquad (5.3)$$

where $B_i,\ A_i,\ A$ *and* B *are real constants.*

Proof: The principal curvatures of the torus and coefficients of the metric of the torus are

$$\lambda_1 = \frac{\cos u_2}{a - r\cos u_2}, \qquad \lambda_2 = \frac{1}{r}, \qquad L_{11} = \big(a + r\cos u_2\big)^2, \qquad L_{22} = r^2.$$

Using (3.10), we obtain

$$\Omega = \big(a + r\cos u_2\big)f_1 + rf_2 + A, \quad W = -\cos u_2\, f_1 - f_2 + B,$$

where A, B are constants and f_1, f_2 are differentiable functions of u_1 and u_2, respectively.

Consider V_{ii} given by (5.1). Thus

$$V_{11} = \frac{1}{\big(a + r\cos u_2\big)\big(W + \Omega\lambda_1\big)}\Big[f_1'' + f_1 - \sin u_2 f_2' - \cos u_2\big(B - f_2\big)\Big], \qquad (5.4)$$

$$V_{22} = \frac{1}{r\big(W + \Omega\lambda_2\big)}\Big[f_2'' + f_2 - B\Big]. \qquad (5.5)$$

From Corollary 1, $\mathcal{M}$ parametrized by (3.8) is a type II Dupin hypersurface relative to torus, if and only if, $V_{ii,i} = 0$ for all $1 \leq i \leq 2$.

Since $\big(W + \lambda_i\Omega\big)_{,i} = 0$, we conclude from $V_{ii,i} = 0$ that the functions f_i are given by

$$f_1(u_1) = A_1\cos u_1 + A_2\sin u_1 + A_3, \qquad f_2 = B_1\cos u_2 + B_2\sin u_2 + B_3.$$

Finally, considering the unit vector field normal to Y

$$N = \Big(\cos u_1\cos u_2\,,\ \sin u_1\cos u_2\,,\ \sin u_2\Big),$$

and substituting f_1, f_2, Ω and W in $X = Y + \frac{\Omega}{W}N$ we obtain (5.3).

□

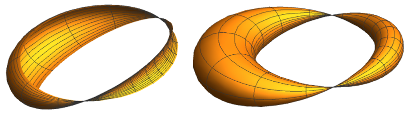

Figure 5.1: On the surfaces above we have a type II Dupin surface relative to torus, with $a = 4,\ r = 1,\ A = 10,\ B = -3,\ A_2 = B_2 = 0,\ B_1 = B_3 = 1,\ A_1 = -1$ and $A_3 = -2$.

Proposition 5.3. *Consider the submanifold* $\mathbb{S}^2 \times \mathbb{R}^{n-2}$ *in* $\mathbb{R}^{n+1}$, *parametrized by*

$$Y(u_1, ..., u_n) = \big(\sin\, u_1 \cos\, u_2\,,\, \sin\, u_1 \sin\, u_2\,,\, \cos\, u_1\,,\, u_3\,,\, u_4\,,\, ...\,,\, u_n\big).$$

Then $\mathcal{M}$ *is a type II Dupin hypersurface relative to* Y, *if and only if,* $\mathcal{M}$ *can be parametrized by*

$$X = Y + \frac{\Omega}{W}N,$$

where

$$N(u_1, ..., u_n) = \big(\sin\, u_1 \cos\, u_2\,,\, \sin\, u_1 \sin\, u_2\,,\, \cos\, u_1\,,\, 0\,,\, ...\,,\, 0\big),$$

and $\Omega = \sin\, u_1 f_2 + \sum_{i\neq 2}^{n} f_i$, $W = -\sin\, u_1 f_2 - f_1 + C$, $f_1(u_1) = A_i \cos\, u_i + B_i \sin\, u_i + C_i$, $1 \le i \le 2$ *and* $f_j(u_j) = C_{j2}u_j^2 + C_{j1}u_j + C_{j0}$, $3 \le j \le n$, *with* C, A_i, B_i, C_i, C_{j2}, C_{j1} *and* C_{j0} *are real constants.*
Proof: The principal curvatures and coefficients of the metric of the of the $\mathbb{S}^2 \times \mathbb{R}^{n-2}$ are

$$\lambda_1 = \lambda_2 = 1, \quad \lambda_j = 0,\ \ 3 \le j \le n, \quad L_{ii} = 1, \quad 1 \le i \ne 2 \le n, \quad L_{22} = \sin^2 u_1.$$

