



A Special class of Hypersurfaces parametrized by lines of curvature in \mathbb{R}^4

Una clase especial de Hipersuperficies parametrizadas por líneas de curvatura en \mathbb{R}^4

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Abstract

In this paper we study hypersurfaces in \mathbb{R}^4 parametrized by lines of curvature with three distinct principal curvatures and with Laplace invariants $m_{ji} = m_{ki} = 0$, $m_{jik} \neq 0$ for i, j, k distinct fixed indices. We characterize locally a generic family of such hypersurfaces in terms of the principal curvatures and three vector valued functions of one variable, this family includes a classe of Dupin hypersurfaces. Moreover, we show that these vector valued functions are invariant under inversions and homotheties.

Keywords Dupin hypersurfaces, Laplace invariants, lines of curvature

Resumen

En este artículo estudiamos hipersuperficies en \mathbb{R}^4 parametrizadas por líneas de curvatura con tres curvaturas principales distintas y con invariantes de Laplace $m_{ji} = m_{ki} = 0$, $m_{jik} \neq 0$ para índices fijos i, j, k distintos. Caracterizamos localmente una familia genérica de tales hipersuperficies en términos de las curvaturas principales y tres funciones vectoriales de una variable, esta familia incluye una clase de hipersuperficies de Dupin. Además, mostramos que estas funciones vectoriales son invariantes por inversiones y dilataciones.

Palabras clave. Hipersuperficies de Dupin, Invariantes de Laplace, Líneas de curvatura

1. Introduction. Dupin surfaces were first studied by Dupin in 1822 and more recently by many authors [1]-[6], [9]-[14] and [16],[17], which studied several aspects of Dupin hypersurfaces. The class of Dupin hypersurfaces is invariant under Lie transformations [11]. Therefore, the classification of Dupin hypersurfaces is considered up to these transformations. The local classification of Dupin surfaces in \mathbb{R}^3 is well known. Pinkall [12] gave a complete classification up to Lie equivalence for Dupin hypersurfaces $M^3 \subset \mathbb{R}^4$, with three distinct principal curvatures. Niebergall [10] and more recently Cecil and Jensen [6] studied proper Dupin hypersurfaces with four distinct principal curvatures and constant Lie curvature (the cross-ratio of four principal curvatures).

Riveros [15] obtained a local characterization of the Dupin hypersurfaces in \mathbb{R}^4 parametrized by lines of curvature, with three distinct principal curvatures and $m_{jik} \neq 0$, in terms of the principal curvatures and three vector valued functions in \mathbb{R}^4 which are invariant under inversions and homotheties, in this case the Laplace invariants $m_{ij} = 0$, for $1 \leq i \neq j \leq 3$.

In this paper we study generic hypersurfaces in \mathbb{R}^4 , parametrized by lines of curvature, with three distinct principal curvatures and with Laplace invariants $m_{ji} = m_{ki} = 0$, $m_{jik} \neq 0$. We obtain a local characterization of a generic family of such hypersurfaces (Theorem 3.1), in terms of the principal curvature functions and three vector valued functions of one variable. This family of hypersurfaces includes the Dupin

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hypersurfaces studied by Riveros [15]. The characterization is based on the theory of higher-dimensional Laplace invariants introduced by Kamran-Tenenblat [7]-[8].

In section 2, we give some properties of hypersurfaces with distinct principal curvatures. In section 3, Theorem 3.1 gives a local characterization of generic hypersurfaces in \mathbb{R}^4 with three distinct principal curvatures. Moreover, we show that the vector valued functions, which appear in the characterization of Theorem 3.1 are invariant under inversions and homotheties, but the functions are not invariant under isometries. Therefore, the vector valued functions are not invariant under the full group of Lie transformations of \mathbb{R}^4 .

