



Uniquely list colorability of the graph $K_2^n + O_m$

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Abstract

Given a list $L(v)$ for each vertex v , we say that the graph G is L -colorable if there is a proper vertex coloring of G where each vertex v takes its color from $L(v)$. The graph is uniquely k -list colorable if there is a list assignment L such that $|L(v)| = k$ for every vertex v and the graph has exactly one L -coloring with these lists. In this paper, we characterize uniquely list colorability of the graph $G = K_2^n + O_m$. We shall prove that if $n = 2$ then G is uniquely 3-list colorable if and only if $m \geq 9$, if $n = 3$ and $m \geq 1$ then G is uniquely 3-list colorable, if $n \geq 4$ then G is uniquely k -list colorable with $k = \lfloor \frac{m}{2} \rfloor + 1$ and if $m \geq n - 1$ then $G = K_2^n + O_m$ is UnLC.

Keywords . Vertex coloring (coloring), list coloring, uniquely list colorable graph, complete r -partite graph.

1. Introduction. All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If G is a graph, then $V(G)$ and $E(G)$ (or V and E in short) will denote its vertex-set and its edge-set, respectively. The set of all neighbours of a subset $S \subseteq V(G)$ is denoted by $N_G(S)$ (or $N(S)$ in short). Further, for $W \subseteq V(G)$ the set $W \cap N_G(S)$ is denoted by $N_W(S)$. If $S = \{v\}$, then $N(S)$ and $N_W(S)$ are denoted shortly by $N(v)$ and $N_W(v)$, respectively. For a vertex $v \in V(G)$, the degree of v (resp., the degree of v with respect to W), denoted by $\deg(v)$ (resp., $\deg_W(v)$), is $|N_G(v)|$ (resp., $|N_W(v)|$). The subgraph of G induced by $W \subseteq V(G)$ is denoted by $G[W]$. The empty and complete graphs of order n are denoted by O_n and K_n , respectively. Unless otherwise indicated, our graph-theoretic terminology will follow [2].

A graph $G = (V, E)$ is called r -partite graph if V admits a partition into r classes $V = V_1 \cup V_2 \cup \dots \cup V_r$ such that the subgraphs of G induced by V_i , $i = 1, \dots, r$, is empty. An r -partite graph in which every two vertices from different partition classes are adjacent is called complete r -partite graph and is denoted by $K_{|V_1|, |V_2|, \dots, |V_r|}$. The complete r -partite graph $K_{|V_1|, |V_2|, \dots, |V_r|}$ with $|V_1| = |V_2| = \dots = |V_r| = s$ is denoted by K_s^r .

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Their union $G = G_1 \cup G_2$ has, as expected, $V(G) = V_1 \cup V_2$ and $E(G) = E_1 \cup E_2$. Their join defined is denoted $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all edges joining V_1 with V_2 .

Let $G = (V, E)$ be a graph and λ is a positive integer.

A λ -coloring of G is a mapping $f : V(G) \rightarrow \{1, 2, \dots, \lambda\}$ such that $f(u) \neq f(v)$ for any adjacent vertices $u, v \in V(G)$. The smallest positive integer λ such that G has a λ -coloring is called the chromatic number of G and is denoted by $\chi(G)$. We say that a graph G is n -chromatic if $n = \chi(G)$.

Let $(L(v))_{v \in V}$ be a family of sets. We call a coloring f of G with $f(v) \in L(v)$ for all $v \in V$ is a list coloring from the lists $L(v)$. We will refer to such a coloring as an L -coloring. The graph G is called λ -list-colorable, or λ -choosable, if for every family $(L(v))_{v \in V}$ with $|L(v)| = \lambda$ for all v , there is a coloring of G from the lists $L(v)$. The smallest positive integer λ such that G has a λ -choosable is called the list-chromatic number, or choice number of G and is denoted by $ch(G)$. The idea of list colorings of graphs is due independently to V. G. Vizing [14] and to P. Erdős, A. L. Rubin, and H. Taylor [7].

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Let G be a graph with n vertices and suppose that for each vertex v in G , there exists a list of k colors $L(v)$, such that there exists a unique L -coloring for G , then G is called a *uniquely k -list colorable graph* or a $UkLC$ graph for short. The idea of uniquely colorable graph was introduced independently by Diniz and Martin [6] and by Mahmoodian and Mahdian [10].

The list coloring model can be used in the channel assignment. The fixed channel allocation scheme leads to low channel utilization across the whole channel. It requires a more effective channel assignment and management policy, which allows unused parts of channel to become available temporarily for other usages so that the scarcity of the channel can be largely mitigated [15]. It is a discrete optimization problem. A model for channel availability observed by the secondary users is introduced in [15]. The research of list coloring consists of two parts: the choosability and the unique list colorability. In [9], we characterized list-chromatic number of the graph $G = K_2^n + O_m$.