Using (3.10), we obtain

$$\Omega\ =\ \sin\, u_1 f_2 + \sum_{i\neq 2}^{n} f_i, \qquad W\ =\ -\sin\, u_1 f_2 - f_1 + C,$$

where C is constant and f_i are differentiable functions of u_i, $1 \le i \le n$.
Consider V_{ii} given by (5.1). Thus

$$V_{11} = \frac{1}{W + \Omega}\Big[f_1'' + f_1 - C\Big], \tag{5.6}$$

$$V_{22} = \frac{1}{\sin\, u_1(W + \Omega)}\Big[f_2'' + f_2 + \cos\, u_1 f_1' + \sin\, u_1 f_1 - C \sin\, u_1\Big], \tag{5.7}$$

$$V_{jj} = \frac{f_j''}{W}, \quad 3 \le j \le n. \tag{5.8}$$

From Corollary 1, $\mathcal{M}$ parametrized by (3.8) is a type II Dupin hypersurface relative to $\mathbb{S}^2 \times \mathbb{R}^{n_2}$, if and only if, $V_{ii,i} = 0$ for all $1 \le i \le n$.
Since $\big(W + \Omega\big)_{,j} = 0$, $1 \le j \le 2$ and $W_{,r} = 0$, $3 \le r \le n$ we conclude from $V_{ii,i} = 0$ that the functions f_i are given by

$$f_i(u_i) = A_i \cos\, u_i + B_i \sin\, u_i + C_i, \quad 1 \le i \le 2, \quad f_j(u_j) = C_{j2}u_j^2 + C_{j1}u_j + C_{j0} \quad 3 \le j \le n,$$

where A_i, B_i, C_i, C_{j2}, C_{j1} and C_{j0} are real constants.

□

Proposition 5.4. *Consider the submanifold* $\mathbb{S}^1 \times \mathbb{R}^{n-1}$ *in* $\mathbb{R}^{n+1}$, *parametrized by*

$$Y(u_1, ..., u_n) = \big(\cos\, u_1\,,\, \sin\, u_1\,,\, u_2\,,\, u_3\,,\, ...\,,\, u_n\big).$$

Then $\mathcal{M}$ *is a type II Dupin hypersurface relative to* Y, *if and only if,* $\mathcal{M}$ *can be parametrized by*

$$X = Y + \frac{\Omega}{W}N,$$

where

$$N(u_1, ..., u_n) = \big(\cos, u_1\,,\, \sin\, u_1\,,\, 0\,,\, ...\,,\, 0\big),$$

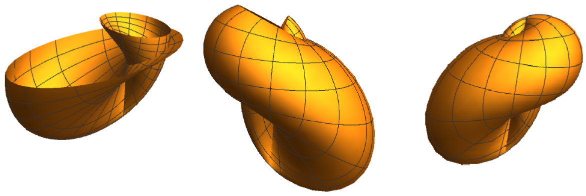

Figure 5.2: On the surfaces above we have a type II Dupin surface relative to $\mathbb{S}^2$, with $A_2 = B_2 = 0$, $B_1 = 2$, $A_3 = 4\, A_1 = -1$, $B_3 = -2$ and $C = 1$.

and $\Omega = \sum_{i=1}^{n} f_i$, $W = -f_1 + C$, $f_1(u_1) = A_1 \cos\, u_1 + B_1 \sin\, u_1 + C_1$ *and* $f_j(u_j) = C_{j2}u_j^2 + C_{j1}u_j + C_{j0}$, $2 \le j \le n$, *with* C, A_1, B_1, C_1, C_{j2}, C_{j1} *and* C_{j0} *are real constants.*
Proof: The principal curvatures and coefficients of the metric of the of the $\mathbb{S}^1 \times \mathbb{R}^{n-1}$ are

$$\lambda_1 = 1, \quad \lambda_j = 0, \;\; 2 \le j \le n, \quad L_{ii} = 1, \quad 1 \le i \le n.$$

Using (3.10), we obtain

$$\Omega \;=\; \sum_{i \neq 1}^{n} f_i, \qquad W \;=\; -f_1 + C,$$

where C is constant and f_i are differentiable functions of u_i, $1 \le i \le n$.