2. Preliminaries. Let Ω be an open subset of \mathbb{R}^n and $x = (x_1, x_2, \dots, x_n) \in \Omega$. Let $X : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$, $n \geq 3$, be a hypersurface parametrized by lines of curvature, with distinct principal curvatures λ_i , $1 \leq i \leq n$ and $N : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ be a unit normal vector field of X . Then

$$(2.1) \quad \begin{aligned} \langle X_{,i}, X_{,j} \rangle &= \delta_{ij} g_{ii}, \quad 1 \leq i, j \leq n, \\ N_{,i} &= -\lambda_i X_{,i}, \end{aligned}$$

where the subscript $,i$ denotes the derivative with respect to x_i . Moreover,

$$(2.2) \quad X_{,ij} - \Gamma_{ij}^i X_{,i} - \Gamma_{ij}^j X_{,j} = 0, \quad 1 \leq i \neq j \leq n,$$

$$(2.3) \quad \Gamma_{ij}^i = \frac{\lambda_{i,j}}{\lambda_j - \lambda_i}, \quad 1 \leq i \neq j \leq n,$$

where Γ_{ij}^k are the Christoffel symbols.

We now consider the higher-dimensional Laplace invariants of the system of equations (2.2) (see [7]-[8] for the definition of these invariants),

$$(2.4) \quad \begin{aligned} m_{ij} &= -\Gamma_{ij,i}^i + \Gamma_{ij}^i \Gamma_{ij}^j, \\ m_{ijk} &= \Gamma_{ij}^i - \Gamma_{kj}^k, \quad k \neq i, j, \quad 1 \leq k \leq n. \end{aligned}$$

As a consequence of (2.3) and the un-numbered lemma appearing in [8], we obtain the following identities, valid for distinct i, j, k, l , $1 \leq i, j, k, l \leq n$:

$$(2.5) \quad \begin{aligned} m_{ijk} + m_{kji} &= 0, \\ m_{ijk,k} - m_{ijk} m_{jki} - m_{kj} &= 0, \\ m_{ij,k} + m_{ijk} m_{ik} + m_{ikj} m_{ij} &= 0, \\ m_{ijk} - m_{ijl} - m_{ljk} &= 0, \\ m_{lik,j} + m_{ijl} m_{kil} + m_{ljk} m_{kij} &= 0. \end{aligned}$$

From Remark 2.2 in [16], follows that for $n \geq 3$, the higher-dimensional Laplace invariants do not change under inversions in spheres centered at the origin and homotheties.

For hypersurfaces with distinct principal curvatures, the *Möbius curvature* is defined, for distinct i, j, k , by

$$(2.6) \quad C^{ijk} = \frac{\lambda_i - \lambda_j}{\lambda_k - \lambda_j}.$$

Since all λ_i are distinct we conclude that $C^{ijk} \neq 0$ and $C^{ijk} \neq 1$. Möbius curvatures are invariant under Möbius transformations.

The following result was obtained in [14], which provides some properties which are satisfied by the principal curvatures of a hypersurface in \mathbb{R}^{n+1} parametrized by lines of curvature.

LEMMA 1. Let $\lambda_r : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$, $n \geq 3$, be smooth functions distinct at each point. Consider functions m_{ijk} defined by (2.3) and (2.4). Then for i, j fixed, $1 \leq i \neq j \leq n$, the following properties hold

$$(2.7) \quad [C^{kji} m_{jki}]_{,i} = m_{jki,i} + \left[\frac{\lambda_{i,i}}{\lambda_j - \lambda_i} \right]_{,k},$$

$$(2.8) \quad [C^{kji} m_{jki}]_{,j} = - \left[\frac{\lambda_{j,j}}{\lambda_j - \lambda_i} \right]_{,k},$$

$$(2.9) \quad [C^{kji} m_{jki}]_{,l} = [C^{lji} m_{jli}]_{,k},$$

where C^{kji} is the Möbius curvature and $1 \leq k \neq l \leq n$ are distinct from i and j . The following Lemma is an application of Theorem 1 in [8].

LEMMA 2. Let $X : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$, $n \geq 3$, be a hypersurface parametrized by lines of curvature, with n distinct principal curvatures λ_r , $1 \leq r \leq n$. For i, j, k fixed, $1 \leq i \neq j \neq k \leq n$, the transformation

$$(2.10) \quad X = V \bar{X}, \quad \text{where} \quad V = \frac{e^{-\int C^{kji} m_{jki} dx_k}}{\lambda_j - \lambda_i},$$

