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# On somewhat near continuity and some applications

# Sobre la continuidad algo cercana y algunas aplicaciones

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**Abstract**

*We give more properties and applications to the class of somewhat nearly continuous functions introduced by Piotrowski. Among the applications, we show that the Baire property is preserved under somewhat nearly continuous quasiopen injection images and Baire spaces are preserved under somewhat nearly continuous quasiopen countable fiber complete preimages. The later statement generalizes the results given by Noll, and Mirmostafaee and Piotrowski.*

**Keywords** . Quasi-continuous, almost quasi-continuous, semi-continuous, $\beta$-continuous, somewhat nearly continuous, Baire space.

**Resumen**

*Damos más propiedades y aplicaciones a la clase de funciones casi continuas introducidas por Piotrowski. Entre las aplicaciones, mostramos que la propiedad de Baire se conserva bajo imágenes de inyección casi abiertas casi continuas y los espacios de Baire se conservan bajo preimágenes completas de fibra contables casi continuas casi continuas. La última declaración generaliza los resultados dados por Noll y Mirmostafaee y Piotrowski.*

**Palabras clave**. Cuasicontinuo, casi cuasicontinuo, semicontinuo, $\beta$-continuo, algo casi continuo, espacio Baire.

**1. Introduction.** One of the most important tools of all mathematics is the notion of continuity. Beginning from the early stage of modern mathematics, many classes of almost continuity were introduced. The most well-known classes are: quasicontinuity, near continuity, somewhat continuity, $\alpha$-continuity and almost quasicontinuity. In 1932, Kempisty [12] introduced the notion of quasicontinuity for extending some classical results of Hahn and Baire concerning separately continuous real-valued functions of many variables. In the same year, Banach considered near continuity while proving Closed Graph Theorem [7, Theorem 4, p40] under the name of almost continuity. Somewhat continuous functions [10] are given by Frolik while investigating the invariance of Baire spaces under mappings, see also [11]. In 1965, a stronger notion to both quasicontinuity and near continuity was introduced by Njastad called $\alpha$-continuity [18]. It is known that a function is $\alpha$-continuous if and only if it is both nearly continuous and quasicontinuous. Later, Borsik [8] defined the class of almost quasicontinuous functions, this class is implied by near continuity and quasicontinuity. As a direct generalization of somewhat continuity and almost quasicontinuity, Piotrowski [22] introduced a new class, named somewhat nearly continuous functions, while working on separate versus joint continuity problems as well as on the Closed Graph Theorem. Due to the importance of this class, we continue the work of Piotrowski and give further properties and characterizations.

**2. Preliminaries.** Throughout this paper, the letters $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{P}$ and $\mathbb{R}$, respectively, stand for the set of natural, rational, irrational and real numbers. The word "space" means an arbitrary topological space. For

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a subset $A$ of a space $(X, \tau)$, the closure and interior of $A$ with respect to $X$ respectively are denoted by $\mathrm{Cl}_X(A)$ and $\mathrm{Int}_X(A)$ (or simply $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$).

**Definition 2.1.** *A subset $A$ of a space $X$ is said to be*

*(1) co-dense if* $\mathrm{Int}(A) = \emptyset$*,*

*(2) preopen [15] if* $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$*,*

*(3) semiopen [13] if* $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$*,*

*(4) $\alpha$-open [18] if* $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$*,*

*(5) $\beta$-open [1] or semipreopen [5] if* $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$*,*

*(6) somewhat open (briefly $sw$-open ) [22] if* $\mathrm{Int}(A) \neq \emptyset$ *or* $A = \emptyset$*,*

The complement of a preopen (resp. semiopen, $\alpha$-open, $\beta$-open, $sw$-open) set is preclosed (resp. semiclosed, $\alpha$-closed, $\beta$-closed, $sw$-closed).

The intersection of all preclosed (resp. semiclosed, $\alpha$-closed, $\beta$-closed) sets in $X$ containing $A$ is called the preclosure (resp. semi-closure, $\alpha$-closure, $\beta$-closure) of $A$, and is denoted by $\mathrm{Cl}_p(A)$ (resp. $\mathrm{Cl}_s(A)$, $\mathrm{Cl}_\alpha(A)$, $\mathrm{Cl}_\beta(A)$).

The union of all preopen (resp. semiopen, $\alpha$-open, $\beta$-open) sets in $X$ contained in $A$ is called the preinterior (resp. semi-interior, $\alpha$-interior, $\beta$-interior) of $A$, and is denoted by $\mathrm{Int}_p(A)$ (resp. $\mathrm{Int}_s(A)$, $\mathrm{Int}_\alpha(A)$, $\mathrm{Int}_\beta(A)$).

The family of all preopen (resp. semiopen, $\alpha$-open, $\beta$-open) subsets of $X$ is denoted by $PO(X)$ (resp. $SO(X)$, $\alpha O(X)$, $\beta O(X)$).

**Remark 2.1.** *It is well-known that for a space $X$,* $\tau \subseteq \alpha O(X) \subseteq PO(X) \cup SO(X) \subseteq \beta O(X)$.

**Definition 2.2.** *Let $X$ be a space and let $A \subseteq X$. A point $x \in X$ is said to be in the preclosure (resp. semi-closure, $\alpha$-closure, $\beta$-closure, $\gamma$-closure) of $A$ if $U \cap A \neq \phi$ for each preopen (resp. semiopen, $\alpha$-open, $\beta$-open, $\gamma$-open) set $U$ containing $x$.*

**Lemma 2.1.** *[6, Proposition 1.1]For a subset $A$ of a space $X$, we have* $\mathrm{Cl}_\beta(A) = A \cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$.

**Lemma 2.2.** *[5, Theorem 3.22] For a subset $A$ of a space $X$, we have* $\mathrm{Cl}_\beta(\mathrm{Int}(A)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$.

**Lemma 2.3.** *Let $A$ be a subset of a space $X$.*

*(i) $A$ is semiopen if and only if* $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Int}(A))$.

*(ii) $A$ is $\beta$-open if and only if* $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$.

*Proof:* (i) If $A$ is semiopen, then $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$ and so $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. For other side of the inclusion, we always have $\mathrm{Int}(A) \subseteq A$. Therefore $\mathrm{Cl}(\mathrm{Int}(A)) \subseteq \mathrm{Cl}(A)$. Thus $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Int}(A))$.

Conversely, assume that $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Int}(A))$, but $A \subseteq \mathrm{Cl}(A)$ always, so $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. Hence $A$ is semiopen.

(ii) Theorem 2.4 in [5].

