Journal homepage http://revistas.unitru.edu.pe/index.php/SSMM

SELECCIONES MATEMÁTICAS
Universidad Nacional de Trujillo
ISSN: 2411-1783 (Online)
2022; Vol. 9(1): 34-43.

Conformable fractional derivatives and applications to Newtonian dynamic and cooling body law

Derivadas fraccionales conformables y aplicaciones a la dinámica Newtoniana y Ley de enfriamiento de los cuerpos

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Received, Mar. 22, 2022 *Accepted, May. 18, 2022*

How to cite this article:
Párraga P, Vivas-Cortez M, Larreal O. *Conformable fractional derivatives and applications to Newtonian dynamic and cooling body law*. Selecciones Matemáticas. 2022;9(1):34–43. http://dx.doi.org/10.17268/sel.mat.2022.01.03

Abstract
This article presents a rigorous review of the conformable fractional derivative given by J. E. Napoles in [11], studying its classical properties, as a differential operator. Likewise, applications are given to Physics, specifically to the free fall of bodies and Newton's law of cooling.
Keywords . Fractional derivatives, Nápoles' fractional derivative, free fall of bodies, cooling bodies.

Resumen
En el presente artículo se presenta una revisión rigurosa de la derivada fraccionaria conformable dada por J. E. Nápoles en [11], estudiando las propiedades clásicas, como operador diferencial, que posee, así mismo se dan aplicaciones a la Física, específicamente a la caída libre de los cuerpos y la ley de enfriamiento de Newton.
Palabras clave. Derivadas fraccionales, derivada fraccional de Nápoles, caída libre de los cuerpos, enfriamiento de los cuerpos.

1. Introduction. In the area of Mathematics, specifically the well-known Fractional Calculus, various models of fractional derivatives appear, understanding here those derivatives whose order is determined by a real number and sometimes by complex numbers.

In 1985, the Riemann-Liouville fractional derivative appears [12] as a result of studies based on controversial discussions among great mathematicians, including Leibniz, L'Hôpital, Laplace, Lacroix Fourier and others. This fractional derivative takes the following form:

Definition 1.1. *Let $\alpha > 0, x > a, \alpha, a, b, x \in \mathbb{R}$. Then*

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^x \frac{f(t)}{(x-t)^{\alpha+1-n}}dt, \quad n-1 < \alpha < n \in \mathbb{N}, \tag{1.1}$$

$$D^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_x^b \frac{f(t)}{(x-t)^{\alpha+1-n}}dt, \quad n-1 < \alpha < n, \in \mathbb{N}, \tag{1.2}$$

and in the case $\alpha = n \in \mathbb{N}$,

$$D^{n}_{a+}f(x) = D^{n}_{b-}f(x) = \frac{d^n}{dt^n}f(x).$$

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The operators (1.1) and (1.2) are called the left and right Riemman-Liouville fractional derivatives, respectively. Other fractional derivative models have been introduced since the latter and in the first half of the last century, among these are the fractional derivative of Caputo, Riesz, Grunwald-Letnikov, and others that use special functions as the integration kernel, such as the function of Mittag-Leffler, the Hypergeometric function and functions of the Legendre type. All these definitions have shown good applications both in Pure Mathematics and in other areas of science and technology. By the middle of the 20th century, Michelle Caputo introduced a definition of a fractional derivative that differs from the Riemman-Liouville definition in the position of the operator of ordinary derivative, in addition to the fact that the considered function has demanding differentiability conditions.

Definition 1.2. *Let $\alpha \in \mathbb{C}$ with $\mathcal{R}e(\alpha) \geq 0$. Let f be a real valued and n times differentiable function defined on an interval $[a, b]$. Then*

$$({}^C D^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt,$$
$$({}^C D^{\alpha}_{b-} f)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt.$$

In the case of the Riemman-Liouville derivative, the derivative of a constant function is not zero, however it has shown its usefulness in problems related to viscoelasticity, viscoelastic deformations, and viscous fluids [7, 8, 17]. Similarly, and because Caputo's model allows including initial and boundary conditions in fractional differential equations, it has been considered one of the models that best models real life problems [3, 4, 15, 9].

