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Generalized Helmholtz equation.

Equación de Helmholtz generalizada.

Carlos M. C. Riveros * and Armando M. V. Corro †

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Abstract

In this paper we introduce the generalized Helmholtz equation and present explicit solutions to this generalized Helmholtz equation, these solutions depend on three holomorphic functions. As an application we present explicit solutions to the Helmholtz equation. We note that these solutions are not necessarily limited to certain domains of the complex plane $\mathbb{C}$.

Keywords Helmholtz equation, holomorphic functions

Resumen

En este artículo introducimos la ecuación de Helmholtz generalizada y presentamos soluciones explícitas para esta ecuación de Helmholtz generalizada, estas soluciones dependen de tres funciones holomorfas. Como aplicación presentamos soluciones explícitas para la ecuación de Helmholtz. Observamos que estas soluciones no necesariamente estan limitadas a ciertos dominios del plano complejo $\mathbb{C}$.

Palabras clave. Ecuación de Helmholtz, funciones holomorfas

1. Introduction. The reduced Helmholtz equation is an elliptic differential equation allowing to describe physical phenomena related to oscillatory problems. As such, it is a classical problem for diverse fields of physics and engineering such as vibration mechanics, electro-magnetics, acoustics or quantum mechanics [2], [5], [8]. Solution of the two-dimensional (2D) Helmholtz equation allows to identify vibration modes for a two-dimensional domain. Analytical solutions are limited to domains with a particular shape such as a rectangle or circle [5], [7]. In general, solving this differential equation relies on numerical methods, see e.g. [12]-[14].

In [4], the Helmholtz equation is transformed to account for a conformal map between the shape of the physical domain and the unit disk as canonical domain. This way, the transformed Helmholtz equation is solved exploiting well known analytical solutions for a circular domain and the solution in the physical domain is obtained by applying the conformal map. In [5] the authors study the Helmholtz equation by the method of fundamental solutions (MFS) using Bessel and Neumann functions. The bounds of errors are derived for bounded simply-connected domains, while the bounds of condition number are derived only for disk domains.

In [11], by using conformal mappings of a plane with elliptic hole and a plane with cross-shaped hole into the exterior of the unit disk, we construct functions playing the role of bases in the spaces of functions analytic in these domains. In addition, on the basis of expansions of analytic functions in series in these bases, we construct solutions of the Helmholtz equation in a plane with holes whose boundary values coincide with the boundary values of these functions.

*ID ORCID: https://orcid.org/0000-0002-1206-7072, Departamento de Matemática, Universidade de Brasília, 70910-900, Brasília-DF, Brazil (carlos@mat.unb.br).

†ID ORCID: https://orcid.org/0000-0002-6864-3876, Instituto de Matemática e Estatística, Universidade Federal de Goiâs, 74001-970, Goiânia-GO, Brazil, (avcorro@gmail.com).

In [7], is proposed the Galerkin boundary element method for exterior problems of 2D Helmholtz equation with arbitrary wavenumber. In [10], the authors present a new boundary integral method for solving the general Helmholtz equation is developed. This new formulation is developed for the two dimensional Helmholtz equation with the method of moments Laplacian solution. The main feature of this new formulation is that the boundary conditions are satisfied independent of the region node discretizations.

In the papers [4], [6] and [7] is considered the Helmholtz equation given by

$$\Delta u(x,y) + (K(x,y))^2 u(x,y) = 0,$$

where K is a wavenumber.

In this paper we introduce the generalized Helmholtz equation and present explicit solutions to this generalized Helmholtz equation. As an application we present explicit solutions to the Helmholtz equation, these solutions depend on three holomorphic functions. We note that these solutions are not necessarily limited to certain domains of the complex plane $\mathbb{C}$, as in [4], [5], [7] and [11].

2. Preliminaries. In this section we present the definitions and results that will be used in the work. In this paper the inner produt $\langle , \rangle : \mathbb{C}\times\mathbb{C} \to \mathbb{R}$ is defininded by

$$\langle f, g\rangle = f_1 g_1 + f_2 g_2, \ \text{where } f = f_1 + i f_2, \ g = g_1 + i g_2,$$

are holomorphic functions.
In the computation we use the following properties: if $f, g : \mathbb{C} \to \mathbb{C}$ are holomorphic functions of $z = u_1 + i u_2$, then

$$\langle f, g\rangle_{,1} = \langle f', g\rangle + \langle f, g'\rangle, \ \ \langle f, g\rangle_{,2} = \langle i f', g\rangle + \langle f, i g'\rangle,$$

$$\langle f, g\rangle = \langle 1, \bar{f} g\rangle, \ \ \Delta\langle f, g\rangle = 4\langle f', g'\rangle. \tag{2.1}$$

Here $\langle f, g\rangle_{,i}$ denotes the derivative of $\langle f, g\rangle$ with respect to $u_i, \ i = 1, 2$.

