Journal homepage http://revistas.unitru.edu.pe/index.php/SSMM

SELECCIONES MATEMÁTICAS
Universidad Nacional de Trujillo
ISSN: 2411-1783 (Online)
2020; Vol. 7(2): 214-221.

Hypersurfaces of the spherical type degenerated

Hipersuperficies de tipo esférico degenerado

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Received, Oct. 13, 2020 *Accepted, Dec. 04, 2020*

How to cite this article:
Corro A, Dias D, Riveros C. *Hypersurfaces of the spherical type degenerated*. Selecciones Matemáticas. 2020;7(2):214–221. http://dx.doi.org/10.17268/sel.mat.2020.02.03

Abstract

In this work, we define the hypersurfaces of the spherical type degenerated *(in short DST-hypersurfaces), these hypersurfaces has the geometric property that the middle spheres pass through the origin of the Euclidean space. We present a representation for these hypersurfaces in the case where the stereographic projection of the Gauss map N is given by the identity application. We characterize the DST-hypersurfaces through a differential equation and we give an explicit example of a two-parameter family of DST-hypersurfaces with planar lines of curvature foliated by $(n-1)$-dimensional spheres. Moreover, we classify the DST-hypersurfaces of rotation.*

Keywords . EDSGW-surfaces, surfaces of the spherical type, support function. planar lines of curvature.

Resumen

En este artículo, definimos las hipersuperficies de tipo esférico degenerado(en abreviatura DST-hipersuperficies), estas hipersuperficies tienen la propriedad geométrica de que las esferas medias pasan por el origen del espacio Euclidiano. Presentamos una representación para estas hipersuperficies para el caso en que la proyección estereográfica de la aplicación de Gauss N es dada por la aplicación identidad. Caracterizamos las DST-hipersuperficies através de una ecuación diferencial y damos un ejemplo explícito de una familia a dos parámetros de DST-hipersuperficies con líneas de curvatura planas foliadas por esferas de dimensión $(n-1)$. Además, clasificamos las DST-hipersuperficies de rotación.

Palabras clave. Superficies EDSGW, superficies de tipo esférico, función soporte, líneas de curvatura planas.

1. Introduction. Let $M \subset \mathbb{R}^3$ be a surface oriented by its normal Gauss map N. Given $\nu \in \mathbb{R}^3$, the functions $\Psi_\nu, \Lambda_\nu : M \to \mathbb{R}^3$ given by $\Psi_\nu(p) = \langle p-\nu, N(p)\rangle$, $\Lambda_\nu(p) = \langle p-\nu, p-\nu\rangle$, $p \in M$, where $\langle,\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^3$, are called *support function* and quadratic *distance function* with respect to $\nu \in \mathbb{R}^3$, respectively. Geometrically, $\Psi_\nu(p)$ measures the signed distance from ν to the tangent plane H_pM of M in p and $\Lambda_\nu(p)$ measures the square of the distance from p to ν.

In 1888, Appell [1] studied a class of oriented surfaces in $\mathbb{R}^3$ associated with area preserving transformations in the sphere. After, Ferreira and Roitman [5] showed that these surfaces are such that there is a fixed point $\nu \in \mathbb{R}^3$ such that the mean curvature H, the Gaussian curvature K and the support function Ψ_ν satisfy $H + \Psi_\nu K = 0$.

In [3], the authors study a special class of oriented surfaces $M \subset \mathbb{R}^3$ that satisfy a relation of the form $2\Psi_\nu H + \Lambda_\nu K = 0$, for some fixed point $\nu \in \mathbb{R}^3$. Called *generalized special Weingarten surfaces depending on support function and the distance function* (in short, EDSGW-surface), this class of surfaces has the geometric property that all the middle spheres pass through a fixed point. Also, they show that these surfaces

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are invariant under dilations and inversions. Moreover, is obtained a Weierstrass type representation for this surfaces that depend on two holomorphic functions. As applications classify the EDSGW-surface of rotation and present a 2-parameter family of complete cyclic EDSGW-surfaces with an isolated singularity and foliated by non-parallel planes, which shows that one can not generalize the result of López [9] valid for linear Weingarten surfaces. Also, in [2], the authors show that the EDSGW-surfaces are in correspondence with the class of surfaces in $\mathbb{S}^2 \times \mathbb{R}$ whose Gaussian curvature K and extrinsic curvature K_E satisfy $K = 2K_E$.