Recently Atangana [2] shows certain properties that differential operators must fulfill. Among these are: the operator defined with order zero is the same function, linearity, describe the rate of change in the neighborhood of a given value, satisfy the product rule and also the chain rule, the quotient rule, and others. Given that the Riemman-Liouville fractional derivative and the Caputo fractional derivative fail under some of these conditions, Atangana states that they are not derivatives, in the strict sense, but must be considered as fractional operators, Khalil [10], for his part, then introduces a definition of fractional derivative with the qualifier "conformable", which meets the properties mentioned above. Similarly Atangana introduces a new model and likewise Abdeljawad [1].

Several investigations have been carried out by other authors in this direction [13]. Other investigations have directed their goals towards fractional derivatives with kernels involving special functions [5, 6, 14, 16]

Recently Juan E. Nápoles [11] proposes certain fractional derivatives, different from those of Khalil and Abdeljawad, with certain kernels that classify them as conformable and not conformable.

The purpose of this paper is to develop in detail such definition, its properties and applications to Newtonian Dynamics and cooling bodies law . In addition some graphic comparisons are shown between the solutions obtained from the ordinary differential model and those obtained by means of the conformable fractional derivatives.

2. Conformable fractional derivative. From Differential Calculus we know that the derivative of a function f at a given point $x = a$ of its domain is defined by the following limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if it limits exists, and we say that f is differentiable function on $x = a$, using the notation $f'(a)$.

Khalil, in [10], propose a new definition of fractional derivative as follows.

Definition 2.1. *Given a function $f : [0, \infty) \to \mathbb{R}$ then the conformable fractional derivative of order $\alpha \in (0, 1]$ is defined by*

$$(T_\alpha f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{2.1}$$

for all $t > 0$. We will use the notation $f^{(\alpha)}$ instead of $T_\alpha f$. If the limit exist for some $x = t > 0$ then it is said that f is $\alpha-$differentiable in that point, similarly, is said to be $\alpha-$differentiable on an interval $[a, b] \subset [0, \infty)$ if it is $\alpha-$differentiable in each point of $[a, b]$, and it is said that f is $\alpha-$differentiable if its $\alpha-$derivative exists in each point of its domain. In particular, if f is $\alpha-$differentiable on some interval $(0, a)$ with $a > 0$ and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ then

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

Abdeljawad [1] give the following definition.

Definition 2.2. *The left fractional derivative of order $\alpha \in (0,1]$, starting in a point $a \in \mathbb{R}$ of a function $f : [a,\infty) \to \mathbb{R}$, is defined by*

$$(T_\alpha^a f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon}. \tag{2.2}$$

Whwn $a = 0$ it will be written T_α. If $(T_\alpha^a f)(t)$ exists in some neighborhood of a then

$$(T_\alpha^a f)(a) = \lim_{t \to a^+} (T_\alpha^a f)(t). \tag{2.3}$$

Similarly, the right fractional derivative of order $\alpha \in (0,1]$, starting in a point $b \in \mathbb{R}, b > a$, is defined by

$$({}^bT_\alpha f)(t) = -\lim_{\varepsilon \to 0} \frac{f(t + \varepsilon(b-t)^{1-\alpha}) - f(t)}{\varepsilon}. \tag{2.4}$$

If $({}^bT_\alpha f)(t)$ exists in some neighborhood of b then

$$({}^bT_\alpha f)(b) = \lim_{t \to b^\cdot} ({}^bT_\alpha f)(t). \tag{2.5}$$

Even the aforementioned authors have introduced conformable fractional operators. In the case of Khalil, the author introduces the following definition.

Definition 2.3. *(Ver [10]) The conformable fractional integral $(I_\alpha^a f)(t)$ of the function f starting in a point $a \geq 0$ is defined by*

$$(I_\alpha^a f)(t) = \int_a^t x^{\alpha-1} f(x)dx.$$

Abdeljawad propose his fractional integral operator as follows.