**Lemma 2.4.** *Let $A$ be a nonempty subset of a space $X$.*

*(i) If $A$ is semiopen, then* $\mathrm{Int}(A) \neq \emptyset$.

*(ii) If $A$ is $\beta$-open, then* $\mathrm{Int}(\mathrm{Cl}(A)) \neq \emptyset$.

*Proof:* (i) Suppose otherwise that if $A$ is a semiopen set such that $\mathrm{Int}(A) = \emptyset$, by Lemma 2.3 (i), $\mathrm{Cl}(A) = \emptyset$ which implies that $A = \emptyset$. Contradiction!

(ii) Similar to (i).

**Lemma 2.5.** *Let $A$ be a subset of a space $X$. Then* $\mathrm{Int}(A) \neq \emptyset$ *if and only if* $\mathrm{Int}_s(A) \neq \emptyset$. *Proof:* Given a subset $A$ of $X$. If $\mathrm{Int}(A) \neq \emptyset$, then $\emptyset \neq \mathrm{Int}(A) \subseteq \mathrm{Int}_s(A)$ and so $\mathrm{Int}_s(A) \neq \emptyset$.

Conversely, if $\mathrm{Int}_s(A) \neq \emptyset$, then for some $x \in A$, there exists a semiopen set $G$ containing $x$ such that $G \subseteq A$. By Lemma 2.4 (i), $\emptyset \neq \mathrm{Int}(G) \subseteq \mathrm{Int}(A)$. Therefore $\mathrm{Int}(A) \neq \emptyset$.

**Lemma 2.6.** *[14, Theorem 2.4] Let $Y$ be a subspace of a space $X$ and let $A \subset Y$. If $Y$ is semiopen in $X$, then $A$ is semiopen in $Y$ if and only if $A$ is semiopen in $X$.*

**Lemma 2.7.** *Let $A, B$ be subsets of $X$. If $A$ is $\alpha$-open and $B$ is preopen (resp. $\beta$-open), then $A \cap B$ is preopen (resp. $\beta$-open) in $X$.*

*Proof:* Lemma 2.1 in [19] (resp. Theorem 2.7 in [1]).

**Lemma 2.8.** *Let $A, B$ be subsets of $X$. If $A$ is $\alpha$-open and $B$ is preopen (resp. $\beta$-open), then $A \cap B$ is preopen (resp. $\beta$-open) in $A$.*

*Proof:* Lemma 2.1 in [16] (resp. Lemma 2.5 in [1]).

**Lemma 2.9.** *Let $A, D$ be subsets of space $X$.*

*(i) If $A$ is open and $D$ is dense, then* $\mathrm{Cl}(A \cap D) = \mathrm{Cl}(A)$

*(ii) If $D$ is open dense, then* $\mathrm{Cl}(A) \cap D = Cl_D(A \cap D)$.

*Proof:* Standard.

**Lemma 2.10.** *Let $A$ be a subset of a space $X$. If $A$ is semiopen, then it has the Baire property.*

*Proof:* Given a semiopen set $A$. By [13, Theorem 7], $A = O \cup N$, where $O$ is open and $N$ is nowhere dense such that $O \cap N = \emptyset$. Therefore $A = O\Delta N$. Hence $A$ has the Baire property.

**3. Somewhat nearly open sets.** This section is committed to studying the main properties of somewhat nearly open sets. This type of open sets was defined by Z. Piotrowski without given many details.

**Definition 3.1.** *[22] A subset $A$ of a space $X$ is said to be somewhat nearly open (briefly $swn$-open) if $\mathrm{Int}(\mathrm{Cl}(A)) \neq \emptyset$ or $A = \emptyset$. A similar concept appeared in [2] under the name of somewhere dense sets except the empty set.*

The complement of each $swn$-open set is called $swn$-closed. That is, a set $F$ is $swn$-closed if $\mathrm{Cl}(\mathrm{Int}(F)) \neq X$ or $F = X$.

**Remark 3.1.** *Let $X$ be a space.*

*(a) a nonempty subset $A$ of $X$ is $swn$-open if and only if $\mathrm{Int}_s(\mathrm{Cl}(A)) \neq \emptyset$, see Lemma 2.5.*

*(b) a nonempty subset $A$ of $X$ is $swn$-open if and only if there is an open (or a semiopen) set $U$ such that $\emptyset \neq U \subseteq \mathrm{Cl}(A)$.*

*(c) a proper subset $B$ of $X$ is $swn$-closed if and only if there is a closed (or a semiclosed) set $F$ such that $\mathrm{Int}(B) \subseteq F \subsetneq X$.*

**Proposition 3.1.** *Any union of $swn$-open sets is $swn$-open.*

*Proof:* Let $\{A_\alpha : \alpha \in \Delta\}$ be any collection of $swn$-open subsets of a space $X$. Now

$$\begin{aligned}\mathrm{Int}(\mathrm{Cl}(\bigcup_{\alpha\in\Delta} A_\alpha)) &\supseteq \mathrm{Int}(\bigcup_{\alpha\in\Delta} \mathrm{Cl}(A_\alpha)) \\ &\supseteq \bigcup_{\alpha\in\Delta} \mathrm{Int}(\mathrm{Cl}(A_\alpha)) \neq \emptyset.\end{aligned}$$

Thus $\bigcup_{\alpha\in\Delta} A_\alpha$ is $swn$-open.

**Remark 3.2.** *The intersection of two $swn$-open sets need not be $swn$-open. For instance, take $swn$-open sets $A = [0, 1]$ and $B = [1, 2]$ in $\mathbb{R}$. Then $\mathrm{Int}(\mathrm{Cl}[A \cap B]) = \emptyset$.*

We remark that, in general, the family of $swn$-open subsets of a space $X$ does not from a topology.

Next we put Remark 2.1 and Lemma 2.4 into the following diagram which shows the relation between $swn$-open and the most well-known types of open sets.