transforms system (2.2) into

$$(2.11) \quad \begin{aligned} \bar{X}_{,ij} + A \bar{X}_{,j} - m_{ij} \bar{X} &= 0, \\ \bar{X}_{,ir} + (A + m_{jir}) \bar{X}_{,r} - m_{ir} \bar{X} &= 0, \\ \bar{X}_{,jr} + m_{irj} \bar{X}_{,j} + m_{ijr} \bar{X}_{,r} &= 0, \\ \bar{X}_{,rl} + m_{ilr} \bar{X}_{,r} + m_{irl} \bar{X}_{,l} &= 0, \end{aligned}$$

where l and r are such that $1 \leq r \neq l \neq i \neq j \leq n$ and

$$(2.12) \quad A = - \int m_{jki,i} dx_k.$$

Moreover,

$$(2.13) \quad A_{,j} = m_{ji} - m_{ij}, \quad A_{,r} = -m_{jri,i}.$$

REMARK 1. For subsequent use, we will compute the derivatives of the function V given by (2.10). It follows from Lemma 1 that,

$$(2.14) \quad \begin{aligned} V_{,i} &= \left(A + \Gamma_{ji}^j \right) V, \\ V_{,j} &= \Gamma_{ij}^i V, \\ V_{,k} &= \Gamma_{ik}^i V, \\ V_{,l} &= \Gamma_{il}^i V, \end{aligned}$$

where A is given by (2.12) and l is distinct from i, j, k .

3. Main results. In this section, we prove our main result which provides a local characterization of generic hypersurfaces parametrized by lines of curvature in \mathbb{R}^4 , with three distinct principal curvatures. We remark that in the case of Dupin hypersurfaces parametrized by lines of curvature the Laplace invariants $m_{ij} = 0$, for $1 \leq i \neq j \leq 3$ and therefore, this family of hypersurfaces includes the class of Dupin hypersurfaces studied by Riveros in [15].

THEOREM 1. Let $X : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^4$, be a hypersurface parametrized by lines of curvature, with three distinct principal curvatures λ_r . For i, j, k distinct fixed indices, suppose $m_{ji} = m_{ki} = 0$ and $m_{jik} \neq 0$ then

$$(3.1) \quad X = V [B_j - B_k],$$

where

$$(3.2) \quad V = \frac{e^{-\int C^{kji} m_{jki} dx_k}}{\lambda_j - \lambda_i}, \quad B_s = \frac{1}{Q_s} \left[\int \frac{Q_s G_i(x_i)}{m_{jik}} dx_i + G_s(x_s) \right], \quad s \neq i,$$

$G_r(x_r)$, $r = i, j, k$, are vector valued functions of \mathbb{R}^4 , $A_{,j} = -m_{ij}$, $A = - \int m_{jki,i} dx_k$ and

$$(3.3) \quad Q_s = \begin{cases} e^{\int A dx_i} & \text{if } s = j, \\ e^{\int (A + m_{jis}) dx_i} & \text{if } s = k. \end{cases}$$

Moreover, considering

$$(3.4) \quad \alpha^i = \left(A + \frac{\lambda_{j,i}}{\lambda_i - \lambda_j} \right) M + M_{,i}, \quad \alpha^s = \frac{\lambda_{i,s}}{\lambda_s - \lambda_i} M + M_{,s}, \quad s \neq i,$$

where $M = B_j - B_k$, the functions $G_r(x_r)$ satisfy the following properties in Ω , for $1 \leq r \neq t \leq 3$:

- a) $\alpha^r \neq 0$,
- b) $\langle \alpha^r, \alpha^t \rangle = 0, \quad r \neq t$,
- c) $\lambda_r = \frac{\langle \alpha_{,r}^r, \alpha^i \times \alpha^j \times \alpha^k \rangle}{V |\alpha^r|^2 |\alpha^i| |\alpha^j| |\alpha^k|}$.

Conversely, let $\lambda_r : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}, r = 1, 2, 3$ be real functions, distinct at each point. Assume that the functions m_{rts} and m_{rt} defined by

$$(3.5) \quad \begin{aligned} m_{rts} &= \frac{\lambda_{r,t}}{\lambda_t - \lambda_r} - \frac{\lambda_{s,t}}{\lambda_t - \lambda_s}, \quad 1 \leq r \neq t \neq s \leq 3, \\ m_{rt} &= - \left[\frac{\lambda_{r,t}}{\lambda_t - \lambda_r} \right]_{,r} - \frac{\lambda_{r,t} \lambda_{t,r}}{(\lambda_t - \lambda_r)^2}, \quad 1 \leq r \neq t \leq 3, \end{aligned}$$

satisfy (2.5), and for i, j, k distinct fixed indices, $m_{ji} = m_{ki} = 0$, and $m_{jik} \neq 0$. Then for any vector valued functions $G_r(x_r)$ satisfying properties a) b) c), where α^r is defined by (3.4), the function $X : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^4$ given by (3.1) describes a hypersurface parametrized by lines of curvature whose principal curvatures are the functions λ_r .

