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# List-Chromatic Number and Chromatically Unique of the Graph $K_2^r + O_k$.

# Número de lista cromática y cromáticidad única del grafo $K_2^r + O_k$.

Le Xuan Hung*

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**Abstract**

*In this paper, we determine list-chromatic number and characterize chromatically unique of the graph $G = K_2^r + O_k$. We shall prove that $ch(G) = r + 1$ if $1 \leq k \leq 2$, $G$ is $\chi$-unique if $1 \leq k \leq 3$.*

**Keywords**. Chromatic number, list- chromatic number, chromatic polynomial, chromatically unique graph, complete r-partite graph.

**Resumen**

*En este artículo, determinamos el número de lista cromática y caracterizamos cromáticamente el grafo $G = K_2^r + O_k$. Probaremos que $ch(G) = r + 1$ si $1 \leq k \leq 2$, $G$ es $\chi$-único si $1 \leq k \leq 3$.*

**Palabras clave**. Número Cromático, polinomio cromático, grafo único cromaticmente grafo completo r-partido.

**1. Introduction.** All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If $G$ is a graph, then $V(G)$ and $E(G)$ (or $V$ and $E$ in short) will denote its vertex-set and its edge-set, respectively. The set of all neighbours of a subset $S \subseteq V(G)$ is denoted by $N_G(S)$ (or $N(S)$ in short). Further, for $W \subseteq V(G)$ the set $W \cap N_G(S)$ is denoted by $N_W(S)$. If $S = \{v\}$, then $N(S)$ and $N_W(S)$ are denoted shortly by $N(v)$ and $N_W(v)$, respectively. For a vertex $v \in V(G)$, the degree of $v$ (resp., the degree of $v$ with respect to $W$), denoted by $\deg(v)$ (resp., $\deg_W(v)$), is $|N_G(v)|$ (resp., $|N_W(v)|$). The subgraph of $G$ induced by $W \subseteq V(G)$ is denoted by $G[W]$. The empty and complete graphs of order $n$ are denoted by $O_n$ and $K_n$, respectively. Unless otherwise indicated, our graph-theoretic terminology will follow [2].

A graph $G = (V, E)$ is called *r-partite graph* if $V$ admits a partition into $r$ classes $V = V_1 \cup V_2 \cup \ldots \cup V_r$ such that the subgraphs of $G$ induced by $V_i$, $i = 1, \ldots, r$, is empty. An $r$-partite graph in which every two vertices from different partition classes are adjacent is called complete $r$-partite graph and is denoted by $K_{|V_1|,|V_2|,\ldots,|V_r|}$ . The complete $r$-partite graph $K_{|V_1|,|V_2|,\ldots,|V_r|}$ with $|V_1| = |V_2| = \ldots = |V_r| = s$ is denoted by $K_s^r$

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Their *union* $G = G_1 \cup G_2$ has, as expected, $V(G) = V_1 \cup V_2$ and $E(G) = E_1 \cup E_2$. Their *join* defined is denoted $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all edges joining $V_1$ with $V_2$.

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs. We call $G_1$ and $G_2$ *isomorphic*, and write $G_1 \cong G_2$, if there exists a mapping $f : V_1 \to V_2$ with $uv \in E_1$ if and only if $f(u)f(v) \in E_2$ for all $u, v \in V_1$.

Let $G = (V, E)$ be a graph and $\lambda$ is a positive integer.

A *$\lambda$-coloring* of $G$ is a mapping $f : V(G) \to \{1, 2, \ldots, \lambda\}$ such that $f(u) \neq f(v)$ for any adjacent vertices $u, v \in V(G)$. The smallest positive integer $\lambda$ such that $G$ has a $\lambda$-coloring is called the *chromatic number* of $G$ and is denoted by $\chi(G)$. We say that a graph $G$ is *$n$-chromatic* if $n = \chi(G)$.