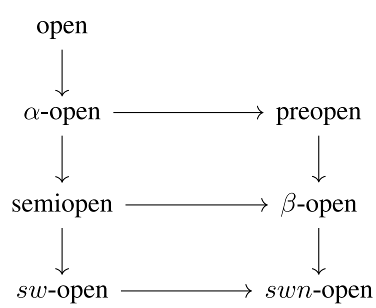


Diagram I:Connections between generalized open sets

In general, none of these implications can be replaced by equivalence as shown below:

**Example 3.1.** *Consider $\mathbb{R}$ with the usual topology. Let $A = [0, 1] \cap \mathbb{Q}$. Then $A$ is $swn$-open but not $sw$-open. If $B = \mathcal{C} \cup [2, 3]$, where $\mathcal{C}$ is the ternary Cantor set, then $B$ is $swn$-open but not $\beta$-open. Examples for other none implications can be found in [4] or the literature.*

**Remark 3.3.** *From the above example, one can conclude that the intersection of an $swn$-open set with an open, a closed or a dense set may not be $swn$-open.*

**Proposition 3.2.** *Let $A, D$ be subsets of a space $X$. If $A$ is open and $D$ is dense, then $A \cap D$ is $swn$-open in $X$.*

*Proof:* If $A = \emptyset$, then $A \cap D = \emptyset$ which is $swn$-open. Let $A$ be nonempty and let $D$ be dense. By Lemma 2.9 (i), $\mathrm{Int}(\mathrm{Cl}(A \cap D)) = \mathrm{Int}(\mathrm{Cl}(A)) \neq \emptyset$. That is $A \cap D$ is $swn$-open.

**Proposition 3.3.** *Let $A, D$ be subsets of a space $X$. If $A$ is $swn$-open and $D$ is open dense, then $A \cap D$ is $swn$-open in $D$.*

*Proof:* Let $A$ be $swn$-open and let $D$ be open dense. By using Lemma 2.9 (ii), one can get $\mathrm{Int}_D(\mathrm{Cl}_D(A\cap D)) = \mathrm{Int}(\mathrm{Cl}_D(A \cap D)) \cap D = \mathrm{Int}[\mathrm{Cl}(A) \cap D] \cap D = \mathrm{Int}(\mathrm{Cl}(A)) \cap D$. Since $\mathrm{Int}(\mathrm{Cl}(A))$ is nonempty open and $D$ is dense, they must have nonempty intersection. Therefore $\mathrm{Int}_D(\mathrm{Cl}_D(A \cap D)) \neq \emptyset$ and so $A \cap D$ is $swn$-open in $D$.

**Proposition 3.4.** *Let $A$ be a subset of a space $X$. Then either $A$ is $sw$-open or $swn$-closed.*

*Proof:* Given $A \subseteq X$. If $A$ is not $sw$-open, then $\mathrm{Int}(A) = \emptyset$. Therefore $\mathrm{Cl}(\mathrm{Int}(A)) = \emptyset \neq X$ and so $A$ is $swn$-closed.

On the other hand, if $A$ is not $swn$-closed, then $\mathrm{Cl}(\mathrm{Int}(A)) = X$. Surely $\mathrm{Int}(A) \neq \emptyset$. Thus $A$ is $sw$-open.

**Corollary 3.1.** *Each subset of a space $X$ is either $swn$-open or $swn$-closed.*

**Remark 3.4.**

*(i) Each nowhere dense is $swn$-closed. The converse is false.*

*(ii) A set is $swn$-open if and only of its closure is $swn$-open.*

*(iii) The interior of an $swn$-open set need not be $swn$-open.*

**Proposition 3.5.** *Let $A$ be a semiclosed subset of a space $X$. Then $A$ is $swn$-open if and only if it is $sw$-open.*

*Proof:* By Lemma 2.3 (i), $A$ is semiclosed if and only if $\mathrm{Int}(\mathrm{Cl}(A)) = \mathrm{Int}(A)$. The rest is clear.

**Proposition 3.6.** *Let $Y$ be a semiopen subspace of a space $X$ and let $A \subseteq Y$. Then $A$ is $swn$-open in $Y$ if and only if it is $swn$-open in $X$.*

*Proof:* Let $A$ be $swn$-open in $Y$. There exist a semiopen subset $H$ in $Y$ such that $H \subseteq \mathrm{Cl}_Y(A)$. Now, $H = H \cap Y \subseteq \mathrm{Cl}(A) \cap Y \subseteq \mathrm{Cl}(A)$. Since $Y$ is semiopen, by Lemma 2.6, $H$ is semiopen in $X$ and $H \subseteq \mathrm{Cl}(A)$. Hence $A$ is $swn$-open in $X$.

Conversely, assume that $A$ is $swn$-open in $X$. Suppose otherwise that $A$ is not $swn$-open in $Y$. Let $G$ be a semiopen set in $X$ with $G \cap Y \neq \emptyset$. Then there is a nonempty semiopen set $H \subseteq G \cap Y$ such that $A \cap H = \emptyset$. Since $H$ is semiopen in $Y$, there is a semiopen set $U$ in $X$ such that $H = U \cap Y$. Now, we have $U \subseteq G$ and $A \cap U = \emptyset$. This means that $A$ is not $swn$-open in $X$, which is impossible. Hence the result.

**Proposition 3.7.** *Let $Y$ be a dense subspace of a space $X$ and let $A \subseteq Y$. Then $A$ is $swn$-open in $Y$ if and only if it is $swn$-open in $X$.*

*Proof:* Similar to Proposition 3.6

**Proposition 3.8.** *Let $X$ be a space. A subset $A$ of $X$ is $\beta$-open if and only if $A \cap U$ is $swn$-open for each open set $U$ in $X$.*

*Proof:* Given a $\beta$-open set $A$ and an arbitrary open set $U$. By Lemma 2.7, $A \cap U$ is $\beta$-open and consequently it is $swn$-open by Lemma 2.4 (ii).

Conversely, let $x \in A$ and assume that $A \cap U$ is $swn$-open for each open set $U$ in $X$. That is $\mathrm{Int}(\mathrm{Cl}(A \cap U)) \neq \emptyset$. Now we have $\emptyset \neq \mathrm{Int}(\mathrm{Cl}(A \cap U)) \subseteq \mathrm{Int}(\mathrm{Cl}(A)) \cap Int(\mathrm{Cl}(U)) = \mathrm{Int}(\mathrm{Cl}(A)) \cap U$, which implies that $x \in \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$ and so $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. This proves that $A$ is $\beta$-open.

**Proposition 3.9.** *Let $X$ be a space. A subset $A$ of $X$ is preopen if and only if $A \cap U$ is $swn$-open for each $\alpha$-open set $U$ in $X$.*

*Proof:* Let $A$ be preopen and let $U$ be any $\alpha$-open. By Lemma 2.8, $A \cap U$ is preopen and so it is $swn$-open, (see Diagram I).

