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A theorem on zero cycles on surfaces

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Abstract
In this paper we prove a result on 0*-cycles on surfaces as an application of the theorem on the kernel of the Gysin homomorphism of Chow groups of* 0*-cycles of degree zero induced by the embedding of a curve into a surface, and we study the connection of this result with Bloch's conjecture and constant cycles curves.*
Keywords . Zero-cycles, Chow groups, surfaces, constant cycle curves, general type surface.

1. Introduction. Let S be a smooth projective connected surface over $\mathbb{C}$.
Let Σ be the linear system of a very ample divisor D on S, let $d = \dim(\Sigma)$ the dimension of Σ, and let

$$\phi_\Sigma : S \hookrightarrow \mathbb{P}^d$$

be the closed embedding of S into $\mathbb{P}^d$, induced by Σ.

For any closed point $t \in \Sigma = \mathbb{P}^{d*}$, let H_t be the hyperplane in $\mathbb{P}^d$ defined by t, let $C_t = H_t \cap S$ be the corresponding hyperplane section of S, and let

$$r_t : C_t \hookrightarrow S,$$

be the closed embedding of the curve C_t into S.

Let Δ be the discriminant locus of Σ, that is,

$$\Delta = \{t \in \Sigma = \mathbb{P}^{d*} : C_t \text{ is singular}\}.$$

Let

$$U = \Sigma \setminus \Delta = \{t \in \Sigma = \mathbb{P}^{d*} : C_t \text{ is smooth}\},$$

be the complement of the discriminant locus of Σ parametrizing smooth curves on S.
For any closed point $t \in \Sigma$, let

$$r_{t*} : H^1(C_t, \mathbb{Z}) \to H^3(S, \mathbb{Z}),$$

be the Gysin homomorphism on cohomology groups induced by r_t, whose kernel $H^1(C_t, \mathbb{Z})_{\mathsf{van}}$ is called the *vanishing cohomology of* C_t (see [5],§3.2.3).

For any closed point $t \in \Sigma$, let $J_t = J(C_t) = J^1(H^1(C_t, \mathbb{Z}))$ be the intermediate Jacobian associated to the Hodge structure of odd weight $H^1(C_t, \mathbb{Z})$, called the *Jacobian of the curve* C_t.

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For any closed point $t \in \Sigma$, let $B_t = J^1(H^1(C_t, \mathbb{Z})_{\mathsf{van}})$ be the abelian subvariety of the abelian variety J_t corresponding to the Hodge substructure $H^1(C_t, \mathbb{Z})_{\mathsf{van}}$ of $H^1(C_t, \mathbb{Z})$.

Let $\mathsf{CH}_0(S)_{\deg=0}$ the Chow group of 0-cycles of degree zero.

For any closed point $t \in \Sigma$, let $\mathsf{CH}_0(C_t)_{\deg=0}$ be the Chow group of 0-cycles of degree zero on C_t.

For any closed point $t \in \Sigma$, let

$$r_{t*} : \mathsf{CH}_0(C_t)_{\deg=0} \to \mathsf{CH}_0(S)_{\deg=0},$$

be the Gysin (pushforward) homomorphism on the Chow groups of 0-cycles of degree zero of C_t and S respectively, induced by the closed embedding r_t, whose kernel

$$G_t = \mathsf{Ker}(r_{t*}),$$

will be called the *Gysin kernel* associated with the hyperplane section C_t.

In our previous paper (see [2]) we proved the following result which we called "A theorem on the Gysin kernel".

Theorem 1.1. *(A theorem on the Gysin kernel) Let $U = \Sigma \setminus \Delta$.*

a) For each $t \in U$ there is an abelian variety $A_t \subset B_t$ such that

$$G_t = \bigcup_{countable} \textit{translates of } A_t \textit{ in } J_t\ .$$

b) For a very general $t \in U \subset \Sigma$ either, $A_t = 0$ and then G_t is countable or $A_t = B_t$ and then G_t is uncountable.

In this paper we will prove a result on Chow group of 0-cycles on surfaces as an application of Theorem 1.1 which is related to Bloch's conjecture and the notion of constant cycle curves.

Bloch conjecture is the converse of a criterion given by Mumford to determine when the Chow groups of 0-cycles on surfaces are not representable or equivalently when the Chow groups of 0-cycles on surfaces are not finite dimensional. More precisely Bloch's conjecture states

Conjecture 1 (Bloch's conjecture). *Let S be a smooth projective surface over $\mathbb{C}$. If $p_g(S) = 0$, then*

$$alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S),$$

is an isomorphism.