In this paper, we characterize uniquely list colorability of the graph $G = K_2^n + O_m$. We shall prove that if $n = 2$ then G is uniquely 3-list colorable if and only if $m \geq 9$, if $n = 3$ and $m \geq 1$ then G is uniquely 3-list colorable, if $n \geq 4$ and $m \geq 1$ then G is uniquely k -list colorable with $k = \lfloor \frac{n}{2} \rfloor + 1$ and if $m \geq n - 1$ then $G = K_2^n + O_m$ is UnLC.

2. Preliminaries. If a graph G is not uniquely k -list colorable, we also say that G has property $M(k)$. So G has the property $M(k)$ if and only if for any collection of lists assigned to its vertices, each of size k , either there is no list coloring for G or there exist at least two list colorings. The least integer k such that G has the property $M(k)$ is called the m -number of G , denoted by $m(G)$. This conception was originally introduced by Mahmoodian and Mahdian in [10].

Example. Set the graph $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$. It is not difficult to see that G is U2LC, G has the property $M(3)$ and $m(G) = 3$

We need the following Lemmas 1–8 to prove our results.

Lemma 1 ([10]). *A connected graph G has the property $M(2)$ if and only if every block of G is either a cycle, a complete graph, or a complete bipartite graph.*

If $n = 1$ then $G = K_2^n + O_m$ is a complete bipartite graph, by Lemma 1, G has the property $M(2)$.

Lemma 2 ([10]). *For every graph G we have $m(G) \leq |E(\overline{G})| + 2$.*

Lemma 3. *If $G = K_2^n + O_m$ is $UkLC$ then $k \leq 2n + 1$.*

Proof: Suppose that G is $UkLC$. Then there exists a list of k colors $L(v)$, such that there exists a unique L -coloring f for G . Let $V(G) = V_1 \cup V_2 \cup \dots \cup V_{n+1}$ is a partition of $V(G)$ such that $|V_1| = |V_2| = \dots = |V_n| = 2, |V_{n+1}| = m$ and for every $i = 1, 2, \dots, n + 1$ the subgraphs of G induced by V_i , is empty graph. Set $V_{n+1} = \{v_1, v_2, \dots, v_m\}$.

It is clear that $|f(V_1 \cup V_2 \cup \dots \cup V_n)| \leq 2n$ and $|L(v_1) \setminus \{f(v_1)\}| = k - 1$. If $k > 2n + 1$ then $|L(v_1) \setminus \{f(v_1)\}| > 2n$. So there exists $c \in L(v_1) \setminus \{f(v_1)\}$ such that $c \notin f(V_1 \cup V_2 \cup \dots \cup V_n)$. It follows that there exists a unique L -coloring f' for G : $f(v) = f'(v)$ if $v \in V(G) \setminus \{v_1\}$ and $f(v_1) = c$. It is not difficult to see that $f \neq f'$, a contradiction. Thus, $k \leq 2n + 1$. ■

Lemma 4 ([10]). *Every $UkLC$ graph has at least $3k - 2$ vertices.*

Lemma 5. *If $G = K_2^n + O_m$ is a graph with $n = 2$ and $1 \leq m \leq 2$ then $m(G) = 3$.*

Proof: By Lemma 1, G is U2LC. Suppose that G is U3LC. By Lemma 4, $|V(G)| \geq 7$, a contradiction. So $m(G) = 3$. ■

Lemma 6 ([10]). *If $G = K_2^n + O_m$ is a graph with $n = 2$ and $m = 3$ then $m(G) = 3$.*

Lemma 7 ([17]). *If $G = K_2^n + O_m$ is a graph with $n = 2$ and $4 \leq m \leq 8$ then $m(G) = 3$.*

Lemma 8. *If $G = K_2^n + O_m$ is a graph with $n = 2$ and $1 \leq m \leq 8$ then $m(G) = 3$.*

Proof: It follows from Lemma 5, Lemma 6 and Lemma 7. ■

Lemma 9 ([17]). *If $n = 2$ and $m = 9$ then $G = K_2^n + O_m$ is U3LC.*

3. Main Results. Set the graph $G = K_2^n + O_m$. Let $V(G) = V_1 \cup V_2 \cup \dots \cup V_{n+1}$ is a partition of $V(G)$ such that $|V_1| = |V_2| = \dots = |V_n| = 2, |V_{n+1}| = m$ and for every $i = 1, 2, \dots, n + 1$ the subgraphs of G induced by V_i , is empty graph.

Theorem 1. *If $n = 2$ then $G = K_2^n + O_m$ is U3LC if and only if $m \geq 9$.*

Proof: First we prove the necessity. Suppose that $G = K_2^n + O_m$ is U3LC. If $1 \leq m \leq 8$ then by Lemma 8, $m(G) = 3$, a contradiction. Therefore, $m \geq 9$.

Now we prove the sufficiency. We prove $G = K_2^n + O_m$ is U3LC by induction on m . If $m = 9$ then by Lemma 9, G is U3LC. So let $m > 9$ and assume the assertion for smaller values of m .

Set $V_3 = \{v_1, v_2, \dots, v_m\}$ and $G' = G - v_m$. It is clear that $G' = K_2^n + O_{m-1}$. By the induction hypothesis, for each vertex v in G' , there exists a list of 3 colors L'_v , such that there exists a unique coloring f' for G' .