Two $\lambda$-colorings $f$ and $g$ are considered different if and only if there exists $u \in V(G)$ such that $f(u) \neq g(u)$. Let $P(G, \lambda)$ (or simply $P(G)$ if there is no danger of confusion) denote the number of distinct $\lambda$-colorings of $G$. It is well-known that for any graph $G$, $P(G, \lambda)$ is a polynomial in $\lambda$, called the *chromatic polynomial* of $G$. The notion of chromatic polynomials was first introduced by Birkhoff [4] in 1912 as a quantitative approach to tackle

*HaNoi University for Natural Resources and Environment 41 A, Phu Dien Road, Phu Dien precinct, North Tu Liem district, Hanoi, Vietnam(lxhung@hunre.edu.vn).

the four-color problem. Two graphs $G$ and $H$ are called *chromatically equivalent* or in short *$\chi$-equivalent*, and we write in notation $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. A graph $G$ is called *chromatically unique* or in short *$\chi$-unique* if $G' \cong G$ (i.e., $G'$ is isomorphic to $G$) for any graph $G'$ such that $G' \sim G$. For examples, all cycles are $\chi$-unique [10]. The notion of $\chi$-unique graphs was first introduced and studied by Chao and Whitehead [7] in 1978. The readers can see the surveys [10], [11] and [13] for more informations about $\chi$-unique graphs.

Let $(S_v)_{v\in V}$ be a family of sets. We call a coloring $f$ of $G$ with $f(v) \in S_v$ for all $v \in V$ is a *list coloring from the lists $S_v$*. The graph $G$ is called *$\lambda$-list-colorable*, or *$\lambda$-choosable*, if for every family $(S_v)_{v\in V}$ with $|S_v| = \lambda$ for all $v$, there is a coloring of $G$ from the lists $S_v$. The smallest positive integer $\lambda$ such that $G$ has a $\lambda$-choosable is called the *list-chromatic number*, or *choice number* of $G$ and is denoted by $ch(G)$.

In this paper, we shall determine list-chromatic number and characterize chromatically unique for the graph $G = K_2^r + O_k$. Namely, we shall prove that $ch(G) = r + 1$ if $1 \le k \le 2$ (Section 2), $G$ is $\chi$-unique if $1 \le k \le 3$ (Section 3).

**2. List colorings.** We need the following lemmas 1–4 to prove our results.

**Lemma 1 ([3]).** *If $K_n$ is a complete graph on $n$ vertices then $\chi(K_n) = n$.*

**Lemma 2.** *If $G = K_{n_1,n_2,\ldots,n_r}$ is a complete $r$-partite graph then $\chi(G) = r$.*

*Proof.* It is clear that the complete graph $K_r$ is a subgraph of $G = K_{n_1,n_2,\ldots,n_r}$. So $\chi(G) \ge r$. Let $V(G) = V_1 \cup V_2 \cup \ldots \cup V_r$ is a partition of $V(G)$ such that for every $i = 1,\ldots,r$, $|V_i| = n_i$ and the subgraphs of $G$ induced by $V_i$, is empty graph. Set mapping

$$f : V(G) \to \{1, 2, \ldots, r\}$$

such that $f(v) = i$ if $v \in V_i$ for every $i = 1, 2, \ldots, r$. Then $f$ is a $r$-coloring of $G$, ie., $\chi(G) \le r$. Thus, $\chi(G) = r$. □

**Lemma 3 ([9]).** *If $G$ is a graph then $ch(G) \ge \chi(G)$.*

**Lemma 4 ([9]).** *If $G_1$ is a subgraph of $G_2$ then $ch(G_1) \le ch(G_2)$.*

We determine list-chromatic number for complete graphs.

**Lemma 5.** *If $K_n$ is a complete graph on $n$ vertices then $ch(K_n) = n$.*

*Proof.* By Lemma 1 and Lemma 3, $ch(K_n) \ge n$. Set $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $S_{v_i}$ is a list of colors of $V_i$ such that $|S_{v_i}| = n$ for every $i = 1, 2, \ldots, n$. Let $f$ be a coloring of $K_n$ such that

$$f(v_1) \in S_{v_1}, f(v_2) \in S_{v_2} \setminus \{f(v_1)\}, \ldots, f(v_n) \in S_{v_n} \setminus \{f(v_1), f(v_2), \ldots, f(v_{n-1})\}.$$

Then $f$ is a $n$-choosable for $K_n$, ie., $ch(K_n) \le n$. Thus, $ch(K_n) = n$. □

Now we determine list-chromatic number for the graph $G = K_2^r$.

**Theorem 6.**

*List-chromatic number of $G = K_2^r$ is*

$$ch(G) = r.$$

*Proof.* By Lemma 2 and Lemma 3, we have $ch(G) \ge r$. Now we prove $ch(G) \le r$ by induction on $r$. For $r = 1$ the assertion holds, so let $r > 1$ and assume the assertion for smaller values of $r$.