Conversely, let $x \in A$. Suppose that $A \cap U$ is $swn$-open for each $\alpha$-open $U$ in $X$. Then $\emptyset \neq \mathrm{Int}(\mathrm{Cl}(A \cap U) \subseteq \mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Int}(\mathrm{Cl}(U)) = \mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U)))$. Since $U$ is $\alpha$-open, by Lemma 2.1, $\mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Cl}_\beta(U) \neq \emptyset$ for each $\alpha$-open $U$. This means that $\mathrm{Int}(\mathrm{Cl}(A)) \cap U \cap V \neq \emptyset$ for each $\beta$-open set $V$ in $X$ containing $x$. Therefore $x \in \mathrm{Cl}_\beta(\mathrm{Int}(\mathrm{Cl}(A)) \cap U) \subseteq \mathrm{Cl}_\beta(\mathrm{Int}(\mathrm{Cl}(A)))$ and so, by Lemma 2.2, $x \in \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))) = \mathrm{Int}(\mathrm{Cl}(A))$. Hence $A$ is preopen.

**Definition 3.2.** *[24] A space $X$ is said to be*

*(1) irresolvable if any two dense subsets intersect.*

*(2) strongly irresolvable if each open subspace is irresolvable.*

**Theorem 3.1.** *Let $X$ be a space. The following are equivalent:*

*(1) strongly irresolvable,*

*(2) each open subspace is irresolvable,*

*(3) each preopen subset of $X$ is $\alpha$-open,*

*(4) each $\beta$-open subset of $X$ is semiopen,*

*(5) each preopen subset of $X$ is semiopen,*

*(6) each dense subset of $X$ is semiopen,*

*(7) each dense subset of $X$ has an interior dense,*

*(8) each co-dense subset of $X$ is nowhere dense,*

*(9) each $swn$-open subset of $X$ is $sw$-open,*

*(10) each subset of $X$ has a nowhere dense boundary.*

*(11) each subset is the union of an open set and a nowhere dense set.*

*Proof:* [24, Theorem 1.7] and [4, Lemma 2.33]

**Lemma 3.1.** *Let $X$ be a space. The following are equivalent:*

*(1) $X$ is strongly irresolvable,*

*(2) $\operatorname{Int}(\operatorname{Cl}(A \cap B)) = \operatorname{Int}(\operatorname{Cl}(A)) \cap \operatorname{Int}(\operatorname{Cl}(B))$ for each subsets $A, B$ of $X$.*

*Proof:* (1) $\Longrightarrow$ (2) The first direction is clear. That is $\operatorname{Int}(\operatorname{Cl}(A \cap B)) \subseteq \operatorname{Int}(\operatorname{Cl}(A)) \cap \operatorname{Int}(\operatorname{Cl}(B))$.

On the other hand, given any two sets $A, B$ in $X$. By Theorem 3.1 (1), $\operatorname{Int}(\operatorname{Cl}(A)) = \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(A)))$. Now, we have

$$\begin{aligned}
\operatorname{Int}(\operatorname{Cl}(A)) \cap \operatorname{Int}(\operatorname{Cl}(B)) &= \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(A))) \cap \operatorname{Int}(\operatorname{Cl}(B)) \\
&\subseteq \operatorname{Cl}(\operatorname{Int}(A)) \cap \operatorname{Int}(\operatorname{Cl}(B)) \\
&\subseteq \operatorname{Cl}[\operatorname{Int}(A) \cap \operatorname{Int}(\operatorname{Cl}(B))] \\
&\subseteq \operatorname{Cl}[A \cap B].
\end{aligned}$$

Then taking the interior of both sides, we get $\operatorname{Int}(\operatorname{Cl}(A)) \cap \operatorname{Int}(\operatorname{Cl}(B)) \subseteq \operatorname{Int}(\operatorname{Cl}(A \cap B))$. Thus $\operatorname{Int}(\operatorname{Cl}(A \cap B)) = \operatorname{Int}(\operatorname{Cl}(A)) \cap \operatorname{Int}(\operatorname{Cl}(B))$.

(2) $\Longrightarrow$ (1) Assume that (2) is true. Let $A$ be a subset of $X$. Now

$$\begin{aligned}
\operatorname{Int}(\partial(A)) &= \operatorname{Int}[\operatorname{Cl}(A) \cap \operatorname{Cl}(X \setminus A)] \\
&= \operatorname{Int}(\operatorname{Cl}(A)) \cap \operatorname{Int}(\operatorname{Cl}(X \setminus A)) \\
&= \operatorname{Int}[\operatorname{Cl}[A \cap (X \setminus A)]] \\
&= \emptyset.
\end{aligned}$$

By Theorem 3.1 (10), $X$ is strongly irresolvable.

**Theorem 3.2.** *Let $X$ be a space. If $X$ is strongly irresolvable, the family $SN(X)$ of all $swn$-open subsets of $X$ forms a topology.*

*Proof:* By Proposition 3.1, $SN(X)$ is closed under arbitrary unions and it contains the empty set by definition. Therefore, it is enough to prove that $SN(X)$ is closed under finite intersections. Let $A, B \in SN(X)$. By Lemma 3.1, $\operatorname{Int}(\operatorname{Cl}(A \cap B)) = \operatorname{Int}(\operatorname{Cl}(A)) \cap \operatorname{Int}(\operatorname{Cl}(B))$. By the choice of $A$, $B$, $\operatorname{Int}(\operatorname{Cl}(A))$ and $\operatorname{Int}(\operatorname{Cl}(B))$ are nonempty open sets in $X$. Since $X$ is strongly irresolvable, $\operatorname{Int}(\operatorname{Cl}(A)) \cap \operatorname{Int}(\operatorname{Cl}(B)) \neq \emptyset$. Thus $\operatorname{Int}(\operatorname{Cl}(A \cap B)) \neq \emptyset$. Hence the proof.