Proof: We observe that from (2.5), the conditions $m_{ji} = m_{ki} = 0$, implies that $m_{jk} = m_{kj} = 0$. From equation (2.2) we have,

$$(3.6) \quad X_{,sr} - \Gamma_{sr}^s X_{,s} - \Gamma_{sr}^r X_{,r} = 0, \quad 1 \leq s \neq r \leq 3.$$

For fixed distinct indices i, j, k , we consider the transformation

$$(3.7) \quad X = V \bar{X},$$

as in Lemma 2, where V is given by (2.10). Then system (3.6) reduces to

$$(3.8) \quad \begin{aligned} \bar{X}_{,ij} + A \bar{X}_{,j} - m_{ij} \bar{X} &= 0, \\ \bar{X}_{,ik} + (A + m_{jik}) \bar{X}_{,k} - m_{ik} \bar{X} &= 0, \\ \bar{X}_{,jk} + m_{ikj} \bar{X}_{,j} + m_{ijk} \bar{X}_{,k} &= 0, \end{aligned}$$

where

$$(3.9) \quad A_{,j} = -m_{ij}, \quad A_{,k} = -m_{jki,i}.$$

It follows from the third and second equations of (2.5) and (3.9) that

$$(3.10) \quad (A + m_{jik})_{,k} = -m_{ik}.$$

Using (3.9) and (3.10) in the first two equations of (3.8), we have that

$$(3.11) \quad \bar{X}_{,i} + A \bar{X} = W^j(x_i, x_k),$$

$$(3.12) \quad \bar{X}_{,i} + (A + m_{jik}) \bar{X} = W^k(x_i, x_j),$$

where W^j and W^k are functions that do not depend on x_j and x_k , respectively. Since $m_{jik} \neq 0$, from (3.11) and (3.12) we have

$$(3.13) \quad \bar{X} = \frac{1}{m_{jik}} [W^k - W^j].$$

Differentiating (3.13) and using (2.5) we have

$$(3.14) \quad \bar{X}_{,i} = -\frac{m_{jik,i}}{(m_{jik})^2} [W^k - W^j] + \frac{1}{m_{jik}} [W^k - W^j]_{,i}$$

$$(3.15) \quad \bar{X}_{,j} = \frac{m_{kji}}{m_{jik}} [W^k - W^j] + \frac{1}{m_{jik}} W^k_{,j}$$

$$(3.16) \quad \bar{X}_{,k} = \frac{m_{jki}}{m_{jik}} [W^k - W^j] - \frac{1}{m_{jik}} W^j_{,k}$$

$$(3.17) \quad \bar{X}_{,ij} = \frac{1}{m_{jik}} W^k_{,ij} - \frac{m_{jik,i}}{(m_{jik})^2} W^k_{,j} + \left[m_{kji} + \frac{m_{ij}}{m_{jik}} + \frac{m_{ijk}m_{jik,i}}{(m_{jik})^2} \right] [W^k - W^j] +$$

$$- \frac{m_{ijk}}{m_{jik}} [W^k - W^j]_{,i}$$

$$(3.18) \quad \bar{X}_{,ik} = -\frac{1}{m_{jik}} W^j_{,ik} + \frac{m_{jik,i}}{(m_{jik})^2} W^j_{,k} + \left[m_{ikj} + \frac{m_{ik}}{m_{jik}} + \frac{m_{ikj}m_{jik,i}}{(m_{jik})^2} \right] [W^k - W^j] +$$

$$- \frac{m_{ikj}}{m_{jik}} [W^k - W^j]_{,i}$$

$$(3.19) \quad \bar{X}_{,jk} = \frac{2m_{ijk}m_{ikj}}{m_{jik}} [W^k - W^j] + \frac{m_{ijk}}{m_{jik}} W^j_{,k} - \frac{m_{ikj}}{m_{jik}} W^k_{,j}$$