The classification of certain classes of surfaces parametrized by planar lines of curvature has been object of study by some authors, for example see [7], [8], [11]-[13]. In [14], the authors characterize the Dupin hypersurfaces which have a principal curvature of constant multiplicity one, as a manifold foliated by $(n-1)$- dimensional Dupin submanifolds associated by Ribaucour transformations. In [10], the author estudies constant mean curvature n-submanifold foliated by spheres in one of the Euclidean, hyperbolic and Lorentz-Minkowski spaces($\mathbb{E}^{n+1}, \mathbb{H}^{n+1}$ or $\mathbb{L}^{n+1}$). Also, in [6] the author show that if a Minimal hypersurface $M \subset \mathbb{R}^{n+1}$ is foliated by round spheres lying in parallel planes then M is a hypersurface of revolution.

In this work, motivated by the works above, we generalize the concept of EDSGW-surfaces studied in [3] for the case of hypersurfaces in $\mathbb{R}^{n+1}$, namely we define the *hypersurfaces of the spherical type degenerated* (in short DST-hypersurfaces), these hypersurfaces include the EDSGW-surfaces. We present a representation for these hypersurfaces in the case where the stereographic projection of the Gauss map N is given by the identity application. We characterize the DST-hypersurfaces through a differential equation and we give an explicit example of a two-parameter family of DST-hypersurfaces with planar lines of curvature foliated by $(n-1)$-dimensional spheres. Finally, we classify the DST-hypersurfaces of rotation.

2. Preliminaries. In this section we give some definitions and results that will serve to show our results.

Definition 2.1. Let $M \subset \mathbb{R}^{n+1}$ be an oriented hypersurface with normal Gauss map N. Given $\nu \in \mathbb{R}^{n+1}$, the functions $\Psi_\nu, \Lambda_\nu : M \to \mathbb{R}^{n+1}$ given by $\Psi_\nu(p) = \langle p-\nu, N(p)\rangle$, $\Lambda_\nu(p) = \langle p-\nu, p-\nu\rangle$, $p \in M$, where $\langle , \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^{n+1}$, are called *support function* and quadratic *distance function* with respect to $\nu \in \mathbb{R}^{n+1}$, respectively.
Geometrically, $\Psi_\nu(p)$ measures the signed distance from ν to tangent hyperplane H_pM of M in p and $\Lambda_\nu(p)$ measures the square of the distance from p to ν.

In the next, without loss of generality we will consider ν as being the origin of the Euclidean space.

Definition 2.2. The *Mean curvature* and the *Gauss-Kronecker curvature* of M are given by

$$H = \frac{1}{n}\sum_{i=1}^{n} k_i \ , \ K = \prod_{i=1}^{n} k_i,$$

where k_i are the principal curvatures of M.

Remark 1. If $X : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$, $n \geq 2$, is a local parametrization of M and $N : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ is the Gauss map of M. Then in local coordinates we get

$$N_{,i} = \sum_{j=1}^{n} W_{ij} X_{,j}, \ 1 \leq i \leq n,$$

where $W = (W_{ij})$ is called *Weingarten matrix of* M.

Definition 2.3. The *rth-mean curvature* H_r of M is defined by

$$H_r = \frac{S_r(W)}{\binom{n}{r}},$$

where, for intergers $0 \leq r \leq n$, $S_r(W)$ is defined by

$$S_0(W) = 1,$$
$$S_r(W) = \sum_{1 \leq i_1 < \ldots < i_r \leq n} k_{i_1} \ldots k_{i_r}.$$

We observe that $H = H_1$ and $K = H_n$.