**4. Somewhat near continuity.** In this section, we further study the class of somewhat nearly continuous functions.

**Definition 4.1.** *[22] Let $X, Y$ be spaces. A function $f : X \to Y$ is said to be somewhat nearly continuous (briefly $swn$-continuous) if the inverse image of each open set in $Y$ is $swn$-open in $X$. We remark that an $swn$-continuous function is an $SD$-continuous surjection in [3].*

The above definition can be stated as:

**Remark 4.1.** *A function $f : X \to Y$ is $swn$-continuous if for each $x \in X$ and each open set $V$ in $Y$ containing $f(x)$, there exists an $swn$-open set $U$ in $X$ containing $x$ such that $f(U) \subset V$.*

**Definition 4.2.** *For a subset $A$ of a space $X$, we introduce the following:*

*(i) $\operatorname{Cl}_{swn}(A) = \bigcap\{F : F$ is $swn$-closed in $X$ and $A \subseteq F\}$.*

*(ii) $\operatorname{Int}_{swn}(A) = \bigcup\{O : O$ is $swn$-open in $X$ and $O \subseteq A\}$.*

**Proposition 4.1.** *Let $X, Y$ be spaces. For a function $f : X \to Y$, the following are equivalent:*

*(1) $f$ is $swn$-continuous,*

*(2) $f^{-1}(F)$ is $swn$-closed set in $X$, for each closed set $F$ in $Y$,*

*(3) $f(\operatorname{Cl}_{swn}(A)) \subset \operatorname{Cl}(f(A))$, for each subset $A$ of $X$,*

*(4) $\operatorname{Cl}_{swn}(f^{-1}(B)) \subset f^{-1}(Cl(B))$, for each subset $B$ of $Y$,*

*(5) $f^{-1}(\operatorname{Int}(B)) \subset \operatorname{Int}_{swn}(f^{-1}(B))$, for each subset $B$ of $Y$,*

*Proof:* Follows from the definition of $swn$-continuity.

**Theorem 4.1.** *Let $X, Y$ be spaces. For a function $f : X \to Y$, the following are equivalent:*

*(1) $f$ is $swn$-continuous,*

*(2) for each open subset $V$ of $Y$ with $f^{-1}(V) \neq \emptyset$, there exists a nonempty open set $U$ in $X$ such that $U \subseteq \operatorname{Cl}(f^{-1}(V))$,*

*(3) for each closed subset $F$ of $Y$ with $f^{-1}(F) \neq X$, there exists a proper closed $E$ in $X$ such that $\operatorname{Int}(f^{-1}(F)) \subseteq E$,*

*(4) for each open dense subset $D$ of $X$, then $f(D)$ is dense in $f(X)$.*

*Proof:* (1) $\Longrightarrow$ (2) Remark 3.1 and the definition of $swn$-continuity.

(2) $\Longrightarrow$ (3) Let $F$ be a closed set in $Y$ such that $f^{-1}(F) \neq X$. Then $Y \setminus F$ is open in $Y$ with $f^{-1}(Y \setminus F) \neq \emptyset$. By (2), there exists an open set $U$ in $X$ such that $\emptyset \neq U \subseteq \operatorname{Cl}(f^{-1}(Y \setminus F)) =$

$X \setminus \mathrm{Int}(f^{-1}(F))$. This implies that $\mathrm{Int}(f^{-1}(F)) \subseteq X \setminus U \neq X$. If $E = X \setminus U$, then $E$ is a proper closed set that satisfies the required property.

(3) $\Longrightarrow$ (4) Let $D$ be open dense in $X$. We need to prove that $f(D)$ is dense in $f(X)$. Suppose otherwise that $f(D)$ is not dense in $f(X)$. There exists a proper closed set $F$ such that $f(D) \subseteq F \subset f(X)$. Therefore $D \subseteq f^{-1}(F)$. By (3), there exist a proper closed set $E$ in $X$ such that $D \subseteq \mathrm{Int}(f^{-1}(F)) \subseteq E \subset X$. This contradicts that $D$ is dense in $X$. Thus (4) holds.

(4) $\Longrightarrow$ (1) W.l.o.g, let $H$ be an open set in $Y$ with $f^{-1}(H) \neq \emptyset$, because if $f^{-1}(H) = \emptyset$, then it is trivially $swn$-open. Suppose that $f^{-1}(H)$ is not $swn$-open. That is $\mathrm{Int}(\mathrm{Cl}(f^{-1}(H)) = \emptyset$. Therefore $\mathrm{Cl}(\mathrm{Int}(X \setminus f^{-1}(H)) = X$. This implies that $\mathrm{Int}(X \setminus f^{-1}(H))$ is dense in $X$. By (4), $f(X \setminus f^{-1}(H))$ is dense in $f(X)$, i.e., $\mathrm{Cl}(f(X \setminus f^{-1}(H))) = f(X)$. This yields that $\mathrm{Cl}(f(X) \setminus H) = f(X) \setminus H = f(X)$ and so $H = \emptyset$. Contradiction to the choice of $H$. It follows that $\mathrm{Int}(\mathrm{Cl}(f^{-1}(H))$ must not be empty. Thus $f^{-1}(H)$ is $swn$-open in $X$.

**Theorem 4.2.** *For a one to one function $f$ from a space $X$ onto a space $Y$, the following are equivalent:*

*(1) $f$ is $swn$-continuous,*
*(2) for each (closed) nowhere dense subset $N$ of $X$, then $f(N)$ is co-dense in $Y$.*

*Proof:* (1) $\Longrightarrow$ (2) Let $N$ be a (closed) nowhere dense set in $X$. We need to show that $f(N)$ is co-dense in $Y$. Suppose otherwise, then there is a nonempty open set $H$ in $Y$ such that $H \subseteq f(N)$ and so $f^{-1}(H) \subseteq f^{-1}(f(N)) = N$. By (1), $\emptyset \neq \mathrm{Int}(\mathrm{Cl}(f^{-1}(H)) \subseteq \mathrm{Int}(\mathrm{Cl}(N) = \mathrm{Int}(N)$. This proves that is not (closed) nowhere dense in $X$, which is contradiction. Hence (2) is established.

(2) $\Longrightarrow$ (1) Let $H$ be an open set in $Y$. If $f^{-1}(H) = \emptyset$, then $f^{-1}(H)$ is $swn$-open by the definition. Let $f^{-1}(H) \neq \emptyset$. If $f^{-1}(H)$ is not $swn$-open, then it is nowhere dense in $X$. By (2), $f(f^{-1}(H))$ is co-dense in $Y$. That is, $\emptyset = \mathrm{Int}(f(f^{-1}(H))) = H$. This is impossible. Therefore $f$ is $swn$-continuous.

**Theorem 4.3.** *A function $f$ from a space $X$ onto a space $Y$ is $swn$-continuous if and only $f^{-1}(A)$ is $swn$-open for each $sw$-open set $A$ in $Y$.*

*Proof:* Assume that $f$ is $swn$-continuous. Let $A$ be $sw$-open in $Y$. If $A = \emptyset$, then $\emptyset = f^{-1}(H)$ is clearly $swn$-open. Let $A \neq \emptyset$. Then there is a nonempty open set $H$ in $Y$ such that $H \subseteq A$. Therefore $f^{-1}(H) \subseteq f^{-1}(A)$. By assumption, $\emptyset \neq \mathrm{Int}(\mathrm{Cl}(f^{-1}(H))) \subseteq \mathrm{Int}(\mathrm{Cl}(f^{-1}(A)))$. This proves that $f^{-1}(A)$ is $swn$-open.