Or equivalently (see [5], Theorem 10.11),

Conjecture 2 (Bloch's conjecture). *Let S be a smooth projective surface over $\mathbb{C}$. If $p_g(S) = 0$, then $\mathsf{CH}_0(S)_{\deg=0}$ is representable.*

Also equivalently (see [5], Proposition 10.10),

Conjecture 3 (Bloch's conjecture). *Let S be a smooth projective surface over $\mathbb{C}$. If $p_g(S) = 0$, then $\mathsf{CH}_0(S)_{\deg=0}$ is finite dimensional.*

On the other hand, the notion of constant cycles curves was introduced by Huybrechts on $K3$ surfaces, see [1], but it can be defined for arbitrary surfaces. The most important examples of constant cycles curves are provided by rational curves, but not all constant cycle curve is rational so it is still not known how much weaker the notion of constant cycles curves really is.

As constant cycles curves of bounded order resemble rational curves in many ways, Huybrechts restated two conjectures on rational curves for constant cycles curves. These two conjectures for constant cycles curves due to Huybrechts are

Conjecture 4. *For any $K3$ surface S there exists a positive integer $n > 0$ such that the union $\bigcup C \subset S$ of all constant cycles curves $C \subset S$ of order $\leq n$ is dense.*

Conjecture 5. *Let S be a complex $K3$ surface. Then any point $x \in S$ with $[x] = c_S$ is contained in a constant cycle curve.*

2. A Theorem on Zero Cycles on Surfaces. In this section we will state and prove the main result of the paper.

Definition 2.1. *Given an integral algebraic scheme T, a (Zariski)* c-open *subset in T is the complement of a (Zariski)* c-closed *subset of T (a countable union of Zariski closed irreducible subsets in T).*

We will use the following terminology: a *property Q holds for a very general point of T* if there exists a c-open subset in T such that Q holds for each closed point in this c-open.

The main result of this section is the following which is obtained as an application of the Theorem 1.1.

Theorem 2.1 (A theorem on 0-cycles on surfaces). *If*

$$alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S),$$

is not an isomorphism, for a very general $t \in U$, then G_t is countable.

Proof: By hypothesis we have $alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S)$ is not an isomorphism.

And we should prove that there exits an c-open in U such that for all t in this c-open G_t is countable.

Since by item a) of Theorem 1.1 we have that for every $t \in U$

$$G_t = \mathsf{Ker}(r_{t*}) = \bigcup_{\text{countable}} \text{translates of } A_t \text{ in } J_t\ ,$$

it is enough to prove that there exits an c-open in U such that for all t in this c-open $A_t = 0$.

By item b) of Theorem 1.1, we know that there exists an c-open U_0 such that for every $t \in U_0$ we have that $A_t = 0$ or on the other hand for every $t \in U_0$ we have that $A_t = B_t$.

So it is enough to prove that for every $t \in U_0$ we have that $A_t = 0$.

Let $t \in U_0$ be any element of U_0. We will prove that $A_t = 0$. By contradiction, suppose that it is not true, i.e. that $A_t = B_t$. Then, we have that

$$G_t = \mathsf{Ker}(r_{t*}) = \bigcup_{\text{countable}} \text{translates of } B_t \text{ in } J_t\ .$$

Now, remember that B_t is the abelian subvariety in J_t corresponding to the Hodge substructure

$$H^1(C_t,\mathbb{Z})_{\mathsf{van}} = \mathsf{Ker}(H^1(C_t,\mathbb{Z}) \to H^3(S,\mathbb{Z})).$$

By definition we have a short exact sequence of Hodge structures

$$0 \to H^1(C_t,\mathbb{Z})_{\mathsf{van}} \to H^1(C_t,\mathbb{Z}) \to H^3(S,\mathbb{Z}) \to 0,$$

which give rise to a short exact sequence of abelian varieties

$$0 \to B_t \to J_t \to \mathrm{Alb}(S) \to 0,$$

by definition this means that $B_t \to J_t$ is injective, $J_t \to \mathrm{Alb}(S)$ is surjective and

$$\mathrm{im}(B_t \to J_t) = \mathsf{Ker}(J_t \to \mathrm{Alb}(S)).$$

So

$$\mathrm{Alb}(S) \cong \frac{J_t}{B_t}.$$

Now consider

$$\pi_t : J_t = \mathsf{CH}_0(C_t)_{\deg=0} \to \mathrm{Alb}(S),$$

and

$$r_{t*} : J_t = \mathsf{CH}_0(C_t)_{\deg=0} \to \mathsf{CH}_0(S)_{\deg=0}.$$

Since $\pi_t : J_t \to \mathrm{Alb}(S)$ is surjective, for a fixed $z \in \mathrm{Alb}(S)$ there exits $x \in J_t$ such that $\pi_t(x) = z$. Then under $r_{t*} : J_t = \mathsf{CH}_0(C_t)_{\deg=0} \to \mathsf{CH}_0(S)_{\deg=0}$ we get $r_{t*}(x) \in \mathsf{CH}_0(S)_{\deg=0}$. So we get map

$$f : \mathrm{Alb}(S) \to \mathsf{CH}_0(S)_{\deg=0},$$

defined by $f(z) = r_{t*}(x)$.