Definition 2.4. *The conformable fractional integral operator of the function f starting in the point $a > 0$ is defined by*

$$(I_\alpha^a f)(t) = \int_a^t (x-a)^{\alpha-1} f(x)dx,$$

and the conformable fractional integral operator ending in $b > 0$ is defined by

$$(I_\alpha^a f)(t) = \int_t^b (b-x)^{\alpha-1} f(x)dx.$$

J. E. Nápoles warns that both the increase added to the argument of the derivatives of Khalil and Abdeljawad, as well as the kernel of the integrals defined by both authors, are functions that depend on the parameter α and the starting or terminal point considered. So he introduces a definition involving the exponential function as follows

Definition 2.5. *Let $f : [0,\infty) \to \mathbb{R}$ be a function. Then the $N_e^\alpha-$derivative of f, of order $\alpha \in (0,1]$ is defined by*

$$(N_e^\alpha f)(t) = lim_{\varepsilon \to 0} \frac{f(t + \varepsilon e^{(\alpha-1)t}) - f(t)}{\varepsilon}. \tag{2.6}$$

With the change $h = \varepsilon e^{(\alpha-1)t}$, we can observe that $h \to 0$ when $\varepsilon \to 0$ and so, we have

$$\begin{aligned} (N_e^\alpha f)(t) &= lim_{\varepsilon \to 0} \frac{f(t + \varepsilon e^{(\alpha-1)t}) - f(t)}{\varepsilon} = lim_{h \to 0} \frac{f(t+h) - f(t)}{h/e^{(\alpha-1)t}} \\ &= e^{(\alpha-1)t} lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = e^{(\alpha-1)t} f'(t). \end{aligned} \tag{2.7}$$

Now we will study some classical properties of the differential operators.

For a constant function we have.

Theorem 2.1. *Let $f:(a,b)\to\mathbb{R}$ a function such that $f(t)=k$ for all $t\geq 0$, where k is an arbitrary real number. Then*

$$(N_e^\alpha f)(t)=0.$$

Proof: Using Definition 2.6 we have that

$$(N_e^\alpha f)(t)=lim_{\varepsilon\to 0}\frac{f(t+\varepsilon e^{(\alpha-1)t})-f(t)}{\varepsilon}=lim_{\varepsilon\to 0}\frac{k-k}{\varepsilon}=0.$$

The proof is complete. □

Theorem 2.2. *Let $f:(a,b)\to\mathbb{R}$ be a function such that $f(t)=t^r$ for all $t\geq 0$, where $r\geq 0$ is any real positive number. Then*

$$(N_e^\alpha f)(t)=e^{(\alpha-1)t}rt^{r-1}.$$

Proof: Using (2.7) we have

$$(N^\alpha f)(t)=e^{(\alpha-1)t}rt^{r-1}.$$

Then the proof is complete. □

A similar result is obtained for the case $r\leq 0$ for functions defined on $(0,\infty)$.

Now we will see that $N_e^\alpha-$diferenciability implies continuity.

Theorem 2.3. *If a function f is $N_e^\alpha-$differentiable in its domain then f is continuous on it.*

Proof: We will proof the continuity of f from the expression

$$\lim_{h\to 0}f(x+h)-f(x)=0.$$

Let observe that

$$f(x+h)-f(x)=e^{(\alpha-1)t}\frac{f(t+h)-f(t)}{h}\cdot\frac{h}{e^{(\alpha-1)t}},$$

so

$$\lim_{h\to 0}f(x+h)-f(x)=e^{(\alpha-1)t}\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}\cdot\lim_{h\to 0}\frac{h}{e^{t-\alpha}}=(N_e^\alpha f)(t)\cdot 0=0.$$

The proof is complete. □

Now we will see the linearity of conformable fractional derivative.