Let $V(G) = V_1 \cup V_2 \cup \ldots \cup V_r$ is a partition of $V(G)$ such that for every $i = 1,\ldots,r$, $|V_i| = 2$ and the subgraphs of $G$ induced by $V_i$, is empty graph. Set

$$V_i = \{v_{i1}, v_{i2}\}$$

for every $i = 1,\ldots,r$. Let $S_{v_{ij}}$ be the lists of colors of $v_{ij}$ such that $|S_{v_{ij}}| = r$ for every $i = 1, 2, \ldots, r; j = 1, 2$. Now we consider separately two cases.

*Case 1: There exists $i \in \{1, 2, \ldots, r\}$ such that $S_{v_{i1}} \cap S_{v_{i2}} \ne \emptyset$.*

Without loss of generality we may assume that $S_{v_{11}} \cap S_{v_{12}} \ne \emptyset$ and $a \in S_{v_{11}} \cap S_{v_{12}}$. set $G' = G - V_1$. It is clear that $G'$ is a graph $K_2^{r-1}$. Again set

$$S'_{v_{ij}} \subseteq S_{v_{ij}} \setminus \{a\}$$

such that $|S'_{v_{ij}}| = r - 1$ for every $i = 2, 3, \ldots, r; j = 1, 2$.

By the induction hypothesis, there exists $(r-1)$-choosable $g$ of $G'$ with the lists of colors $S'_{v_{ij}}$ for every $i = 2, 3, \ldots, r; j = 1, 2$.
Let $f$ be the coloring of $G$ such that
$f(v_{ij}) = g(v_{ij})$ for every $i = 2, 3, \ldots, r; j = 1, 2$,
$f(v_{1j}) = a$ for every $j = 1, 2$.
Then $f$ is a $r$-choosable for $G$, ie., $ch(G) \leq r$.
*Case 2:* $S_{v_{i1}} \cap S_{v_{i2}} = \emptyset$ *for every* $i = 1, 2, \ldots, r$.
Let $b \in S_{v_{11}}$. Set $G' = G - V_1 = K_2^{r-1}$ and

$$S'_{v_{ij}} \subseteq S_{v_{ij}} \setminus \{b\}$$

such that $|S'_{v_{ij}}| = r-1$ for every $i = 2, 3, \ldots, r; j = 1, 2$.
By the induction hypothesis, there exists $(r-1)$-choosable $g$ of $G'$ with the lists of colors $S'_{v_{ij}}$ for every $i = 2, 3, \ldots, r; j = 1, 2$. Since $|S_{v_{11}} \cup S_{v_{12}}| = 2r$ and $|V(G'| = 2(r-1)$, it follows that

$$|(S_{v_{11}} \cup S_{v_{12}}) \setminus g(V(G'))| \geq 2.$$

We again divide this case into two subcases.
*Subcase 2.1:* $((S_{v_{11}} \cup S_{v_{12}}) \setminus g(V(G'))) \cap S_{v_{12}} \neq \emptyset$.
Let $c \in ((S_{v_{11}} \cup S_{v_{12}}) \setminus g(V(G'))) \cap S_{v_{12}}$. Let $f$ be the coloring of $G$ such that
$f(v_{ij}) = g(v_{ij})$ for every $i = 2, 3, \ldots, r; j = 1, 2$,
$f(v_{11}) = b$, $f(v_{12}) = c$.
Then $f$ is a $r$-choosable for $G$, ie., $ch(G) \leq r$.
*Subcase 2.2:* $((S_{v_{11}} \cup S_{v_{12}}) \setminus g(V(G'))) \cap S_{v_{12}} = \emptyset$.
By $|(S_{v_{11}} \cup S_{v_{12}}) \setminus g(V(G'))| \geq 2$, there exists $d \in (S_{v_{11}} \cup S_{v_{12}}) \setminus g(V(G'))$, $d \neq b$. It is clear that $b, d \in S_{v_{11}}$. Since $|S_{v_{12}}| = r$ and $|g(V(G'))| \leq 2(r-1)$, there exists $i \in \{2, 3, \ldots, r\}$ such that $g(v_{i1}), g(v_{i2}) \in S_{v_{12}}$. Without loss of generality we may assume that $g(v_{21}), g(v_{22}) \in S_{v_{12}}$. Let $e \in (S_{v_{21}} \cup S_{v_{22}}) \setminus g(V(G'))$. First assume that $e \in S_{v_{21}}$. If $e \neq b$ then coloring $f$ of $G$ such that
$f(v_{ij}) = g(v_{ij})$ for every $i = 3, 4, \ldots, r; j = 1, 2$,
$f(v_{22}) = g(v_{22})$, $f(v_{21}) = e$,
$f(v_{11}) = b$, $f(v_{12}) = g(v_{21})$.
is a $r$-choosable for $G$. If $e = b$ then coloring $f$ of $G$ such that
$f(v_{ij}) = g(v_{ij})$ for every $i = 3, 4, \ldots, r; j = 1, 2$,
$f(v_{22}) = g(v_{22})$, $f(v_{21}) = e$,
$f(v_{11}) = d$, $f(v_{12}) = g(v_{21})$.
is a $r$-choosable for $G$. By symmetry, we can show that $ch(G) \leq r$ if $e \in S_{v_{22}}$. □