Since $\mathsf{Ker}(\pi_t) \subset G_t = \ker(r_{t*})$ this map is well defined. Indeed, suppose that there are x_1, x_2 such that $\pi_t(x_1) = z$ and $\pi_t(x_2) = z$, then we have that $\pi_t(x_1) = \pi_t(x_2)$, then since π_t is an homomorphism we have $\pi_t(x_1 - x_2) = 0$, it follows that $x_1 - x_2 \in B_t$ and since, by our assumption, $B_t \subset \ker(r_{t*})$ we have that $r_{t*}(x_1 - x_2) = 0$, then $r_{t*}(x_1) = r_{t*}(x_2)$ because r_{t*} is a homomorphism. So, f is well defined since it does not depend on the choice of the preimage of z.

The composition

$$\mathsf{CH}_0(S)_{\deg=0} \xrightarrow{alb_S} \mathrm{Alb}(S) \xrightarrow{f} \mathsf{CH}_0(S)_{\deg=0},$$

is the identity. Indeed, since $\mathsf{CH}_0(S)_{\deg=0}$ is generated by points of the form $a - b$ where a, b lies in some C_t of our fixed family, it suffices to check that $a - b$ maps to $a - b$ under the composition. It follows that alb_S is injective.

On the other hand, $alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S)$ is a surjection by definition, so $\mathsf{CH}_0(S)_{\deg=0}$ is isomorphic to $\mathrm{Alb}(S)$ which contradicts to the hypothesis, so we are done.

3. Relation with Bloch's Conjecture. The main result of the previous section on 0-cycles on surfaces is useful to prove Bloch's conjecture because its contrapositive form, and hence equivalent form is as follows

Corollary 3.1. *If G_t is uncountable, for a very general $t \in U$, then $alb : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S)$ is an isomorphism.*

So, if a surface has $p_g = 0$ and we want to prove that Bloch's conjecture holds for this surface, i.e. that alb_S is an isomorphism, it is enough to prove that G_t is not countable, for a very general $t \in U$.

Bloch's conjecture is proved for surfaces of special type i.e. for surfaces with Kodaira dimension less than 2, but for surfaces of general type, i.e. for surfaces with Kodaira dimension 2 it is not proved yet, except for some particular cases.

For surfaces of general type the relation between the theorem on 0-cycles of surfaces and Bloch's conjecture is expressed as follows.

Let S be a surface of general type with $p_g(S) = 0$. In this case $p_g(S) = 0$ implies that $q(S) = 0$, i.e. $\mathrm{Alb}(S) = 0$, and the Bloch's conjecture for surfaces of general type can be stated as follows

Conjecture 6 (Bloch's conjecture for surfaces of general type). *Let S be a smooth projective surface over $\mathbb{C}$ of general type. If $p_g(S) = 0$, then*

$$\mathsf{CH}_0(S)_{\deg=0} = 0,$$

i.e. any two closed points in S are rationally equivalent.

On the other hand, from Theorem 1.1 applied to regular surfaces S, i.e., surfaces with $q(S) = 0$ we get the following corollary

Corollary 3.2. *If S is regular, i.e. $q(S) = 0$.*

a) For every $t \in U$ there is an Abelian variety $A_t \subset B_t$ such that

$$G_t = \bigcup_{countable} \textit{translates of } A_t \,.$$

b) For a very general $t \in U$ either, $A_t = 0$, i.e. G_t is countable or $G_t = J_t$, i.e. $r_{t*} = 0$.

Proof: For item a) there is nothing to prove.

For item b) note that if $q(S) = 0$, then $H^3(S, \mathbb{Z}) = 0$, it follows that $H^1(C_t, \mathbb{Z}) = H^1(C_t, \mathbb{Z})_{van}$ which is equivalent to $B_t = J_t$, for $t \in U$.

By Theorem 1.1 there exists a c-open U_0 in U such that for every $t \in U_0$ we have that $A_t = 0$ or for every $t \in U_0$ we have that $A_t = B_t$.

In the first case we have that

$$G_t = \cup_{countable} \text{ translates of } 0,$$

i.e., G_t is countable.

In the second case, we have $A_t = B_t$ and, combined with the remark at the beginning of the proof, we have that $A_t = J_t$, then

$$G_t = J_t,$$

i.e. $r_{t*} = 0$. In particular, in this case G_t is uncountable.