Theorem 2.4. *Let f,g be real valued functions defined on an interval (a,b), both $N_e^\alpha-$diferentiable, and β,δ arbitries real numbers. Then*

$$(N_e^\alpha[\beta f+\delta g])(t)=\beta(N_e^\alpha f)(t)+\delta(N_e^\alpha g)(t).$$

Proof: Let f,g as in the statement and β,δ arbitries real numbers. Then, using Definition 2.6 we have

$$\begin{aligned}(N_e^\alpha[\beta f+\delta g])(t)&=lim_{\varepsilon\to 0}\frac{[\beta f+\delta g](t+\varepsilon e^{(\alpha-1)t})-[\beta f+\delta g](t)}{\varepsilon}\\&=lim_{\varepsilon\to 0}\frac{\beta f(t+\varepsilon e^{(\alpha-1)t})-\beta f(t)+\delta g(t+\varepsilon e^{(\alpha-1)t})-\delta g(t)}{\varepsilon}\\&=lim_{\varepsilon\to 0}\frac{\beta f(t+\varepsilon e^{(\alpha-1)t})-\beta f(t)}{\varepsilon}+lim_{\varepsilon\to 0}\frac{\delta g(t+\varepsilon e^{(\alpha-1)t})-\delta g(t)}{\varepsilon}\\&=\beta(N_e^\alpha f)(t)+\delta(N_e^\alpha g)(t).\end{aligned}$$

The proof is complete. □

The rule of the product of functions is established as follows.

Theorem 2.5. *Let f,g be two $N_e^\alpha-$differentiable in a point $t>0$ and $\alpha\in(0,1]$. Then*

$$(N_e^\alpha[fg])(t)=g(t)(N_e^\alpha f)(t)+f(t)(N_e^\alpha g)(t).$$

Proof: Let f and g be two functions as in the statement and $\alpha \in (0,1]$. Then, using Definition 2.6 we get

$$\begin{aligned}
\left(N_e^{\alpha}\left[fg\right]\right)(t) &= \lim_{\varepsilon\to 0}\frac{fg(t+\varepsilon e^{(\alpha-1)t})-fg(t)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon e^{(\alpha-1)t})g(t+\varepsilon e^{(\alpha-1)t})-fg(t)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon e^{(\alpha-1)t})g(t+\varepsilon e^{(\alpha-1)t})-f(t)g(t+\varepsilon e^{(\alpha-1)t})+f(t)g(t+\varepsilon e^{(\alpha-1)t})-fg(t)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{\left[f(t+\varepsilon e^{(\alpha-1)t})-f(t)\right]g(t+\varepsilon e^{(\alpha-1)t})-f(t)\left[g(t+\varepsilon e^{(\alpha-1)t})-g(t)\right]}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{\left[f(t+\varepsilon e^{(\alpha-1)t})-f(t)\right]g(t+\varepsilon e^{(\alpha-1)t})}{\varepsilon}+\lim_{\varepsilon\to 0}\frac{\left[g(t+\varepsilon e^{(\alpha-1)t})-g(t)\right]f(t)}{\varepsilon}\\
&= g(t)\left(N_e^{\alpha}f\right)(t)+f(t)\left(N_e^{\alpha}g\right)(t).
\end{aligned}$$

The proof is complete. □

The rule for the $N_e^{\alpha}-$derivative of the multiplicative inverse function is established as follows.

Theorem 2.6. *Let $h(x)=f^{-1}(x)$ defined in those points where f is different from zero. Then*

$$\left(N_e^{\alpha}f^{-1}\right)(t) = -\frac{N_e^{\alpha}f(t)}{f^2(t)}.$$

Proof: Using Definition 2.6 we have that

$$\begin{aligned}
\left(N_e^{\alpha}f^{-1}\right)(t) &= \lim_{\varepsilon\to 0}\frac{f^{-1}(t+\varepsilon e^{(\alpha-1)t})-f^{-1}(t)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{f(t)-f(t+\varepsilon e^{(\alpha-1)t})}{\varepsilon\left[f(t)-f(t+\varepsilon e^{(\alpha-1)t})\right]}\\
&= -\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon e^{(\alpha-1)t})-f(t)}{\varepsilon\left[f(t)-f(t+\varepsilon e^{(\alpha-1)t})\right]}\\
&= -\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon e^{(\alpha-1)t})-f(t)}{\varepsilon}\cdot\lim_{\varepsilon\to 0}\frac{1}{f(t)-f(t+\varepsilon e^{(\alpha-1)t})}\\
&= -\frac{N_e^{\alpha}f(t)}{f^2(t)}.
\end{aligned}$$

The proof is complete. □

With this last result it can be prooved the rule for the quotient of two functions.