**Theorem 7.**
*If* $1 \leq k \leq 2$ *then list-chromatic number of* $G = K_2^r + O_k$ *is*

$$ch(G) = r + 1.$$

*Proof.* It is not difficult to see that $G = K_2^r + O_k$ is a complete $(r+1)$-partite graph. By Lemma 2 and Lemma 3, we have $ch(G) \geq r+1$. Now we prove $ch(G) \leq r+1$. By $1 \leq k \leq 2$, it follows that $G = K_2^r + O_k$ is a subgraph of $K_2^{r+1}$. By Lemma 4 and Theorem 6, $ch(G) \leq r+1$. Thus, $ch(G) = r+1$. □

**3. Chromatic uniqueness.** The results of the following lemmas were proved in [12]. So we omit their proofs here.

**Lemma 8 ([12]).** *Let* $G$ *and* $H$ *be two* $\chi$*-equivalent graphs. Then*
*(i)* $|V(G)| = |V(H)|$;
*(ii)* $|E(G)| = |E(H)|$;
*(iii)* $\chi(G) = \chi(H)$;
*(iv)* $G$ *is connected if and only if* $H$ *is connected;*
*(v)* $G$ *is* 2*-connected if and only if* $H$ *is* 2*-connected.*

We need the following lemmas 9–11 to prove our results.

**Lemma 9.** *Let* $G = (V_1 \cup V_2 \cup \ldots \cup V_{r+1}, E)$ *be a* $(r+1)$*-partite graph with* $|V_1| \geq |V_2| \geq \ldots \geq |V_{r+1}|$ *and* $|V_1| + |V_2| + \ldots + |V_{r+1}| = 2r+1$. *Then*

$$|E| \leq 2r^2.$$

$|E| = 2r^2$ *if and only if* $G$ *is a complete* $(r+1)$*-partite graph* $K_{|V_1|,|V_2|,\ldots,|V_{r+1}|}$ *with*

$$|V_1| = |V_2| = \ldots = |V_r| = 2, |V_{r+1}| = 1.$$

*Proof.* We prove lemma by induction on $r$. For $r = 1$ the assertion holds, so let $r > 1$ and assume the assertion for smaller values of $r$. If $|V_{r+1}| \geq 2$ then $|V_1| + |V_2| + \ldots + |V_{r+1}| \geq 2r + 2$, a contradiction. So, $|V_{r+1}| = 1$. If $|V_r| \geq 3$ then $|V_1| + |V_2| + \ldots + |V_{r+1}| \geq 3r + 1$, a contradiction. Therefore, $|V_r| \leq 2$. Now we consider separately two cases.

*Case 1: There exists* $i \in \{1, 2, \ldots, r\}$ *such that* $|V_i| = 2$.

Set $G' = G - V_i$. It is clear that $G'$ is a $r$-partite graph

$$(V_1 \cup V_2 \cup \ldots \cup V_{i-1} \cup V_{i+1} \cup \ldots \cup V_{r+1}, E')$$

.

By the induction hypothesis,

$$|E'| \leq 2(r-1)^2.$$

We have

$$\begin{aligned} |E| &\leq |E'| + |V_i|(|V_1| + \ldots + |V_{i-1}| + |V_{i+1}| + \ldots + |V_{r+1}|) \\ &\leq 2(r-1)^2 + 2(2r-1) \\ &= 2r^2. \end{aligned}$$

It is not difficult to see that $|E| = 2r^2$ if and only if $G$ is a complete $(r+1)$-partite graph $K_{|V_1|,|V_2|,\ldots,|V_{r+1}|}$ with

$$|V_1| = |V_2| = \ldots = |V_r| = 2, |V_{r+1}| = 1.$$

*Case 2:* $|V_i| \neq 2$ *for every* $i = 1, 2, \ldots, r$.