The substitution of \bar{X} and $\bar{X}_{,i}$ into (3.11) and (3.12), gives

$$(3.20) \quad \left(A - \frac{m_{jik,i}}{m_{jik}} \right) [W^k - W^j] + [W^k - W^j]_{,i} = m_{jik} W^j_{,i}$$

$$(3.21) \quad \left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}} \right) [W^k - W^j] + [W^k - W^j]_{,i} = m_{jik} W^k_{,i}$$

Using (3.13), (3.15), (3.17) and (3.21) in first equation of system (3.8), we obtain

$$(3.22) \quad W^k_{,ij} + \left(A - \frac{m_{jik,i}}{m_{jik}} \right) W^k_{,j} + m_{kji}m_{jik} = 0,$$

Using (3.13), (3.16), (3.18) and (3.20) in second equation of system (3.8), we obtain

$$(3.23) \quad W^i_{,ik} + \left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}} \right) W^j_{,k} + m_{ikj}m_{jik} = 0,$$

Using (3.15), (3.16) and (3.19) in the third equation of system (3.8) we obtain an identity.

From (2.5) and (3.9) we obtain

$$(3.24) \quad \left(A - \frac{m_{jik,i}}{m_{jik}} \right)_{,j} = m_{kji}m_{jik}, \quad \left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}} \right)_{,k} = m_{ikj}m_{jik}.$$

It follows from (3.2), (3.3) and (3.24) that the solutions of equations (3.22) and (3.23) are given by

$$(3.25) \quad W^k(x_i, x_j) = \frac{m_{jik}}{Q_j} \left[\int \frac{Q_j \tilde{G}_i(x_i)}{m_{jik}} dx_i + G_j(x_j) \right]$$

$$(3.26) \quad W^j(x_i, x_k) = \frac{m_{jik}}{Q_k} \left[\int \frac{Q_k \tilde{G}_i(x_i)}{m_{jik}} dx_i + G_k(x_k) \right]$$

From (3.20), (3.25) and (3.26) it follows that

$$(3.27) \quad G_i(x_i) = \tilde{G}_i(x_i).$$

The substitution of (3.25), (3.26) and (3.27) in (3.13) gives

$$\bar{X} = B_j - B_k,$$

where we have used (3.2), which substituted into (3.7), implies (3.1).

Considering α^i and α^s , $s = j, k$ defined by (3.4), it follows from (2.7), (2.8), (2.13) and (2.4) that

$$(3.28) \quad X_{,r} = V\alpha^r, \quad r = i, j, k.$$

Differentiating (3.28), we have

$$(3.29) \quad X_{,rr} = V_{,r}\alpha^r + V\alpha_{,r}^r, \quad r = i, j, k.$$

It follows from (3.28) that the metric of $X_{,r}$ is given by

$$(3.30) \quad g_{rr} = (V)^2|\alpha^r|^2, \quad g_{rt} = 0, \quad r \neq t.$$

A unit vector field normal to X is given by

$$(3.31) \quad N = \frac{\alpha^i \times \alpha^j \times \alpha^k \times \alpha^l}{|\alpha^i| |\alpha^j| |\alpha^k| |\alpha^l|}.$$

Since X is a hypersurface parametrized by orthogonal curvature lines, with λ_s , as principal curvature we have, for $1 \leq r \neq s \leq 3$

$$\langle N, X_{,rs} \rangle = 0, \quad \lambda_s = \frac{\langle X_{,rr}, N \rangle}{g_{rr}}.$$

Hence from (3.29) and (3.31) we obtain for $r = i, j, k$,

$$\lambda_r = \frac{\langle \alpha_{,r}^r, \alpha^i \times \alpha^j \times \alpha^k \times \alpha^l \rangle}{V|\alpha^r|^2|\alpha^i| |\alpha^j| |\alpha^k| |\alpha^l|}.$$

Therefore, we conclude that conditions a), b) and c) are satisfied.