Theorem 2.7. *Let f and g be two $N_e^{\alpha}-$differentiable functions in some $t>0$, $\alpha\in(0,1]$, and such that $g(t)\neq 0$. Then*

$$\left(N_e^{\alpha}\left[f/g\right]\right)(t) = \frac{g(t)\left(N_e^{\alpha}f\right)(t)-f(t)\left(N_e^{\alpha}g\right)(t)}{g^2(t)}.$$

Proof: Using Theorems 2.5 and 2.6 we have that

$$\begin{aligned}
\left(N_e^{\alpha}\left[f/g\right]\right)(t) &= \left(N_e^{\alpha}\left[fg^{-1}\right]\right)(t) = g^{-1}(t)\left(N_e^{\alpha}f\right)(t)+f(t)\left(N_e^{\alpha}g^{-1}\right)(t)\\
&= \frac{\left(N_e^{\alpha}f\right)(t)}{g(t)}-\frac{f(t)\left(N_e^{\alpha}g\right)(t)}{g^2(t)}\\
&= \frac{g(t)\left(N_e^{\alpha}f\right)(t)-f(t)\left(N_e^{\alpha}g\right)(t)}{g^2(t)}.
\end{aligned}$$

The proof is complete. □

Naturally, the following integral operator is defined.

Definition 2.6. *Let f be a locally integrable function in the interval $[a,b]$. Then the conformable integral operator is defined by*

$$ {}_aJ_{N_e}^{\alpha}f(t) = \int_a^t\frac{f(s)ds}{e^{(\alpha-1)s}} = \int_a^t e^{-(\alpha-1)s}f(s)ds. \tag{2.8}$$

The linearity of this operator can be observed in the following development.

$$\begin{aligned}{}_{a}J^{\alpha}_{N_e}\left(\beta f+\delta g\right)(t) &= \int_a^t \frac{(\beta f(s)+\delta g(s))ds}{e^{(\alpha-1)s}} \\ &= \beta\int_a^t \frac{f(s)ds}{e^{(\alpha-1)s}} + \delta\int_a^t \frac{g(s)ds}{e^{(\alpha-1)s}} \\ &= \beta\left({}_{a}J^{\alpha}_{N_e} f\right)(t) + \delta\left({}_{a}J^{\alpha}_{N_e} g\right)(t).\end{aligned}$$

In addition, we have that

$${}_{a}J^{\alpha}_{N_e}\left[N^{\alpha}_{e} f\right](t) = \int_a^t e^{-(\alpha-1)s} N^{\alpha}_{e} f(s)ds = \int_a^t e^{-(\alpha-1)s} e^{(\alpha-1)s} f'(s)ds = f(t)-f(a),$$

and

$$N^{\alpha}_{e}\left[{}_{a}J^{\alpha}_{N_e} f\right](t) = N^{\alpha}_{e}\left[\int_a^t \frac{f(s)ds}{e^{(\alpha-1)s}}\right] = e^{(\alpha-1)s}\left(\int_a^t \frac{f(s)ds}{e^{(\alpha-1)s}}\right)' = f(t).$$

.

3. Aplications to Physiscs.

3.1. Falling Bodies. This problem considers the fall of a body of mass m, starting from rest, under the action of gravity. Suppose that the chosen reference system has as its origin the starting point (rest of the body) at a height A from the floor at the moment the fall begins, that is, at $t = 0$. The downward movement will be chosen as positive. At any point P on its trajectory, the distance traveled will be the function y dependent on time t, consequently, using ordinary derivatives, the instantaneous velocity and acceleration snapshot, respectively, are given by

$$v(t) = \frac{dy(t)}{dt} \text{ y } a = \frac{dv(t)}{dt} = \frac{d^2y(t)}{dt^2}.$$