In [10], the authors consider the following two-dimensional elliptic equation for a smooth function $\Psi(u_1, u_2)$ defined in a 2D region defined by $\mathfrak{R}$ which is bounded by a contour C so that

$$\Delta\Psi(u_1, u_2) + \lambda(u_1, u_2)\Psi(u_1, u_2) = F(u_1, u_2), \tag{2.2}$$

where $\lambda(u_1, u_2)$ and $F(u_1, u_2)$ are known functions in the domain $\mathfrak{R}$. The general form of (2.2) includes, as specializations, the following cases:

1. Laplace's equation, with $\lambda = 0$ and $F = 0$,
2. Poisson's equation, with $\lambda = 0$ and $F \neq 0$,
3. Helmholtz's equation, with $\lambda \neq 0$ and $F \neq 0$ $(F = 0)$.

In this paper we consider the two-dimensional *Helmholtz equation* for $h(z)$ defined by

$$\Delta h(z) + c(K(z))^2 h(z) = 0, \tag{2.3}$$

where $K(z)$ indicates the wavenumber and c is a real non-zero constant.

Definition 1. The two-dimensional *generalized Helmholtz equation* for $h(z)$ is defined as

$$\Delta\left\{\frac{1}{(K(z))^2}\left(\Delta h(z) + c(K(z))^2 h(z)\right)\right\} = 0 \tag{2.4}$$

where $K(z)$ is a function and c is a real non-zero constant.

3. Main results. In this section, we present our main results that provide explicit solutions for the generalized Helmholtz equation and explicit solutions are also obtained for the Helmholtz equation, these solutions depend on three holomorphic functions.

Theorem 1. *Let g be a holomorphic function, c a real non-zero constant and $K = \dfrac{2\sqrt{2}|g'|}{c + |g|^2}$. Then the functions $h = \dfrac{\langle 1, A\rangle + \langle g, B\rangle}{c + |g|^2}$ are solutions of the generalized Helmholtz equation, where A, B are holomorphic*

functions.
Proof: Consider

$$h = \frac{f}{T}, \quad \text{where } \; T = c + |g|^2. \tag{3.1}$$

Calculating the Laplacian of h we have

$$\Delta h = \frac{\Delta f}{T} + 2\left\langle \nabla f, \nabla\left(\frac{1}{T}\right)\right\rangle + f\Delta\left(\frac{1}{T}\right).$$

Using the expression of T given in (3.1), we get

$$\begin{aligned}\Delta h &= \frac{\Delta f}{T} - 4\left\langle \nabla f, \frac{g\overline{g'}}{T^2}\right\rangle + f\left(-\frac{4|g'|^2}{T^2} + \frac{8|gg'|^2}{T^3}\right)\\ &= \frac{\Delta f}{T} - 4\left\langle \nabla f, \frac{g\overline{g'}}{T^2}\right\rangle + 4f|g'|^2\left(\frac{1}{T^2} - \frac{2c}{T^3}\right).\end{aligned}$$

This equation can be written as

$$\Delta h + \frac{8c|g'|^2}{T^2}h = \frac{|g'|^2}{T^2}\left(T\frac{\Delta f}{|g'|^2} - 4\left\langle \nabla f, \frac{g}{g'}\right\rangle + 4f\right). \tag{3.2}$$

Therefore, $h = \dfrac{f}{T}$ is a solution of the generalized Helmholtz equation if and only if

$$\Delta\left\{T\frac{\Delta f}{|g'|^2} - 4\left\langle \nabla f, \frac{g}{g'}\right\rangle + 4f\right\} = T\Delta\left(\frac{\Delta f}{|g'|^2}\right) = 0.$$

On the other hand, the solutions of the equation $\Delta\left(\dfrac{\Delta f}{|g'|^2}\right) = 0$ are given by $f = \langle 1, A\rangle + \langle g, B\rangle$, where A, B are holomorphic functions. Thus, the proof is complete. □

Remark 1. We observe that every solution of the Helmholtz equation is the solution of the generalized Helmholtz equation. The following result provides conditions for a solution of the generalized Helmholtz equation to be solved from the Helmholtz equation.