From Theorem 2.1 on 0-cycles of surfaces applied to surfaces S with $q(S) = 0$ we get the following result

Corollary 3.3. *Let S be a surface with $q(S) = 0$. If $\mathsf{CH}_0(S)_{\deg=0} \neq 0$, then, for a very general curve in U, G_t is countable.*

Proof: It is obvious since when $q(S) = 0$ then $\mathrm{Alb}(S) = 0$.

The above corollary is equivalent to the following

Corollary 3.4. *Let S be a surface with $q(S) = 0$. If $r_{t*} = 0$, for a very general $t \in U$, then $\mathsf{CH}_0(S)_{\deg=0} = 0$.*

Proof: The contrapositive form of Corollary 3.3 tell us that if it is not true that G_t is countable, for a very general $t \in U$, then $\mathsf{CH}_0(S)_{\deg=0} = 0$. Then applying item b) of Corollary 3.2 we are done.

So, if a surface of general type has $p_g = 0$ and we want to prove that Bloch's conjecture holds for this surface, i.e. $\mathsf{CH}_0(S)_{\deg=0} = 0$ it is enough to prove that $r_{t*} = 0$, for a very general $t \in U$ i.e. more precisely we have

Corollary 3.5. *Let S be a surface of general type with $p_g(S) = 0$. If $r_{t*} = 0$, for a very general $t \in U$, then Bloch's conjecture holds for S.*

Proof: Since S is a surface of general type with $p_g(S) = 0$, it follows that $q(S) = 0$, then we apply Corollary 3.4 and we are done.

4. Relation with Constant Cycle Curves. Let S be a surface over $\mathbb{C}$.

Definition 4.1 (Pointwise constant cycle curve). *A curve $C \subset S$ is a* pointwise constant cycle curve *if all closed points $x \in C$ define the same class $[x] \in \mathsf{CH}^2(S)$.*

Since we are working over $\mathbb{C}$ which is algebraically closed, we have the following equivalent definition (see [1] §3).

Definition 4.2. *A curve $C \subset S$ is a* pointwise constant cycle curve *if and only if*

$$r_{C*} : \mathrm{Pic}^0(\tilde{C}) = \mathsf{CH}_0(\tilde{C})_{\deg=0} \to \mathsf{CH}_0(S),$$

is the zero map. Here, $r_C : \tilde{C} \to S$ is the composition of the normalization $\tilde{C} \to C$ with the closed embedding $C \hookrightarrow S$.

By Lemma 3.8 in [1], we can define the notion of a constant cycle curve on any surface as follows

Definition 4.3 (Constant cycle curve). *Let S be a surface over $\mathbb{C}$. An integral curve $C \subset S$ is a* constant cycle curve *if and only if there exists a positive integer n such that*

$$n \cdot [\eta_C] \in \mathrm{im}(\mathsf{CH}^2(S) \to \mathsf{CH}^2(S \times_{\mathbb{C}} \mathbb{C}(\eta_C))),$$

where the generic point $\eta_C \in C$ is viewed as a closed point in $S \times_{\mathbb{C}} \mathbb{C}(\eta_C)$. Equivalently, $C \subset S$ is a constant cycle curve if and only if η_C is viewed as a point in the geometric generic fibre $S \times_{\mathbb{C}} \overline{\mathbb{C}(\eta_C)}$, then

$$[\eta_C] \in \mathrm{im}(\mathsf{CH}^2(S) \to \mathsf{CH}^2(S \times_{\mathbb{C}} \overline{\mathbb{C}(\eta_C)})).$$

Definition 4.4. *Let S be a surface over $\mathbb{C}$. We call an arbitrary curve $C \subset S$ a* constant cycle curve *if every integral component of C is a constant cycle curve.*

When the ground field is $\mathbb{C}$ the notion de pointwise constant cycle curve and constant cycle curve coincide thanks to the following proposition, see Proposition 3.7 in [1].

Proposition 4.1. *Let S be a surface over an algebraically closed field k. Then a constant cycle curve $C \subset S$ is also a pointwise constant cycle curve. If k is uncountable, the converse holds true as well.*

The following result shows the relation of the theorem on the 0-cycles on surfaces, i.e. Theorem 2.1 and the notion of constant cycle curves.

Corollary 4.1. *Let S be a surface of general type with $p_g(S) = 0$. If the curve C_t is a constant cycle curve, for a very general $t \in U$, then S satisfies Bloch's conjecture.*

Proof: We apply Corollary 3.5 and definition of constant cycle curve and we are done.

The above corollary says that if a surface of general type has $p_g = 0$ and we want to prove that Bloch's conjecture holds for this surface, i.e. $\mathsf{CH}_0(S)_{\deg=0} = 0$ it is enough to prove that C_t is a constant cycle curve, for a very general $t \in U$.

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