According to Newton's Law, we have

$$F = mg = m\frac{dv(t)}{dt} \text{ ó } \frac{dv(t)}{dt} = g.$$

Then this problem is modeled by the following differential equation with initial condition as follows

$$\frac{dv(t)}{dt} = g,\ v(0) = 0.$$

Then starting with integration we get that

$$v(t) = gt + C,$$

using the initial condition we get

$$v(t) = gt.$$

Expressing in terms of the position function we have that

$$\frac{dy(t)}{dt} = gt,\ y(0) = 0.$$

Another integration will give

$$y(t) = \frac{gt^2}{2} + D,$$

and using the initial condition we obtain

$$y(t) = \frac{gt^2}{2}.$$

Now consider the problem with the conformable fractional derivative defined in Definition 2.6

$$N_e^{\alpha} v(t) = mg, \;\; v(0) = 0.$$

Using (2.7) we have

$$e^{(\alpha-1)t} v'(t) = mg \;\text{ ó }\; v'(t) = mge^{-(\alpha-1)t},$$

integrating we have that

$$v(t) = \frac{-mge^{-(\alpha-1)t}}{(\alpha-1)} + E,$$

using the initial condition we get

$$v(t) = \frac{-mge^{-(\alpha-1)t}}{(\alpha-1)} + \frac{mg}{(\alpha-1)}.$$

Similarly at the case of ordinary derivative

$$y'(t) = e^{-(\alpha-1)t}\left[\frac{-mge^{-(\alpha-1)t}}{(\alpha-1)} + \frac{mg}{(\alpha-1)}\right],$$

again, applying integration

$$y(t) = \int \frac{-mge^{-2(\alpha-1)t}}{(\alpha-1)} dt + \int \frac{mge^{-(\alpha-1)t}}{(\alpha-1)} dt = \frac{mge^{-2(\alpha-1)t}}{2(\alpha-1)^2} - \frac{mge^{-(\alpha-1)t}}{(\alpha-1)^2} + F.$$

Using the initial condition

$$y(t) = \frac{mge^{-2(\alpha-1)t}}{2(\alpha-1)^2} - \frac{mge^{-(\alpha-1)t}}{(\alpha-1)^2} + \frac{mg}{2(\alpha-1)^2}.$$

The following graph in 3.1 shows the solutions obtained by means of the conformable fractional derivative, for different values of α: $\alpha = 0.5$ (verde), $\alpha = 0.75$ (azul) y $\alpha = 0.95$ (magenta), and the solution obtained by means of ordinary derivatives

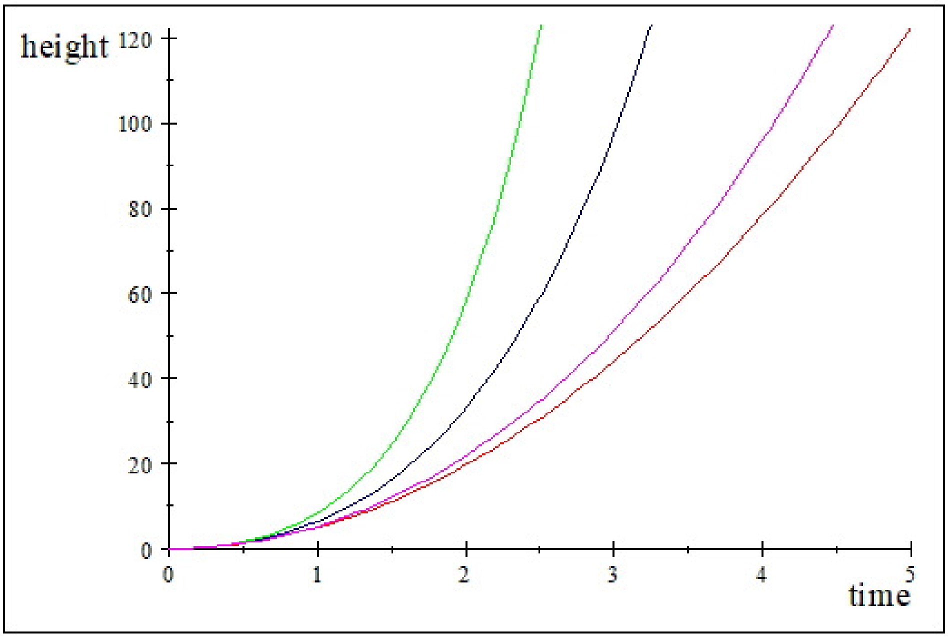

Figure 3.1: Comparative solutions for several values of α.