Corollary 1. *Let g be a holomorphic function, c a real non-zero constant and $K = \dfrac{2\sqrt{2}|g'|}{c+|g|^2}$. Then the functions $h = \dfrac{\langle 1, A\rangle + \langle g, B\rangle}{c+|g|^2}$ are solutions of the Helmholtz equation, where A is a holomorphic function and B is a holomorphic function such that $B = \dfrac{1}{c}\displaystyle\int (A'g - Ag' + ic_1 g')dz$, c_1 is a real constant.*

Proof: From Theorem 1, the solutions of the generalized Helmholtz equation are given by $h = \dfrac{f}{c+|g|^2}$, where

$$f(z) = \langle 1, A\rangle + \langle g, B\rangle. \tag{3.3}$$

From (3.2), h is a solution of the Helmholtz equation if and only if

$$T\frac{\Delta f}{|g'|^2} - 4\left\langle \nabla f, \frac{g}{g'}\right\rangle + 4f = 0. \tag{3.4}$$

Differentiating (3.4) using (2.1), we obtain that the equation (3.4) is equivalent to

$$4\left\langle 1, c\frac{B'}{g'} - \frac{A'g}{g'} + A\right\rangle = 0.$$

Therefore there is a real constant c_1 such that

$$c\frac{B'}{g'} - \frac{A'g}{g'} + A = ic_1,$$

thus, the result follows from this expression. □

Corollary 2. *Let g be a holomorphic function, c a real non-zero constant. Then the equation*

$$(c+|g|^2)\frac{\Delta f}{|g'|^2} - 4\left\langle \nabla f, \frac{g}{g'}\right\rangle + 4f = 0,$$

admits solutions given by

$$f = \langle 1, A\rangle + \langle g, B\rangle, \quad \textit{where} \quad B = \frac{1}{c}\int (A'g - Ag' + ic_1 g')dz$$

onde c_1 *é uma constante real.*
Proof: The proof follows from Corollary 1. □

4. Examples of solutions of the generalized Helmholtz equation. In this section, we present graphics of some solutions of the generalized Helmholtz equation.

Example 1 Considering $g(z) = \dfrac{\cos z}{z}$, $A(z) = z^2$, $B(z) = \log z$ in Theorem 1, we obtain

$$h(z) = \frac{2(u_1^4 - u_2^4) + \log\left(u_1^2 + u_2^2\right) A_1(u_1, u_2) - 2\arctan\left(\frac{u_2}{u_1}\right) A_2(u_1, u_2)}{2c\left(u_1^2 + u_2^2\right) + \cos(2u_1) + \cosh(2u_2)},$$

$$K(z) = \frac{4\sqrt{(1 + u_1^2 + u_2^2)\cosh(2u_2) - (-1 + u_1^2 + u_2^2)\cos(2u_1) + 2u_1\sin(2u_1) - 2u_2\sinh(2u_2)}}{2c\left(u_1^2 + u_2^2\right) + \cos(2u_1) + \cosh(2u_2)}.$$

where

$$A_1(u_1, u_2) = u_1\cos u_1\cosh u_2 - u_2\sin u_1\sinh u_2, \ A_2(u_1, u_2) = u_2\cos u_1\cosh u_2 + u_1\sin u_1\sinh u_2.$$

The graphics of the functions $h(z)$ and $K(z)$ for $c = \frac{1}{2}$ are given in the figures 4.1 and 4.2, respectively.

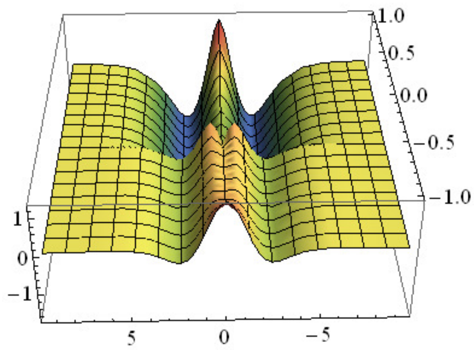

FIGURE 4.1. $c = \frac{1}{2}$

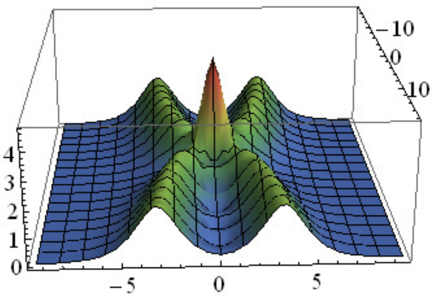

FIGURE 4.2. $c = \frac{1}{2}$

Example 2 Considering $g(z) = z^3$, $A(z) = \sin z$, $B(z) = \cos z$ in Theorem 1, we obtain