Conversely, let λ_r be real functions distinct at each point. Assume that the functions m_{rts} and m_{rt} , defined by (3.5), satisfy (2.5) and suppose $G_r(x_r)$, $1 \leq r \leq 3$, are vector valued functions satisfying properties a), b) and c). Defining X by (3.1), it follows from Lemma 1 and properties a) and b), that X is an immersion, whose coordinates curves are orthogonal. Moreover, the induced metric is given by (3.30) and a unit normal vector field by (3.31).

Differentiating (3.28) with respect to x_t , using Lemma 1, the expressions (2.5), (2.14) and (3.4) we obtain

$$X_{,rt} = V \left(\frac{\lambda_{r,t}}{\lambda_t - \lambda_r} \alpha^r + \frac{\lambda_{t,r}}{\lambda_r - \lambda_t} \alpha^t \right), \quad r \neq t.$$

From (3.31), it follows that $\langle X_{,rt}, N \rangle = 0$. Hence the second fundamental form is diagonal and therefore the coordinates curves are lines of curvature. Moreover, it follows from (3.29) - (3.31) and from property c) that for $r = i, j, k$,

$$\frac{\langle X_{,rr}, N \rangle}{g_{rr}} = \frac{\langle \alpha_{,r}^r, \alpha^i \times \alpha^j \times \alpha^k \times \alpha^l \rangle}{V|\alpha^r|^2|\alpha^i| |\alpha^j| |\alpha^k| |\alpha^l|} = \lambda_r$$

which concludes the proof. \square

Now we show that the vector valued functions which appear in Theorem 1 are invariant under inversions and homotheties.

THEOREM 2. *Let $X : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be a hypersurface with three distinct principal curvatures λ_r , parametrized by lines of curvature as in the Theorem 1. Then the vector valued functions $G_r(x_r)$, $1 \leq r \leq 3$ are invariants under inversions and homotheties.*

Proof: a) Assuming without loss of generality that $0 \notin X(\Omega)$, we consider $\tilde{X} = I^4(X)$ a hypersurface parametrized by lines of curvature, obtained by composing X with the inversion defined by $I^4(X) = \frac{X}{\langle X, X \rangle}$ and whose distinct principal curvatures are given by

$$(3.32) \quad \tilde{\lambda}_r = \langle X, X \rangle \lambda_r + 2\langle X, N \rangle, \quad r = i, j, k.$$

Applying the Theorem 1 to \tilde{X} , we have for i, j, k fixed distinct indices

$$\tilde{X} = \tilde{V} \left[\tilde{B}_j - \tilde{B}_k \right],$$

where

$$(3.33) \quad \begin{aligned} \tilde{B}_s &= \frac{1}{\tilde{Q}_s} \left[\int \frac{\tilde{Q}_s \tilde{G}_i(x_i)}{\tilde{m}_{jik}} dx_i + \tilde{G}_s(x_s) \right], \quad s \neq i, \\ \tilde{V} &= \frac{e^{-\int \tilde{C}^{kji} \tilde{m}_{jki} dx_k}}{\tilde{\lambda}_j - \tilde{\lambda}_i}, \quad \tilde{C}^{kji} = \frac{\tilde{\lambda}_k - \tilde{\lambda}_j}{\tilde{\lambda}_i - \tilde{\lambda}_j} \\ \tilde{A} &= - \int \tilde{m}_{jki,i} dx_k, \end{aligned}$$

\tilde{Q}_s , $s \neq i$ are defined by (3.3) and $\tilde{G}_r(x_r)$, $r = i, j, k$ are vector valued functions in \mathbb{R}^4 .
Since \tilde{X} and X have the same higher-dimensional Laplace invariants, we obtain

$$(3.34) \quad \tilde{A} = A, \quad \tilde{Q}_s = Q_s, \quad s \neq i.$$

Substituting (3.32) in (3.33), we have

$$(3.35) \quad \tilde{V} = \frac{V}{\langle X, X \rangle}.$$

On the other hand,

$$(3.36) \quad \tilde{X} = \frac{X}{\langle X, X \rangle}.$$

We will show that $\tilde{G}_r(x_r) = G_r(x_r)$, $r = i, j, k$. It follows from (3.35) and (3.36) that

$$(3.37) \quad \tilde{B}_j - B_j - (\tilde{B}_k - B_k) = 0.$$

We observe that

$$B_{j,i} = -AB_j + \frac{G_i(x_i)}{m_{jik}}, \quad B_{k,i} = -(A + m_{jik})B_k + \frac{G_i(x_i)}{m_{jik}}.$$