Consider V_{ii} given by (5.1). Thus

$$V_{11} = \frac{1}{W + \Omega}\Big[f_1'' + f_1 - C\Big], \tag{5.9}$$

$$V_{jj} = \frac{f_j''}{W}, \qquad 2 \le j \le n. \tag{5.10}$$

From Corollary 1, $\mathcal{M}$ parametrized by (3.8) is a type II Dupin hypersurface relative to $\mathbb{S}^1 \times \mathbb{R}^{n-1}$, if and only if, $V_{ii,i} = 0$ for all $1 \le i \le n$.
Since $\big(W + \Omega\big)_{,1} = 0$ and $W_{,r} = 0$, $2 \le r \le n$ we conclude from $V_{ii,i} = 0$ that the functions f_i are given by

$$f_1(u_1) = A_1 \cos\, u_1 + B_1 \sin\, u_1 + C_1, \qquad f_j(u_j) = C_{j2}u_j^2 + C_{j1}u_j + C_{j0} \quad 2 \le j \le n,$$

where A_1, B_1, C_1, C_{j2}, C_{j1} and C_{j0} are real constants.

□

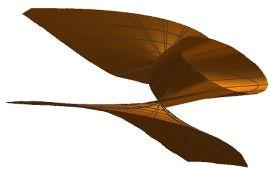

Figure 5.3: On the surface above we have a type II Dupin surface relative to cylinder $\mathbb{S}^1 \times \mathbb{R}$, with $C_{21} = B_1 = 0$, $C_1 = 3$, $A_1 = -2$, $C_{22} = -1$, $C_{20} = 2$ and $C = 1$.

6. Conclusions. From the results obtained in this work we can make the following conclusions:
For each fixed hypersurface M in Euclidean space and we introduce two types of spaces relative to M, of type I and type II. We observe that when M is a hyperplane, the two geometries coincides with the isotropic geometry.

ORCID and License

Armando M. V. Corro https://orcid.org/0000-0002-6864-3876
Marcelo Lopes Ferro https://orcid.org/0000-0001-6832-2274

References

[1] Aydin ME, Ergut M. Isotropic geometry of graph surfaces associated with product production functions in economics. Tamkang J. Math. 2016; 47:433-443.

[2] Berger M. Geometry II. Springer; 1987.

[3] Berger M, Gostiaux B. Differential geometry: manifolds, curves, and surfaces. Springer; 1988.

[4] Bianchi L. Lezioni di geometria Differenziale. Terza Edicione. Nicola Zanichelli Editore; 1927.

[5] Chen BY, Decu S, Verstraelen L. Notes on isotropic geometry of production models. Kragujevac J. Math. 2014; 38:23-33.

[6] Corro AMV, Ferreira WP, Tenenblat K. On Ribaucour transformations for hypersurfaces. Mat. Contemp. 1999; 17:137-160.

[7] Corro AMV, Ferreira WP, Tenenblat K. Ribaucour transformations for Constant mean curvature and linear Weingarten surfaces. Pacific Journal of Mathematics. 2003; 212(2):265-296.

[8] Corro AMV, Ferreira WP, Tenenblat K. Minimal surfaces obtained by Ribaucour transformations. Geometriae Dedicata, Netherlands. 2003; 96(1):117-150.

[9] Corro AMV, Tenenblat K. Ribaucour transformation revisited. Comum. Geom. 2004; 12(5):1055-1082.

[10] Da Silva LCB. The geometry of Gauss map and shape operator in simply isotropic and pseudo-isotropic spaces. J. Geom. 2019; 110:31. DOI: 10.1007/s00022-019-0488-9

[11] Da Silva LCB. Rotation minimizing frames and spherical curves in simply isotropic and pseudo-isotropic 3-spaces. Tamkang Journal Of Mathematics. 2020; 31(1):1-23.

[12] Pottmann H, Opitz K. Curvature analysis and visualization for functions defined on Euclidean spaces or surfaces. Comput. Aided Geom. Des. 1994; 11:655-674.

[13] Strubecker K. Differentialgeometrie des isotropen Raumes, I. Theorie der Raumkurven. Sitzungsber. Akad. Wiss. Wien. Math.-Naturw. Kl. IIa. 1941; 150:1-53.

[14] Strubecker K. Differentialgeometrie des isotropen Raumes II. Die Flächen konstanter Relativkrümmung $K = rt - s^2$. Math. Z. 1942; 47:743-777.

[15] Strubecker K. Differentialgeometrie des isotropen Raumes. III. Flächentheorie. Math. Z. 1942; 48:369-427.