In this case, $|V_1| \geq 3$. Let $h \in \{1, 2, \ldots, r\}$ such that $|V_h| = 1$ and $|V_{h-1}| \geq 3$. Let $G_1 = K_{p_1,p_2,\ldots,p_{r+1}}$ be a complete $(r+1)$-partite graph such that $p_h = |V_h| + 1$, $p_{h-1} = |V_{h-1}| - 1$ and $p_i = |V_i|$ for every $i \in \{1, 2, \ldots, r\} \setminus \{h-1, h\}$. By Case 1,

$$|E(G_1)| \leq 2r^2.$$

We have

$$\begin{aligned} |E(G_1)| &= \sum_{1 \leq i < j \leq r+1} p_i p_j \\ &= \sum_{i,j \in \{1,\ldots,r+1\} \setminus \{h-1,h\}} p_i p_j + \sum_{i \in \{1,\ldots,r+1\} \setminus \{h-1,h\}} p_i p_{h-1} + \\ &+ \sum_{i \in \{1,\ldots,r+1\} \setminus \{h-1,h\}} p_i p_h + p_{h-1} p_h \\ &= \sum_{i,j \in \{1,\ldots,r+1\} \setminus \{h-1,h\}} |V_i||V_j| + \sum_{i \in \{1,\ldots,r+1\} \setminus \{h-1,h\}} |V_i|(|V_{h-1}| - 1) + \\ &+ \sum_{i \in \{1,\ldots,r+1\} \setminus \{h-1,h\}} |V_i|(|V_h| + 1) + (|V_{h-1}| - 1)(|V_h| + 1) \\ &= \sum_{1 \leq i < j \leq r+1} |V_i||V_j| + |V_{h-1}| - |V_h| - 1 \\ &\geq |E| + 1. \end{aligned}$$

It follows that $|E| < 2r^2$. □

By argument similar to Lemma 9, we can prove the lemmas below also are true. We omit their proof here.

**Lemma 10.** *Let* $G = (V_1 \cup V_2 \cup \ldots \cup V_{r+1}, E)$ *be a* $(r+1)$*-partite graph with* $|V_1| + |V_2| + \ldots + |V_{r+1}| = 2r + 2$. *Then*

$$|E| \leq 2r(r+1).$$

$|E| = 2r(r+1)$ *if and only if* $G$ *is a complete* $(r+1)$*-partite graph* $K_{|V_1|,|V_2|,\ldots,|V_{r+1}|}$ *with*

$$|V_1| = |V_2| = \ldots = |V_r| = |V_{r+1}| = 2.$$

**Lemma 11.** *Let* $G = (V_1 \cup V_2 \cup \ldots \cup V_{r+1}, E)$ *be a* $(r+1)$*-partite graph with* $|V_1| \le |V_2| \le \ldots \le |V_{r+1}|$ *and* $|V_1| + |V_2| + \ldots + |V_{r+1}| = 2r+3$*. Then*

$$|E| \le 2r(r+2).$$

$|E| = 2r(r+2)$ *if and only if* $G$ *is a complete* $(r+1)$*-partite graph* $K_{|V_1|,|V_2|,\ldots,|V_{r+1}|}$ *with*

$$|V_1| = |V_2| = \ldots = |V_r| = 2, |V_{r+1}| = 3.$$

Now we characterize chromatically unique for the graph $G = K_2^r + O_k$.

**Theorem 12.** *The graph* $G = K_2^r + O_k$ *is* $\chi$*-unique if* $1 \le k \le 3$.

*Proof.* Suppose that $1 \le k \le 3$. Let $G' = (V', E')$ is a graph such that $G' \sim G$. Since Lemma 2 and (iii) of Lemma 8 we have

$$\chi(G') = \chi(G) = r+1.$$

Let $G'$ has a coloring $f$ using $r+1$ colors $1, 2, \ldots, r+1$. Set

$$V'_i = \{u \in V' \mid f(u) = i\}.$$

for every $i = 1, 2, \ldots, r+1$. It follows that $G'$ is a $(r+1)$-partite graph $(V'_1 \cup V'_2 \cup \ldots \cup V'_{r+1}, E')$. By (i) and (ii) of Lemma 8 we have

$$|V(G')| = |V(G)|, |E(G')| = |E(G)|.$$

By Lemma 9, Lemma 10 and Lemma 11, it is not difficult to see that $G' \cong G$. Thus $G$ is $\chi$-unique.
□

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