Definition 2.4. Let M be an orientable hypersurface in $\mathbb{R}^{n+1}$ with $\Psi(p) \neq 0$ for all $p \in M$, the *radius function* is given by $\widetilde{R}(p) = -\frac{\Lambda(p)}{2\Psi(p)}$.

Remark 2. It follows from definition 2.4 that the sphere with center in $p + \widetilde{R}(p)N(p)$ and radius $\widetilde{R}(p)$ has the geometric property that it passes through the origin, where $N(p)$ is Gauss map of M.

Definition 2.5. Let M be an orientable hypersurface in $\mathbb{R}^{n+1}$, for all $p \in M$, the sphere with center in $p + \frac{H_{n-1}(p)}{H_n(p)}N(p)$ and radius $\frac{H_{n-1}(p)}{H_n(p)}$ is called *middle sphere*.
In the next Theorem we present a way to obtain hypersurfaces as envelope of a sphere congruence related to systems of hydrodynamic type, this result was obtained in [4].

Theorem 2.1. *Let M be an orientable hypersurface in $\mathbb{R}^{n+1}$ with Gauss-Kronecker curvature K non-zero everywhere and Gauss map $N \neq -e_{n+1}$. There exists a local orthogonal parametrization $Y : U \to \Pi$, where U is connected open subset of $\mathbb{R}^n$, $\Pi = \{(u_1, u_2, ..., u_{n+1}) \in \mathbb{R}^{n+1} : u_{n+1} = 0\}$ and differentiable function $h : U \to \mathbb{R}$, such that M is locally parametrized by*

$$X(u) = \left(Q(u) - \frac{2R(u)}{T(u)}Y(u), -\frac{2R(u)}{T(u)}\right) \tag{2.1}$$

where $u = (u_1, ..., u_n) \in U \subset \mathbb{R}^n$,

$$T(u) = 1 + |Y(u)|^2,\ Q(u) = \sum_{i=1}^{n} \frac{h_{,i}}{L_i} Y_{,i},\ L_i = \langle Y_{,i}, Y_{,i}\rangle,\ R(u) = \langle Q(u), Y(u)\rangle - h(u). \tag{2.2}$$

The Gauss map is given by

$$N(u) = \frac{1}{1 + |Y(u)|^2}(2Y(u), 1 - |Y(u)|^2). \tag{2.3}$$

The Weingartem matrix W is given by

$$W = 2(TV - 2RI)^{-1}, \tag{2.4}$$

where I is identity matrix and the matrix $V = (V_{ij})$ is defined by

$$V_{ij} = \frac{1}{L_j}\left(h_{,ij} - \sum_{l=1}^{n} \widetilde{\Gamma}^l_{ij} h_{,l}\right) \tag{2.5}$$

$$\widetilde{\Gamma}^i_{ii} = \frac{L_{i,i}}{2L_i}, \qquad \widetilde{\Gamma}^i_{ij} = \frac{L_{i,j}}{2L_i} = -\frac{L_j}{L_i}\widetilde{\Gamma}^j_{ii}, \quad i \neq j. \tag{2.6}$$

The regularity condition of X is given by

$$P = det(TV - 2RI) \neq 0. \tag{2.7}$$

Moreover, the coefficients of the first, second and third fundamental form of X are given by

$$a_{ii} = \frac{L_i}{T^2}A_i^2 + \sum_{k\neq i}^{n}(TV_{ik})^2 L_k,\ a_{ij} = -\frac{L_j}{T}V_{ij}[A_i + A_j] + \sum_{k\neq i,j}^{n} V_{ik}V_{jk}L_k,\ 1 \leq i \neq j \leq n, \tag{2.8}$$

$$b_{ii} = \frac{2L_i}{T^2}A_i,\ b_{ij} = -\frac{2}{T}L_j V_{ij},\ 1 \leq i \neq j \leq n,\ A_i = 2R - TV_{ii}, \tag{2.9}$$

$$\langle N_{,i}, N_{,i}\rangle = \frac{4}{T^2}L_i,\ \langle N_{,i}, N_{,j}\rangle = 0,\ 1 \leq i \neq j \leq n. \tag{2.10}$$

Conversely, let $Y : U \to \Pi$ be an orthogonal parametrization of Π, where U is a connected open subset of $\mathbb{R}^n$ and a differentiable function $h : U \to \mathbb{R}$. Then (2.1) define an immersion in $\mathbb{R}^{n+1}$ with Gauss-Kronecker curvature non-zero, Gauss map given by (2.3) and (2.4)-(2.10) are satisfied.