The following development allows us to obtain a fractional expression for the total time needed by a body to complete its fall. Making $y(t) = A$ we find

$$A = \frac{mge^{-2(\alpha-1)t}}{2(\alpha-1)^2} - \frac{mge^{-(\alpha-1)t}}{(\alpha-1)^2} + \frac{mg}{2(\alpha-1)^2},$$

with some algebraic operations we get

$$\frac{A(\alpha-1)^2}{mg} - \frac{1}{2} = \frac{e^{-2(\alpha-1)t}}{2} - e^{-(\alpha-1)t}.$$

Using the changes $u = e^{-(\alpha-1)t}$ and $s = A(\alpha-1)^2/mg - 1/2$ we can write

$$\frac{u^2}{2} - u - s = 0.$$

Then, solving the quadratic equation

$$u = \frac{1+\sqrt{1+2s}}{1} \text{ y } u = \frac{1-\sqrt{1+2s}}{1},$$

to avoid $u < 0$, then we choose the following solution

$$e^{-(\alpha-1)t} = 1 + \sqrt{1 + 2\left(\frac{A(\alpha-1)^2}{mg} - \frac{1}{2}\right)}$$

applying the logarithm function we get

$$t = \frac{-\ln\left[1 + \sqrt{1 + 2\left(\frac{A(\alpha-1)^2}{mg} - \frac{1}{2}\right)}\right]}{\alpha - 1}.$$

3.2. Cooling bodies law. Newton's law of cooling states that the rate of heat loss from a body is proportional to the temperature difference between the body and its surroundings. Heat transfer is important in physical processes because it is a type of energy that is in motion due to temperature change, for example, is present in processes of condensation, vaporization, crystallization, climatic changes and others.

Specifically, Newton's law of cooling states that the cooling of a body is directly proportional to the difference between the initial temperature of a body $T(t), t > 0$, and that of the environment T_a, and follows the following differential model

$$\frac{dT}{dt} = -k\left(T(t) - T_a\right), \;\; T(0) = T_0,$$

where T_0 is the initial temperature in $t = 0$. The solution shows how cooling of a body follows a law of exponential decay of the form

$$T(t) = T_a + (T_0 - T_a)e^{-kt}.$$

The graph of this solution is shown below in 3.2.

Using (2.7) we can write fractional differential model as follows

$$\frac{d^\alpha T}{dt^\alpha} = -k\left(T(t) - T_a\right),$$

so, we have

$$e^{\alpha-1}T'(t) = -k\left(T(t) - T_a\right) \Rightarrow \frac{dT(t)}{dt} = -e^{1-\alpha}k\left(T(t) - T_a\right) \Rightarrow \frac{dT}{T(t) - T_a} = -e^{1-\alpha}k,$$

then applying integration

$$\ln(T(t) - T_a) = -e^{1-\alpha}kt + C.$$

Using the initial condition we obtain

$$\ln(T_0 - T_a) = C,$$

so we have

$$\ln(T(t) - T_a) = -e^{1-\alpha}kt + \ln(T_0 - T_a),$$

applying the properties of the logarithm function we get

$$\ln\left(\frac{T(t) - T_a}{T_0 - T_a}\right) = -e^{1-\alpha}kt,$$

and using the exponential function we obtain

$$T(t) = e^{-e^{1-\alpha}kt}\left(T_0 - T_a\right) + T_a.$$

Next, we show the graph of this solution in 3.3 for some values of α: $\alpha : \alpha = 0.5$ (green), $\alpha = 0.75$ (azul), $\alpha = 0.95$ (sienna) , for the comparison with the graph of the obtained solution using ordinary derivative (red).