$$h(z) = \frac{\cosh u_2\left(u_1\left(u_1^2 - 3u_2^2\right)\cos u_1 + \sin u_1\right) + u_2\left(u_2^2 - 3u_1^2\right)\sin u_1\sinh u_2}{c + \left(u_1^2 + u_2^2\right)^3},$$

$$K(z) = \frac{6\sqrt{2}\left(u_1^2 + u_2^2\right)}{c + \left(u_1^2 + u_2^2\right)^3}.$$

i) The graphics of the functions $h(z)$ and $K(z)$ for $c = 6$ are given in the figures 4.3 and 4.4, respectively.

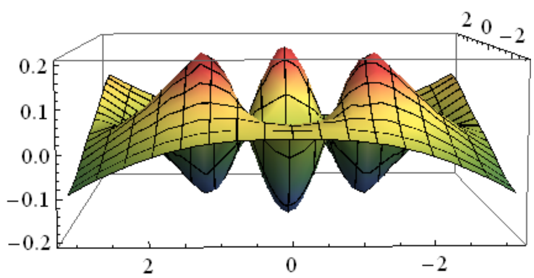

FIGURE 4.3. $c = 6$

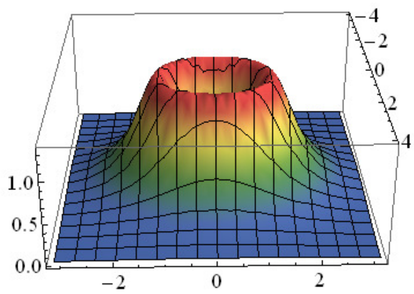

FIGURE 4.4. $c = 6$

ii) The graphics of the functions $h(z)$ and $K(z)$ for $c = -27$ are given in the figures 4.5 and 4.6, respectively.

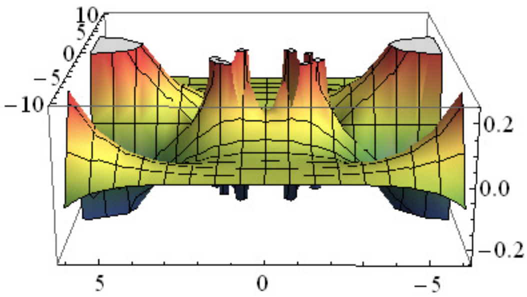

FIGURE 4.5. $c = -27$

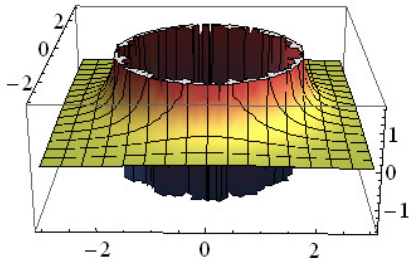

FIGURE 4.6. $c = -27$

Example 3 Considering $g(z) = \cosh^2(2z),\ A(z) = e^z + z,\ B(z) = \cosh z$ in Theorem 1, we obtain

$$h(z) = \frac{4u_1 + 6\cos u_2 \cosh u_1 + \cos(5u_2)\cosh(3u_1) + \cos(3u_2)\cosh(5u_1) + 4\cos u_2 \sinh u_1}{4c + \cos^2(4u_2) + 2\cos(4u_2)\cosh(4u_1) + \cosh^2(4u_1)},$$

$$K(z) = \frac{4\sqrt{\cosh(8u_1) - \cos(8u_2)}}{c + \frac{1}{4}(\cos(4u_2) + \cosh(4u_1))^2}.$$

The graphics of the functions $h(z)$ and $K(z)$ for $c = 2$ are given in the figures 4.7 and 4.8, respectively.