3. DST-hypersurfaces. The following definition generalizes the EDSGW-surfaces studied in [3].

Definition 3.1. An orientable hypersurface $M \subset \mathbb{R}^{n+1}$ is a *hypersurface of the spherical type degenerated* (in short DST-hypersurface) if for all $p \in M$ the middle spheres pass through the origin, that is, they satisfy the following relation

$$2\Psi H_{n-1} + \Lambda H_n = 0. \tag{3.1}$$

The following result provides a characterization of the DST-hypersurfaces.

Theorem 3.1. *Let M be an orientable hypersurface in $\mathbb{R}^{n+1}$ with Gauss-Kronecker curvature $K \neq 0$ as in Theorem 2.6.Then M is a DST-hypersurface if, and only if, there is a differentiable function $h : U \subset \mathbb{R}^n \to \mathbb{R}$ satisfying*

$$h\triangle_L h = \frac{n}{2}|\nabla_L h|^2 \tag{3.2}$$

such that M is locally parametrized by

$$X(u) = \left(\nabla_L h(u) - \frac{2R(u)}{T(u)}u, -\frac{2R(u)}{T(u)}\right), \quad u \in U \tag{3.3}$$

where
$T(u) = 1 + |Y|^2$, $R(u) = \langle \nabla_L h(u), u\rangle - h(u)$, $u \in U$, $L = (L_{ij})$, $L_{ij} = \langle Y_{,i}, Y_{,j}\rangle$.
The Gauss map N of M is given by

$$N(u) = \frac{1}{1 + |Y(u)|^2}\left(2Y(u), 1 - |Y(u)|^2\right), \quad u \in U \subset \mathbb{R}^n. \tag{3.4}$$

Proof: Consider an orientable DST-hypersurface $M \subset \mathbb{R}^{n+1}$, with Gauss-Kronecker curvature $K \neq 0$, and let $Y(u)$, $u \in U$, be a local orthogonal parametrization of hyperplane Π. From Theorem 2.6, it follows that there exist a differentiable function $h : U \to \mathbb{R}$ such that M locally parametrized by (2.1).
From (2.2) and (2.3) we get

$$Q(u) = \sum_{j=1}^{n} \frac{h_{,j}}{L_{jj}} Y_{,j} = \nabla_L h \quad \text{and} \quad R(u) = \langle \nabla_L h, u\rangle - h(u). \tag{3.5}$$

Using (3.5) in (2.1) we obtain (3.3) and from (2.3) we get (3.4).
Also, from (2.4) and (2.5) the Weingarten matrix W of X is given by $W = 2(TV - 2RI)^{-1}$ where $V = (h_{,ij})$.
Since, V is simetric, let $\{a_1, ..., a_n\}$ the eigenvalues of V. It follows that $\lambda_i = \dfrac{2}{Ta_i - 2R}$, $1 \leq i \leq n$, are the eigenvalues of W and consequently, the principal curvatures $k_1, ..., k_n$ of M are given by

$$k_i = -\lambda_i = \frac{2}{2R - Ta_i}, \quad 1 \leq i \leq n. \tag{3.6}$$

The fact that the Gauss-Kronecker curvature K is non-zero at all points of M, guarantees that $k_i \neq 0$ for every $1 \leq i \leq n$.
Thus, from (3.3) and (3.4) we get that the support function Ψ and the quadratic distance function Λ of M are given by

$$\Psi = \frac{2h}{T} \quad \text{and} \quad \Lambda = |\nabla_L h|^2 - \frac{4Rh}{T}. \tag{3.7}$$

We can assume that $h(u) \neq 0$ for every $u \in U$. Indeed, if there is $u_0 \in U$ such that $h(u_0) = 0$ then $\Psi(u_0) = 0$. Consequently, $\Lambda(u_0)K(u_0) = 0$ and therefore $X(u_0) = 0$.