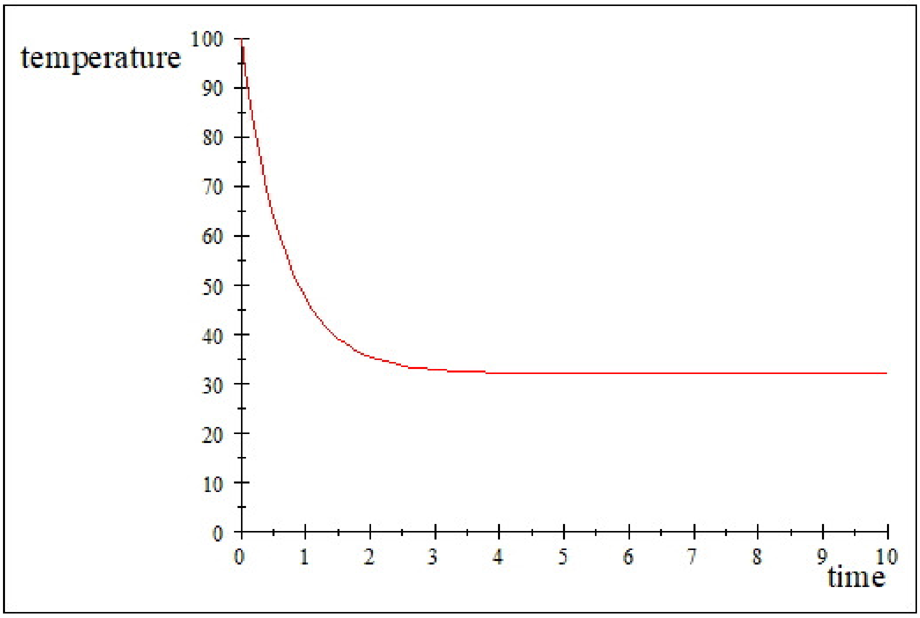

Figure 3.2: Graph of the solution of the differential equations with ordinary derivative.

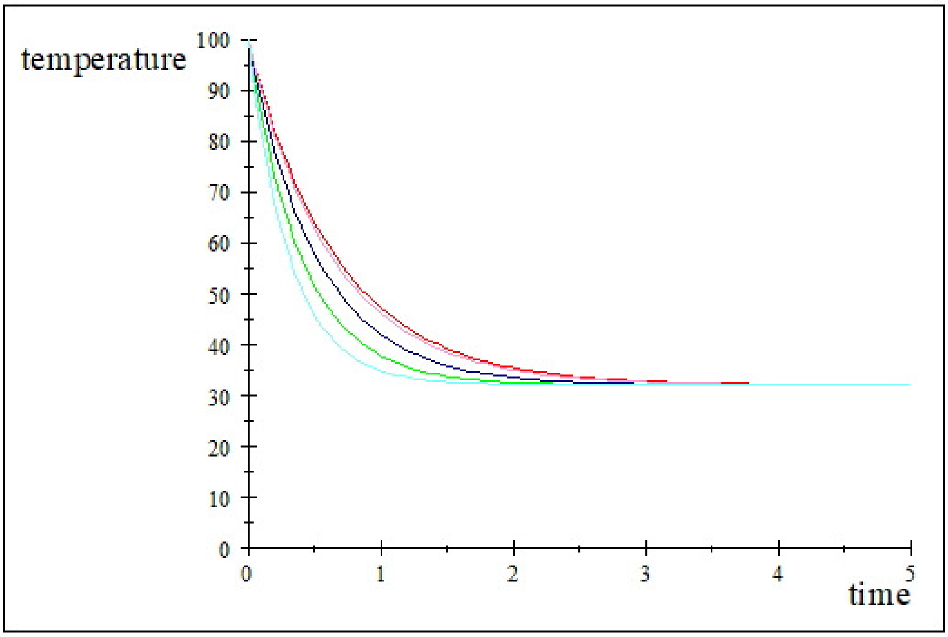

Figure 3.3: Comparative solutions for several values of α.

4. Conclusions. In this article, a brief review of the classical fractional derivatives was made, it also focuses its attention on the conformable derivative given in (2.6), evaluating its basic properties as a differential operator and from these apply said derivative fractional in classical problems of Physics, such as: the free fall of bodies and the cooling of bodies law given by Newton. We find original solutions to these problems and some contrast graph are given.

We hope that the results found will stimulate research in this area of Fractional Calculus.

5. Acknowledgments. The authors thank the Postgraduate Institute of the Universidad Técnica de Manabí for its technical contribution in the preparation of this article. Similarly, thank to the editorial team of the journal Selecciones Matemáticas from University of Trujillo (Perú).

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