Conversely, if $G$ is an open set in $Y$, then it is $sw$-open. By assumption, $f^{-1}(G)$ is $swn$-open. Hence $f$ is $swn$-continuous.

**5. Comparison and applications.** In this section, we study the connection between $swn$-continuous function and other well-known classes of continuity, and then more properties of $swn$-continuity are given.

Let us first recall the following definition:

**Definition 5.1.** *A function $f$ from a space $X$ into a space $Y$ is called*

*(1) quasicontinuous [12] or semicontinuous [13] if the inverse image of each open set in $Y$ is semiopen in $X$,*
*(2) nearly continuous [23] or precontinuous [15] if the inverse image of each open set in $Y$ is preopen in $X$,*
*(3) $\alpha$-continuous [18] if the inverse image of each open set in $Y$ is $\alpha$-open in $X$,*
*(4) almost quasicontinuous [8] or $\beta$-continuous [1] if the inverse image of each open set in $Y$ is $\beta$-open in $X$,*
*(5) somewhat continuous [11] (briefly $sw$-continuous) if the inverse image of each open set in $Y$ is $sw$-open in $X$,*
*(6) contra-semicontinuous [9] if the inverse image of each open set in $Y$ is semiclosed in $X$,*
*(7) quasiopen or semiopen [21] if the image of each open set in $X$ is semiopen in $Y$,*
*(8) quasiclosed or semiclosed [21] if the image of each closed set in $X$ is semiclosed in $Y$.*
*(9) somewhat open [11] (briefly $sw$-open) if the image of each open set in $X$ is $sw$-open in $Y$,*

The following is the consequence of the Diagram I (see also [22, Diagram I]):

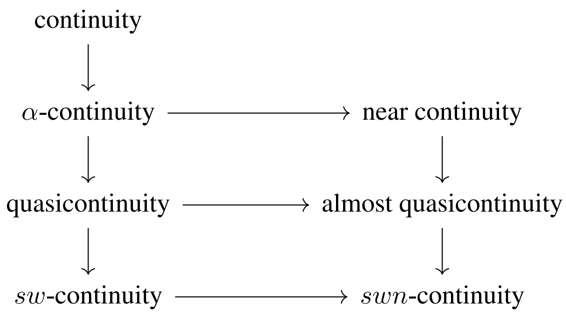


Diagram II:Relations between generalized types of continuity

In general, none of these implications can be replaced by equivalence. We only provide counterexamples for functions directly related to $swn$-continuity. Other examples are available in the literature.

**Example 5.1.** *Let $X = Y = \mathbb{R}$ with usual topology and let $f : X \to Y$ be the Dirichlet function. That is $f$ is defined by*

$$f(x) = \begin{cases} 0, & x \notin \mathbb{Q}; \\ 1, & x \in \mathbb{Q}. \end{cases}$$

*Then $f$ is $swn$-continuous but not $sw$-continuous. The inverse image of any open subset of $Y$ containing only $1$ is $\mathbb{Q}$ which is not $sw$-open in $X$.*

**Example 5.2.** *Let $X = Y = \mathbb{R}$ with usual topology. Define the function $f : X \to Y$ by*

$$f(x) = \begin{cases} x, & \textit{if } x \notin \{0, 1\}; \\ 0, & \textit{if } x = 1; \\ 1, & \textit{if } x = 0. \end{cases}$$

*One can easily show $f$ is $swn$-continuous because the inverse image of any interval always contains some interval, so the interior of its closure cannot be empty. On the other hand $f$ is not almost quasicontinuous. Take the open set $G = (-\varepsilon, \varepsilon)$, where $\varepsilon < 1$. Therefore $f^{-1}(G) = (-\varepsilon, 0) \cup (0, \varepsilon) \cup \{1\}$. But $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(G)))) = [-\varepsilon, \varepsilon]$ and so $f^{-1}(G) \nsubseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(G))))$. In conclusion, $f$ cannot be almost quasicontinuous.*

**Proposition 5.1.** *Let $X, Y$ be spaces and let $D$ be an open dense subspace of $X$. If $f : X \to Y$ is $swn$-continuous on $X$, then $f|_D$ is $swn$-continuous on $D$.*

*Proof:* Follows from Proposition 3.3.

From Remark 3.3 and Propositions 2.21-2.28 in [22], we have

**Remark 5.1.** *The restriction of an $swn$-continuous function to an open (resp. a dense, a closed) subspace need not be $swn$-continuous.*

**Theorem 5.1.** *Let $X, Y$ be spaces. A function $f : X \to Y$ is almost quasicontinuous if and only if $f|_U$ is $swn$-continuous for each open subset $U \subseteq X$.*

*Proof:* Assume that $f$ is almost quasicontinuous. Let $H$ be an open subset of $Y$ and let $U$ be an open subset of $X$. By assumption $f^{-1}(H)$ is $\beta$-open in $X$. By Lemma 2.8, $f^{-1}(H) \cap U$ is $\beta$-open in $U$ and thus, by Lemma 2.4 (ii), $f^{-1}(H) \cap U$ is an $swn$-open subset of $U$. Hence, $f|_U$ is $swn$-continuous.

Conversely, suppose that $f|_U$ is $swn$-continuous for each open subset $U$ of $X$. Let $H$ be an open set in $Y$. Then $f^{-1}|_U(H) = f^{-1}(H) \cap U$ is $swn$-open in $U$. Since $U$ is an open subset of $X$ and each open is semiopen, by Proposition 3.6, $f^{-1}(H) \cap U$ is $swn$-open in $X$ for each open $U$ and consequently, by Proposition 3.8, $f^{-1}(H)$ is $\beta$-open in $X$. Thus $f$ is almost quasicontinuous.

**Theorem 5.2.** *Let $X, Y$ be spaces. A function $f : X \to Y$ is nearly continuous if $f|_U$ is $swn$-continuous for each $\alpha$-open subset $U \subseteq X$.*

*Proof:* By the same steps given in the proof of Theorem 5.1 and using Proposition 3.9, one can obtain the proof.

**Theorem 5.3.** *Let $X, Y$ be spaces and let $f : X \to Y$ be a function. If $X$ is strongly irresolvable, then $f$ $swn$-continuous if and only if it is $sw$-continuous*

*Proof:* From Theorem 3.1 (9).