Thus, unless possibly the origin, from definition 2.3, (3.6) and (3.7) we obtain

$$
\begin{aligned}
2\Psi H_{n-1} + \Lambda H_n &= 0\\
\frac{2H_{n-1}}{H_n} &= -\frac{\Lambda}{\Psi}\\
\frac{2}{nk_1 \ldots k_n} \sum_{j_1 < \ldots < j_{n-1}} k_{j_1} \cdot \ldots \cdot k_{j_{n-1}} &= \frac{T}{2h}\left(-|\nabla_L h|^2 + \frac{4Rh}{T}\right)\\
\frac{2}{n}\sum_{i=1}^{n} \frac{1}{k_i} &= 2R - \frac{T|\nabla_L h|^2}{2h}\\
\frac{1}{n}\sum_{i=1}^{n}(2R - Ta_i) &= 2R - \frac{T|\nabla_L h|^2}{2h}\\
\frac{1}{n}\left(2nR - T\sum_{i=1}^{n} a_i\right) &= 2R - \frac{T|\nabla_L h|^2}{2h}\\
2R - \frac{T}{n}tr(V) &= 2R - \frac{T|\nabla_L h|^2}{2h}\\
\frac{1}{n}\triangle_L h &= \frac{|\nabla_L h|^2}{2h}
\end{aligned}
$$

The last equation is equivalent to (3.2). The proof is complete.
The following result provides solutions to equation (3.2) and in this way we obtain families of DST-hypersurfaces.

Proposition 3.1. *Let $\phi : U \subset \mathbb{R}^n \to \mathbb{R}$, $n \geq 3$ be a harmonic function such that $\phi(u) \neq 0 \; \forall \; u \in U$. Then*

$$h(u) = \phi(u)^{\frac{2}{2-n}}, \quad u \in U$$

is a solution of (3.2).
Proof: Considering $Y = (u, 0)$, we obtain that $\nabla_L h = \nabla h$, $\triangle_L h = \triangle h$ and

$$h_{,i} = \frac{2}{2-n}\phi^{\frac{n}{2-n}}\phi_{,i}, \quad 1 \leq i \leq n \tag{3.8}$$

$$h_{,ii} = \frac{2n}{(2-n)^2}\phi^{\frac{2n-2}{2-n}}\phi_{,i}^2 + \frac{2}{2-n}\phi_{,ii}, \quad 1 \leq i \leq n. \tag{3.9}$$

From (3.8) and (3.9) it follows

$$|\nabla h|^2 = \frac{4}{(2-n)^2}\phi^{\frac{2n}{2-n}}|\nabla\phi|^2, \quad \triangle h = \frac{2n}{(2-n)^2}\phi^{\frac{2n-2}{2-n}}|\nabla\phi|^2.$$

Substituting these expressions in (3.2) we obtain the result.

Proposition 3.2. *Let X given as in Theorem 3.2, $Y = (u, 0)$ and $h(u) = \phi_{ab}(u)^{\frac{2}{2-n}}$, $n \geq 3$, where $\phi_{ab} : \mathbb{R}^n \to \mathbb{R}$ is a harmonic function, $a \neq 0$, b are real constants. Then X is a two-parameter DST-hypersurface with planar lines of curvature, foliated by $(n-1)$-dimensional spheres.*
Proof: From Proposition 3.3, $h(u) = \phi_{ab}(u)^{\frac{2}{2-n}}$ satisfies (3.2) and

$$\nabla h(u) = \frac{2}{2-n}\phi_{ab}(u)^{\frac{n}{2-n}}e_1, \; R(u) = \phi_{ab}(u)^{\frac{n}{2-n}}\left(\frac{2au_1}{2-n} - \phi_{ab}(u)\right). \tag{3.10}$$