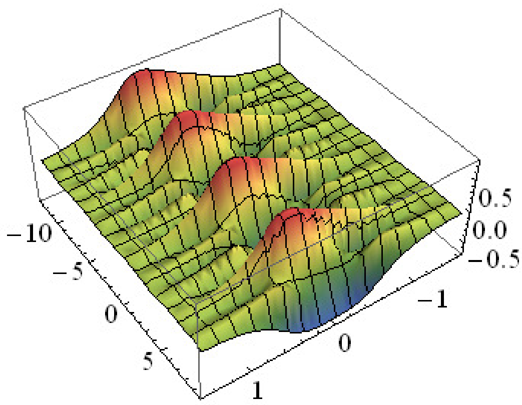

FIGURE 4.7. $c = 2$

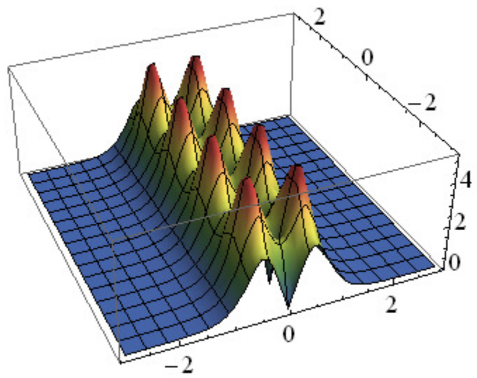

FIGURE 4.8. $c = 2$

5. Examples of solutions of the Helmholtz equation. In this section, we present graphics of some solutions of the Helmholtz equation.

Example 1 Considering $g(z) = z,\ A(z) = e^z,\ B(z) = \frac{1}{c}(e^z(z-2) + ic_1 z)$ in Corollary 2, we obtain

$$h(z) = \frac{e^{u_1}\left(\left(c - 2u_1 + u_1^2 + u_2^2\right)\cos u_2 - 2u_2 \sin u_2\right)}{c\,(c + u_1^2 + u_2^2)},$$

$$K(z) = \frac{2\sqrt{2}}{c + u_1^2 + u_2^2}.$$

The graphics of the functions $h(z)$ and $K(z)$ for $c = \frac{1}{18}$ are given in the figures 5.1 and 5.2, respectively.

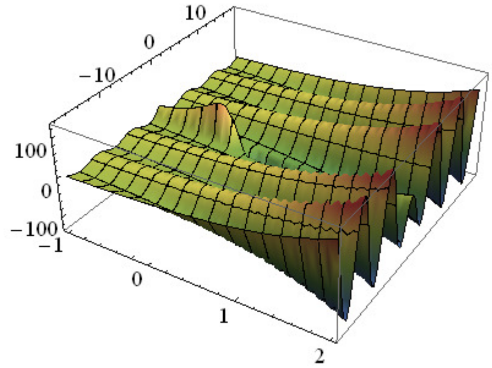

FIGURE 5.1. $c = \frac{1}{18}$

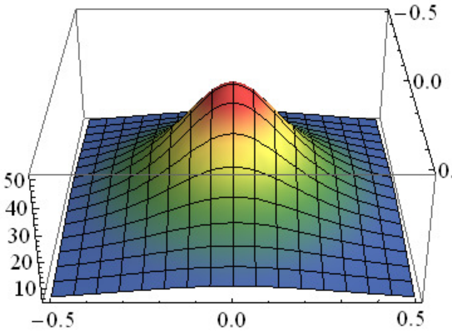

FIGURE 5.2. $c = \frac{1}{18}$

Example 2 Considering $g(z) = \sin z,\ A(z) = z,\ B(z) = \dfrac{1}{c}(z\sin z - 2\cos z + ic_1 z)$ in Corollary 2, we obtain

$$h(z) = \frac{u_1(2c - \cos(2u_1)) + u_1(1 + 2\sinh^2 u_2) - 2\sin(2u_1) + 2c_1(u_1\cos u_1 \sinh u_2 - u_2\cosh u_2 \sin u_1)}{c(2c + \cos(2u_1) + \cosh(2u_2))},$$
$$K(z) = \frac{4\sqrt{\cosh(2u_2) - \cos(2u_1)}}{2c + \cos(2u_1) + \cosh(2u_2)}.$$

The graphics of the functions $h(z)$ with $c = 1, c_1 = -2$ and $K(z)$ with $c = 1$ are given in the figures 5.3 and 5.4, respectively.