This fact follows from the equalities

$$Q_{j,i} = AQ_j, \quad Q_{k,i} = (A + m_{jik})Q_k.$$

Therefore differentiating (3.37) with respect to x_i , we get

$$-A(\tilde{B}_j - B_j) + (A + m_{jik})(\tilde{B}_k - B_k) = 0.$$

From (3.37) and the fact that $m_{jik} \neq 0$, we have

$$(3.38) \quad \tilde{B}_k = B_k.$$

Differentiating with respect to x_i , we get $\tilde{G}_k(x_k) = G_k(x_k)$, hence it follows that $\tilde{G}_i(x_i) = G_i(x_i)$. Substituting (3.38) in (3.37), we obtain

$$\tilde{B}_j = B_j.$$

and hence $\tilde{G}_j(x_j) = G_j(x_j)$, which concludes the proof of a).

b) Let $\bar{X} = aX$ be a homothety of X . Then \bar{X} is a hypersurface parametrized by lines of curvature, with distinct principal curvatures given by

$$(3.39) \quad \bar{\lambda}_r = \frac{\lambda_r}{a}, \quad r = i, j, k.$$

Applying Theorem 1 to \bar{X} , we have for i, j, k distinct fixed indices

$$\bar{X} = \bar{V} [\bar{B}_j - \bar{B}_k].$$

where

$$\begin{aligned}
 \bar{B}_s &= \frac{1}{\bar{Q}_s} \left[\int \frac{\bar{Q}_s \bar{G}_i(x_i)}{\bar{m}_{jik}} dx_i + \bar{G}_s(x_s) \right], \quad s \neq i, \\
 \bar{V} &= \frac{e^{-\int \bar{C}^{kji} \bar{m}_{jki} dx_k}}{\bar{\lambda}_j - \bar{\lambda}_i}, \quad \text{where } \bar{C}^{kji} = \frac{\bar{\lambda}_k - \bar{\lambda}_j}{\bar{\lambda}_i - \bar{\lambda}_j} \\
 \bar{A} &= - \int \bar{m}_{jki,i} dx_k,
 \end{aligned}
 \tag{3.40}$$

$\bar{Q}_s, s \neq i$ are defined by (3.3) and $\bar{G}_r(x_r), r = i, j, k$ are vector valued functions in \mathbb{R}^4 .

Since \bar{X} and X have the same Laplace invariants. Therefore, it follows that

$$\bar{A} = A, \quad \bar{Q}_s = Q_s, \quad s \neq i.
 \tag{3.41}$$

We will show that $\bar{G}_r(x_r) = G_r(x_r)$. Substituting (3.39) in (3.40), we have

$$\bar{V} = aV.
 \tag{3.42}$$

Since

$$X = V [B_j - B_k], \quad \bar{X} = \bar{V} [\bar{B}_j - \bar{B}_k].
 \tag{3.43}$$

Substituting (3.41), (3.42) and (3.43) in $\bar{X} = aX$ we have,

$$\bar{B}_j - B_j - (\bar{B}_k - B_k) = 0.$$

The same argument of item a) proves that $\bar{G}_r(x_r) = G_r(x_r), \forall r. \square$

Example 1: Let $X : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be an immersion

$$X(x_1, x_2, x_3) = ((a + r \cos x_1) \cos x_2, (a + r \cos x_1) \sin x_2, r \sin x_1, x_3),$$

that describes a hypersurface of Dupin.

The principal curvatures are given by

$$k_1 = -\frac{1}{r}, \quad k_2 = -\frac{\cos x_1}{a + r \cos x_1}, \quad k_3 = 0,$$

From (2.4) we have that the laplace invariants are given by

$$m_{ij} = 0, \quad 1 \leq i \neq j \leq 3, \quad m_{213} = -\frac{r \sin x_1}{a + r \cos x_1} \neq 0.$$

From Theorem 1, we get

$$X = V [B_2 - B_3],$$

where

$$A = 0, \quad V = -\frac{(a + r \cos x_1)^2}{a \sin x_1}, \quad Q_2 = 1, \quad Q_3 = a + r \cos x_1.$$