Using (3.10) in (3.3) we obtain that

$$X(u) = \frac{2\phi_{ab}(u)^{\frac{n}{2-n}}}{T(u)}\left(\frac{aT(u)}{2-n}e_1 - \left(\frac{2au_1}{2-n} - \phi_{ab}(u)\right)u, \phi_{ab}(u) - \frac{2au_1}{2-n}\right), \; u \in U \tag{3.11}$$

is a two-parameter DST-hypersurface, where $T = 1 + |u|^2$.
We observe that the Gauss map is given by

$$N(u) = \frac{1}{1+|u|^2}\left(2u, 1 - |u|^2\right) \tag{3.12}$$

The coefficients of the first I and the second II fundamental forms of X are given by

$$I = \langle X_{,i}, X_{,j}\rangle = \begin{cases} \left[\frac{2a^2n}{(2-n)^2}\phi_{ab}(u)^{\frac{2n-2}{2-n}} - \frac{2R(u)}{T(u)}\right]^2, & i=j=1 \\ \frac{4R(u)^2}{T(u)^2}, & i=j\neq 1 \\ 0, & i\neq j \end{cases}$$

$$II = -\langle X_{,i}, N_{,j}\rangle = \begin{cases} \frac{2}{T}\left[\frac{2R(u)}{T(u)} - \frac{2a^2n}{(2-n)^2}\phi_{ab}(u)^{\frac{2n-2}{2-n}}\right], & i=j=1 \\ \frac{4R(u)}{T(u)^2}, & i=j\neq 1 \\ 0, & i\neq j. \end{cases}$$

Also, since $\widetilde{\Gamma}_{ij}^k = 0$ for $1\leq i,j,k\leq n$, from (2.5) we have that

$$V_{11} = h_{,11}$$
$$V_{ij} = 0, 1\leq i\neq j\leq n.$$

Thus, from Theorem 3.1 in [13] X is parametrized by lines of curvature.
On the other hand, since the Laplace invariants $\widetilde{m}_{ij} = -\widetilde{\Gamma}_{ij,i}^i + \widetilde{\Gamma}_{ij}^i\widetilde{\Gamma}_{ij}^j = 0$, $1\leq i\neq j\leq n$, it follows from Theorem 3.3 in [13], that these lines of curvature are planar.
Also, from (2.4) the principal curvatures are given by

$$k_1 = \frac{2}{2R - TV_{11}},\ k_j = \frac{1}{R}, j = 2,\cdots,n.$$

Therefore, the DST-hypersurface X is foliated by $(n-1)$-dimensional spheres. The proof is complete.
Another important class of solutions to the equation (3.2) are the radial solutions, these solutions provide DST-hypersurfaces.
The following result classifies the DST-hypersurfaces of rotation.

Theorem 3.2. *Let M be a DST-hypersurface in $\mathbb{R}^{n+1}, n\geq 3$ with Gauss-Kronecker curvature $K\neq 0$, $Y = (u,0)$ and Gauss map N given locally by (3.12). M is of rotation if, and only if, the function $h : U\subset\mathbb{R}^n\to\mathbb{R}$ is given by*

$$h(u) = c_0\left(|u|^{2-n} + c_1\right)^{\frac{2}{2-n}},\quad u\in U \tag{3.13}$$

where c_0, $c_1\in\mathbb{R}$, $c_0>0$, and
i) $U = \mathbb{R}^n$, if $c_1\geq 0$;
ii) $U = \{u\in\mathbb{R}^n; u\neq 0$ and $|u|^{n-2}\neq -c_1^{-1}\}$, if $c_1 < 0$.
In this case, up to a dilation ($c_0 = 1$), M is locally parametrized by

$$X(u) = \frac{2\left(|u|^{2-n} + c_1\right)^{\frac{n}{2-n}}}{1+|u|^2}\left(\frac{(1+c_1|u|^n)}{|u|^n}u, c_1 - |u|^{2-n}\right), u\in U. \tag{3.14}$$

Moreover, if $c_1 = 0$ then X parameterizes a sphere.