**Theorem 5.4.** *Let $X, Y$ be spaces. For a function $f : X \to Y$ , the following are equivalent:*

*(1) $f$ is $swn$-continuous and contra-semicontinuous,*

*(2) $f$ is $sw$-continuous and contra-semicontinuous,*

*Proof:* From Lemma 3.5.

**Proposition 5.2.** *Let $f$ be a one to one function from a space $X$ onto a space $Y$. Then $f$ quasiopen if and only if it is quasiclosed.*

*Proof:* Obvious.

**Theorem 5.5.** *For a one to one quasiopen function $f$ from a space $X$ onto a space $Y$, the following are equivalent:*

*(1) $f$ is $swn$-continuous,*

*(2) for each (closed) nowhere dense subset $N$ of $X$, then $f(N)$ is nowhere dense in $Y$,*

*(3) for each $swn$-open subset $A$ of $Y$, then $f^{-1}(A)$ is $swn$-open in $X$,*

*(4) $f$ is almost quasicontinuous.*

*Proof:* (1) $\Longrightarrow$ (2) Let $N$ be a closed nowhere dense set in $X$. By quasiopenness of $f$, $\text{Int}(\text{Cl}(f(N))) \subseteq f(N)$ and so $\text{Int}(\text{Cl}(f(N))) = \text{Int}(f(N))$. By Theorem 4.2, $\text{Int}(f(N)) = \emptyset$. Thus $\text{Int}(\text{Cl}(f(N))) = \emptyset$. Hence $f(N)$ is nowhere dense in $Y$.

(2)$\Longleftrightarrow$(3) Suppose (3) is not true. There exists an $swn$-open subset $A$ of $Y$ such that $f^{-1}(A)$ is not $swn$-open, which means that $f^{-1}(A)$ is a nowhere dense in $X$. By (2), $f(f^{-1}(A)) = A$ is nowhere dense, i.e., $A$ is not $swn$-open. This is contradiction. Hence (3) must be true. The converse can be proved similarly.

(2) $\Longrightarrow$ (4) Let $H$ be an open set in $Y$. We want to show that $f^{-1}(H)$ is $\beta$-open in $X$. Let $x \notin \text{Cl}(\text{Int}(\text{Cl}(f^{-1}(H))))$. Then there is an open set $G$ in $X$ containing $x$ such that $\text{Int}(\text{Cl}(f^{-1}(H))) \cap G = \emptyset$ and so $\emptyset = \text{Int}(\text{Cl}(f^{-1}(H))) \cap \text{Int}(\text{Cl}(G)) \supseteq \text{Int}(\text{Cl}(f^{-1}(H) \cap G))$. Therefore $f^{-1}(H) \cap G$ is nowhere dense in $X$. By (2), $f(f^{-1}(H) \cap G) = H \cap f(G)$ is nowhere dense in $Y$. This implies that $\text{Int}(H \cap f(G)) = H \cap \text{Int}(f(G)) = \emptyset$ and so $H \cap \text{Cl}(\text{Int}(f(G))) = \emptyset$. Since $f$ is quasiopen, then $f(G) \subseteq \text{Cl}(\text{Int}(f(G)))$. Therefore $H \cap f(G) = \emptyset$ and then $f^{-1}(H) \cap G = \emptyset$. Thus $x \notin f^{-1}(H)$. This yields that $f^{-1}(H) \subseteq \text{Cl}(\text{Int}(\text{Cl}(f^{-1}(H))))$, which establishes that, $f$ is almost quasicontinuous.

(4) $\Longrightarrow$ (1) Let $V$ be an open set in $Y$. If $V = \emptyset$, clearly its inverse is $swn$-open. Suppose that $V \neq \emptyset$. By (4), $f^{-1}(V) \subseteq \text{Cl}(\text{Int}(\text{Cl}(f^{-1}(V))))$. By Lemma 2.4 (ii), $\text{Int}(\text{Cl}(f^{-1}(V))) \neq \emptyset$. Thus $f$ is $swn$-continuous.

**Proposition 5.3.** *Let $f$ be a one to one quasiopen $swn$-continuous function $f$ from a space $X$ onto a space $Y$. If $M$ is a meager set in $X$, then $f(M)$ is meager in $Y$.*

*Proof:* Let $M$ be a meager set in $X$. Then $M = \bigcup_{i=1}^{\infty} N_i$ such tat $N_i$ are nowhere set in $X$ for $i = 1, 2, \cdots$. Therefore

$$f(M) = f\left(\bigcup_{i=1}^{\infty} N_i\right) = \bigcup_{i=1}^{\infty} f(N_i).$$

By Theorem 5.5 (2), $f(N_i)$ is nowhere dense for each $i$. Hence $f(M)$ is meager in $Y$.

**Theorem 5.6.** *Let $X, Y$ be space and let $f : X \to Y$ be a function. If $f$ is quasiopen, the following are equivalent:*

*(1) $f$ is $swn$-continuous,*

*(2) for each open dense set $D$ in $X$, then $\text{Int}(f(N))$ is dense in $f(X)$.*

*Proof:* (1) $\Longrightarrow$ (2) Let $D$ be open dense in $X$. By Theorem 4.1 (4), $f(D)$ is dense in $f(X)$. Since $f$ is quasiopen, then $f(D) \subseteq \text{Cl}(\text{Int}(f(D)))$ and so $f(X) = \text{Cl}(f(D)) = \text{Cl}(\text{Int}(f(D)))$ by Lemma 2.3 (i). Thus $\text{Int}(f(N))$ is dense in $f(X)$.

(2) $\Longrightarrow$ (1) Straightforward (from Theorem 4.1 (4) $\Longrightarrow$ (1)).

**Theorem 5.7.** *Let $f$ be a one to one quasiopen $swn$-continuous function $f$ from a space $X$ onto a space $Y$. If $A \subseteq X$ has the Baire property, then $f(A)$ has the Baire property in $Y$.*

*Proof:* Let $A \subseteq X$ be a set of Baire property. Then $A = G \Delta N$ for some open $G$ and meager $N$ subsets of $X$. Now, $f(A) = f(G) \Delta f(N)$. By Proposition 5.3, $f(N)$ is meager. It is enough to show that $f(G)$ has the Baire property. Since $G$ in open and $f$ is quasiopen, so $f(G)$ a semiopen set in $Y$, by Lemma 2.10, $f(G)$ has the Baire property. Thus $f(A)$ has the Baire property.