We assign the following lists for the vertices of G : $L(v) = L'(v)$ if $v \in V(G')$, $L(v_m) = L'(v_{m-1})$.

A unique coloring f of G exists from the assigned lists: $f(v) = f'(v)$ if $v \in V(G')$, $f(v_m) = f'(v_{m-1})$. ■

Theorem 2. *If $n = 3$ and $m \geq 1$ then $G = K_2^n + O_m$ is U3LC.*

Proof: Set $V_1 = \{u_1, u_2\}$, $V_2 = \{u_3, u_4\}$, $V_3 = \{u_5, u_6\}$ and $V_4 = \{v_1, v_2, \dots, v_m\}$.

We assign the following lists for the vertices of G : $L(u_1) = \{1, 4, 5\}$, $L(u_2) = \{2, 4, 5\}$, $L(u_3) = \{1, 2, 3\}$, $L(u_4) = \{3, 4, 5\}$, $L(u_5) = \{1, 2, 4\}$, $L(u_6) = \{3, 4, 5\}$, $L(v_1) = L(v_2) = \dots = L(v_m) = \{3, 4, 5\}$.

A unique coloring f of G exists from the assigned lists: $f(u_1) = 1$, $f(u_2) = 2$, $f(u_3) = 3$, $f(u_4) = 3$, $f(u_5) = 4$, $f(u_6) = 4$, $f(v_1) = f(v_2) = \dots = f(v_m) = 5$. ■

Theorem 3. *If $n \geq 4$ and $m \geq 1$ then $G = K_2^n + O_m$ is UkLC with $k = \lfloor \frac{n}{2} \rfloor + 1$.*

Proof: Set $V_i = \{u_{i1}, u_{i2}\}$ for every $i = 1, 2, \dots, n$ and $V_{n+1} = \{v_1, v_2, \dots, v_m\}$. Put $t = \lfloor \frac{n}{2} \rfloor$. We assign the following lists for the vertices of G :

$L(u_{i1}) = \{1, 2, \dots, t+1\}$ for every $i = 1, 2, \dots, t+1$;

$L(u_{i2}) = \{t+2, t+3, \dots, 2t+1, i\}$ for every $i = 1, 2, \dots, t+1$;

$L(u_{(t+i)j}) = \{2, 3, \dots, t+1, t+1+i\}$ for every $i = 1, 2, \dots, n-t-1$, $j = 1, 2$;

$L(v_1) = L(v_2) = \dots = L(v_m) = \{2, 3, \dots, t+1, n+1\}$.

A unique coloring f of G exists from the assigned lists:

$f(u_{i1}) = i$ for every $i = 1, 2, \dots, t+1$;

$f(u_{i2}) = i$ for every $i = 1, 2, \dots, t+1$;

$f(u_{(t+i)j}) = t+1+i$ for every $i = 1, 2, \dots, n-t-1$, $j = 1, 2$;

$f(v_1) = f(v_2) = \dots = f(v_m) = n+1$. ■

Corollary 1. *If $G = K_2^n + O_m$ is a graph with $n = 4$ and $m = 1$ then $m(G) = 4$.*

Proof: By Theorem 3, G is U3LC. If G is U4LC then by Lemma 4, $|V(G)| \geq 10$, a contradiction. So $m(G) = 4$. ■

Now we characterize uniquely list colorability of the graph $G = K_2^n + O_m$ with $m \geq n-1$.

Theorem 4.

If $m \geq n-1$ then $G = K_2^n + O_m$ is UnLC.

Proof:

Set $V_i = \{u_{i1}, u_{i2}\}$ for every $i = 1, 2, \dots, n$ and $V_{n+1} = \{v_1, v_2, \dots, v_m\}$. We assign the following lists for the vertices of G :

$L(u_{i1}) = \{1, 2, \dots, n\}$ for every $i = 1, 2, \dots, n$;

$L(u_{i2}) = \{n+1, n+2, \dots, 2n-1, i\}$ for every $i = 1, 2, \dots, n$;

$L(v_j) = \{1, 2, \dots, n-1, n+j\}$ for every $j = 1, 2, \dots, n-1$;

$L(v_j) = L(v_{n-1})$ for every $j = n, n+1, \dots, m$.

A unique coloring f of G exists from the assigned lists:

$f(u_{i1}) = i$ for every $i = 1, 2, \dots, n$;

$f(u_{i2}) = i$ for every $i = 1, 2, \dots, n$;

$f(v_j) = n+j$ for every $j = 1, 2, \dots, n-1$;

$f(v_j) = f(v_{n-1})$ for every $j = n, n+1, \dots, m$. ■

Finally we determine $m(G)$ number for the graph $G = K_2^n + O_m$ with $n-1 \leq m \leq n$.

Corollary 2. *If $G = K_2^n + O_m$ is a graph with $n-1 \leq m \leq n$ then $m(G) = n+1$.*

Proof: By Theorem 4, G is UnLC. If G is $U(n+1)$ LC then by Lemma 4, $|V(G)| \geq 3n+1$, a contradiction. So $m(G) = n+1$. ■

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