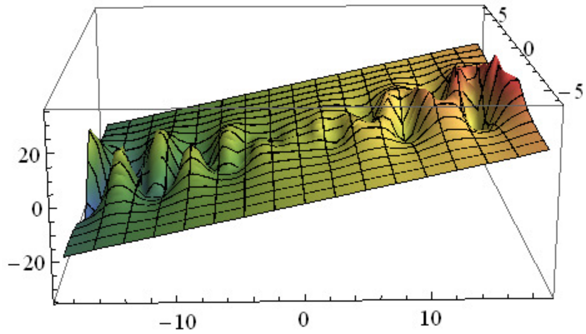

FIGURE 5.3. $c = 1, c_1 = -2$

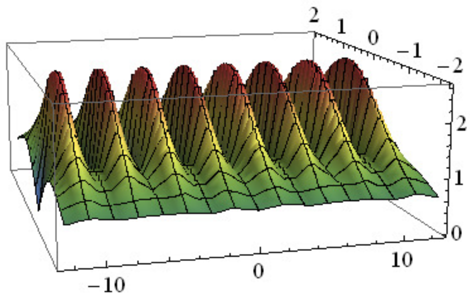

FIGURE 5.4. $c = 1$

Example 3 Considering $g(z) = \cosh z,\ A(z) = \sinh z,\ B(z) = \dfrac{1}{c}(-z + ic_1\cosh z)$ in Corollary 2, we obtain

$$h(z) = -\frac{2(u_1\cos u_2 \cosh u_1 + (-c\cos u_2 + u_2 \sin u_2)\sinh u_1)}{c(2c + \cos(2u_2) + \cosh(2u_1))},$$
$$K(z) = \frac{4\sqrt{\cosh(2u_1) - \cos(2u_2)}}{2c + \cos(2u_2) + \cosh(2u_1)}.$$

i) The graphics of the functions $h(z)$ and $K(z)$ for $c = 2$ are given in the figures 5.5 and 5.6, respectively.
ii) The graphics of the functions $h(z)$ and $K(z)$ for $c = -3$ are given in the figures 5.7 and 5.8, respectively.

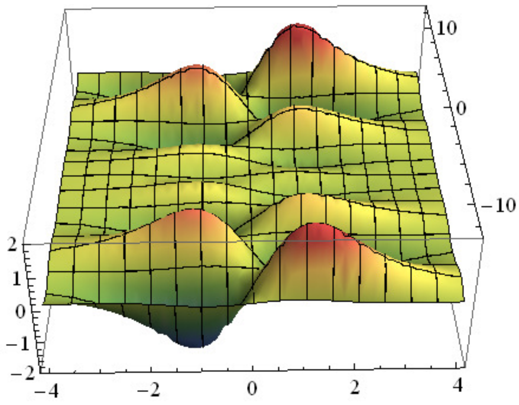

FIGURE 5.5. $c = 2$

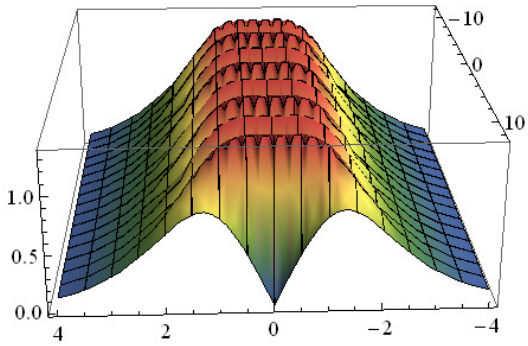

FIGURE 5.6. $c = 2$

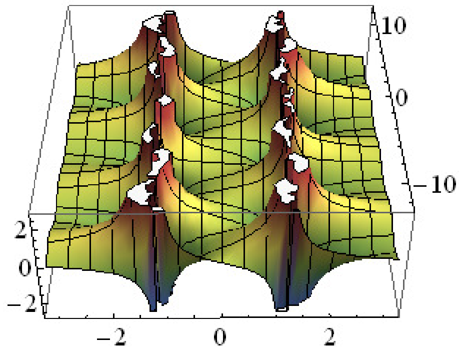

FIGURE 5.7. $c = -3$

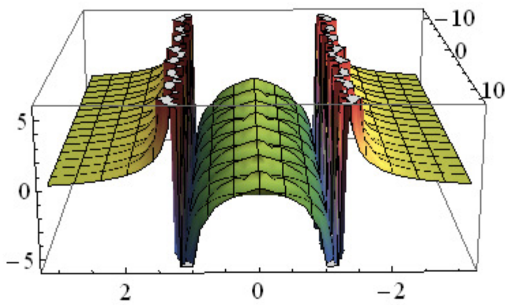

FIGURE 5.8. $c = -3$

6. Conclusions. In this paper, we present a different way of obtaining solutions to the Helmholtz equation without imposing boundary conditions, that is, the solutions are not necessarily limited to certain domains of the complex plane $\mathbb{C}$. These solutions depend on three holomorphic functions. Only when c is negative because of the regularity of the solution, we need to restrict the solution to a certain domain of $\mathbb{C}$ (see Figures 4.5, 4.6, 5.7 and 5.8).

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