**Theorem 5.8.** *Let $f$ be a one to one quasiopen $swn$-continuous function $f$ from a space $X$ onto a Baire space $Y$. Then $X$ is a Baire space.*

*Proof:* Assume that $G$ is an open meager subset of $X$. By Proposition 5.3, $f(G)$ is meager in $Y$. But $f$ is quasiopen, so $f(G)$ is semiopen in $Y$. By Lemma 2.4 (i), $\text{Int}(f(G)) \neq \emptyset$. Contradiction to the assumption that $Y$ is Baire. Hence $X$ is a Baire space.

The above Theorem is a slight generalization of the following result given by Noll:

**Corollary 5.1.** *Let $f$ be a one to one $sw$-open open $sw$-continuous function $f$ from a space $X$ onto a Baire space $Y$. Then $X$ is a Baire space.*

*Proof:* Corollary of Theorem 1 in [20], (*c.f.* Theorem 18 in [11]).

We remark that the ”one to one” condition in Theorem 5.8 can be weakened to ”countably fiber-complete”.

**Definition 5.2.** *[17] Let $X, Y$ be spaces. A function $f : X \to Y$ is called countably fiber-complete if for each centered sequence $\{G_n\}_{n\in\mathbb{N}}$ of open subsets of $X$, $\bigcap_{n\in\mathbb{N}} G_n \neq \emptyset$, if there is $y \in Y$ such that $f^{-1}(y) \cap G_n \neq \emptyset$ for each $n$.*

**Theorem 5.9.** *Let $f$ be a quasiopen $swn$-continuous countably fiber-complete function $f$ from a space $X$ onto a Baire space $Y$. Then $X$ is a Baire space.*

*Proof:* Let $\{D_n\}_{n\in\mathbb{N}}$ be a countable collection of dense open subsets of $X$. We need to show that $\bigcap_{n\in\mathbb{N}} D_n$ is dense in $X$. Let $G$ be any nonempty open subset of $X$. Since $f$ is quasiopen, then $f(G)$ is a semiopen subset of $Y$. By Lemma 2.4 (i), $\mathrm{Int}(f(G)) \neq \emptyset$. Set $H = \mathrm{Int}(f(G))$. By Theorem 5.6, $\{\mathrm{Int}(f(D_n))\}_{n\in\mathbb{N}}$ is a countable collection of dense subsets of $Y$. Since $Y$ is Baire, $\bigcap_{n\in\mathbb{N}} \mathrm{Int}(f(D_n))$ is dense in $Y$. It follows that $\bigcap_{n\in\mathbb{N}} \mathrm{Int}(f(D_n)) \cap H \neq \emptyset$. Let $y \in \bigcap_{n\in\mathbb{N}} \mathrm{Int}(f(D_n)) \cap H$. This implies that $\{y\} \cap f(D_n) \cap H \neq \emptyset$ and therefore $f^{-1}\{y\} \cap D_n \cap G \neq \emptyset$ for each $n$. By countable fiber-completeness of $f$, $\bigcap_{n\in\mathbb{N}} D_n \cap G \neq \emptyset$, which means that $\bigcap_{n\in\mathbb{N}} D_n$ is dense in $X$. This proves that $X$ is a Baire space.

The next example shows that the condition quasiopenness of a function $f$ in Theorems 5.8 and 5.9 cannot be dropped:

**Example 5.3.** *Let $X = \mathbb{R}$ with the right order topology, i.e., the topology generated by the basis $B_a = \{x : x > a\}$, let $Y = \mathbb{R}$ with the finite complement topology and define $f : X \to Y$ to be the identity function. We claim that $f$ is $swn$-continuous but not quasiopen. On the other hand $Y$ is a Baire space but $X$ is not. Clearly $f$ is one to one and onto (consequently $f$ satisfies countable fiber-completeness). First we want to show that $f$ is $swn$-continuous. Given any open set $G$ in $Y$, then it has the form $G = (-\infty, x_1) \cup (x_1, x_2) \cup \cdots \cup (x_n, \infty)$ for $x_1, x_2, \ldots, x_n \in \mathbb{R}$. It follows that $f^{-1}(G)$ always contains an open set, and each open set in $X$ is dense, so $\mathrm{Int}(\mathrm{Cl}(f^{-1}(G))) \neq \emptyset$. Thus $f$ is $swn$-continuous. Now, take the open set $(0, \infty)$ in $X$, then $(0, \infty) = f((0, \infty)) \nsubseteq \mathrm{Cl}(\mathrm{Int}(f(0, \infty))) = \emptyset$. Therefore $f$ is not quasiopen. The nowhere sets in $Y$ are only finite. Then $Y$ cannot be written a countable union of finite sets, so it is a Baire space. While $X = \bigcup_{r\in\mathbb{Q}} N_r$, where $N_r = \{x : x < r\}$ is nowhere dense for each $r$, [25, p74]. Hence $X$ is not a Baire space.*

We shall compare Theorems 5.8 and 5.9 with the following results by Noll and Mirmostafaee and Piotrowski respectively. We claim that our results are superior and the next example proves our claim.

**Theorem 5.10.** *Let $f$ be a one-to-one $sw$-open $sw$-continuous function $f$ from a space $X$ onto a Baire space $Y$. Then $X$ is a Baire space.*

*Proof:* Corollary of Theorem 1 in [20], (*c.f.* Theorem 18 in [11]).

**Theorem 5.11.** *[17, Theorem 1.7] Let $f$ be a $sw$-open and $sw$-continuous countably fiber complete function from a space $X$ onto a Baire space $Y$. Then $X$ is a Baire space.*

**Example 5.4.** *Let $r$ be the usual topology on $\mathbb{R}$ and let $\theta$ be a topology on $\mathbb{R}$ generated by $r \cup \{\mathbb{P}\}$. Suppose that $g : (\mathbb{R}, r) \to (\mathbb{R}, \theta)$ is the identity function. One can easily see that $g$ sends open sets to open, so $g$ is semiopen and consequently $sw$-open. We now check that $g$ is $swn$-continuous. Take any open set $H$ in $\theta$, its inverse either contains an open interval or a subset of $\mathbb{P}$ and in both cases $\mathrm{Int}(\mathrm{Cl}(g^{-1}(H))) \neq \emptyset$. If we take $\mathbb{P}$ as an open set in $\theta$, then $\mathrm{Int}(g^{-1}(\mathbb{P})) = \emptyset$ and so $g$ cannot be $sw$-continuous. On the other hand, by Baire category theorem both $(\mathbb{R}, r)$ and $(\mathbb{R}, \theta)$ are Baire spaces.*

In conclusion, we have given two examples which verified that Theorems 5.8 and 5.9 are natural generalizations of Theorems 5.10 and 5.11.

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