Proof: Since M is a DST-hypersurface in $\mathbb{R}^{n+1}$ with Gauss-Kronecker curvature $K\neq 0$ and Gauss map N given by (3.4), it follows from Theorem 3.1 that there exist a differentiable function $h : U\subset\mathbb{R}^n\to\mathbb{R}$, satisfying (3.2), such that M is locally parametrized by (3.3).
From Corollary 2.9 in [4], we get that M is of rotation if, and only if, the function h is radial.
Thus, consider $h(u) = J(t)$, $u\in U$, where $t = |u|^2$ and J is a differentiable function.
Hence,

$$h_{,i} = 2u_iJ',\ h_{,ii} = 4u_i^2J'' + 2J',\ 1\leq i\leq n \tag{3.15}$$

where J' denote the derivative of J with respect to t.
Using (3.15) we obtain

$$|\nabla h|^2 = 4(J')^2t,\ \triangle h = 4J''t + 2nJ'.$$

Substituting these expressions in (3.2), we get

$$2JJ''t + nJJ' = n(J')^2t. \tag{3.16}$$

We can assume that $J(t) \neq 0$ and $J'(t) \neq 0$ for every $t > 0$. In fact, in both cases $\Lambda = 0$. So, unless $u = 0$, we can divide the above expression by $JJ't$.
Thus, from (3.16) we obtain

$$\left[\ln\left(\frac{(J')^{\frac{2}{n}}t}{J}\right)\right]' = 0.$$

For $n = 2$, the classification was obtained in [3].
For $n \geq 3$, the solutions of above equation are given by

$$J(t) = c_0\left(t^{\frac{2-n}{2}} + c_1\right)^{\frac{2}{2-n}} \tag{3.17}$$

for real constants $c_0, c_1 \in \mathbb{R}$, $c_0 > 0$. Thus, we obtain (3.13).
Observe that if $c_1 \geq 0$, $U = \mathbb{R}^n$.
On the other hand, if $c_1 < 0$, $U = \{u \in \mathbb{R}^n; |u|^{n-2} \neq -c_1^{-1}\}$.

Up to a dilation, we can assume that $c_0 = 1$. Thus, we obtain that the expressions for ∇h and R are given respectively by

$$\nabla h(u) = \frac{2\left(|u|^{2-n} + c_1\right)^{\frac{n}{2-n}}}{|u|^n}u, \quad R(u) = \left(|u|^{2-n} + c_1\right)^{\frac{n}{2-n}}\left(|u|^{2-n} - c_1\right). \tag{3.18}$$

Using (3.18) in (3.3) we get that M is locally parametrized by (3.14).

In the case that $c_1 = 0$, (3.14) becomes

$$X(u) = \frac{2}{1 + |u|^2}(u, -|u|^2), \quad u \in \mathbb{R}^n. \tag{3.19}$$

Therefore, X parameterizes a sphere with center in $(0, ..., 0, -1) \in \mathbb{R}^{n+1}$ and radius 1.

4. Conclusions. From the results obtained in this work we can make the following conclusions: The hypersurfaces of the spherical type degenerated generalize the EDSGW-surfaces, these hypersurfaces has the geometric property that the middle spheres pass through the origin of the Euclidean space. The DST-hypersurfaces are characterized by a differential equation, that is, the solutions of this equation provide families of DST-hypersurfaces and we give an explicit example of a two-parameter family of DST-hypersurfaces with planar lines of curvature foliated by $(n-1)$-dimensional spheres. Also, a special case of solutions of this differential equation are the radial functions, which provide families of DST-hypersurfaces of rotation. As a first step in this work, we classify the DST-hypersurfaces of rotation. Finally, this work can be used to classify the DST-hypersurfaces with planar lines of curvature that satisfy an additional geometric property, future works in this direction we will be presenting.

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