The vectorial functions are given by

$$\begin{aligned}
 G_1(x_1) &= \left(0, 0, -\frac{a}{\sin x_1 (a + r \cos x_1)}, 0 \right), \\
 G_2(x_2) &= \left(\frac{a \cos x_2}{r}, \frac{a \sin x_2}{r}, 0, 0 \right), \\
 G_3(x_3) &= \left(0, 0, 0, -\frac{ax_3}{r} \right).
 \end{aligned}$$

Example 2: Let $Y(x_1, x_2)$ be a surface parametrized by lines of curvature on the sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ defined in $\Omega = I \times \mathbb{R}$, with two distinct principal curvatures \tilde{k}_1, \tilde{k}_2 that only depend on x_1 and satisfying $\tilde{k}_{1,1} \neq 0, \tilde{k}_{2,1} \neq 0, \forall x_1 \in I, I$ is an open interval. We consider the hypersurface $X : \Omega \times J \rightarrow \mathbb{R}^4$

parametrized by lines of curvature given by $X(x_1, x_2, x_3) = Y(x_1, x_2) + x_3 \tilde{N}(x_1, x_2)$, where \tilde{N} is the normal vector of the surface Y and J is an open interval such that $(x_3 \tilde{k}_i - 1) \neq 0, \forall x_3 \in J, i = 1, 2$. We can show that the normal vector to X is Y and the distinct principal curvatures of X are given by

$$(3.44) \quad k_1 = \frac{1}{x_3 \tilde{k}_1 - 1}, \quad k_2 = \frac{1}{x_3 \tilde{k}_2 - 1}, \quad k_3 = 0.$$

By (2.3), we have

$$(3.45) \quad \Gamma_{12}^1 = \Gamma_{31}^3 = \Gamma_{32}^3 = 0, \quad \Gamma_{13}^1 = \frac{\tilde{k}_1}{x_3 \tilde{k}_1 - 1}, \quad \Gamma_{21}^2 = \frac{(x_3 \tilde{k}_1 - 1) \tilde{k}_{2,1}}{(x_3 \tilde{k}_2 - 1)(\tilde{k}_1 - \tilde{k}_2)}, \quad \Gamma_{23}^2 = \frac{\tilde{k}_2}{x_3 \tilde{k}_2 - 1}.$$

From (2.4) and (3.45) we have that the laplace invariants are given by

$$(3.46) \quad m_{12} = m_{21} = m_{23} = m_{31} = m_{32} = 0, \quad m_{13} = \frac{\tilde{k}_{1,1}}{(x_3 \tilde{k}_1 - 1)^2},$$

$$(3.47) \quad m_{213} = \frac{(x_3 \tilde{k}_1 - 1) \tilde{k}_{2,1}}{(x_3 \tilde{k}_2 - 1)(\tilde{k}_1 - \tilde{k}_2)}, \quad m_{231} = \frac{\tilde{k}_1 - \tilde{k}_2}{(x_3 \tilde{k}_1 - 1)(x_3 \tilde{k}_2 - 1)}.$$

Also,

$$C^{321} = \frac{x_3 \tilde{k}_1 - 1}{x_3(\tilde{k}_1 - \tilde{k}_2)}.$$

From Theorem 1, we get

$$X = V [B_2 - B_3],$$

where

$$(3.48) \quad A = -\frac{x_3 \tilde{k}_{2,1}}{x_3 \tilde{k}_2 - 1} + \frac{x_3 \tilde{k}_{1,1}}{x_3 \tilde{k}_1 - 1} + \frac{\tilde{k}_{1,1}}{\tilde{k}_2 - \tilde{k}_1}, \quad V = \frac{x_3 \tilde{k}_1 - 1}{\tilde{k}_1 - \tilde{k}_2},$$

$$Q_2 = \frac{x_3 \tilde{k}_1 - 1}{x_3 \tilde{k}_2 - 1} e^{\int \frac{\tilde{k}_{1,1}}{\tilde{k}_2 - \tilde{k}_1} dx_1}, \quad Q_3 = \frac{x_3 \tilde{k}_1 - 1}{\tilde{k}_1 - \tilde{k}_2}.$$

The vectorial functions are given by

$$G_1(x_1) = \left(-\tilde{k}_{1,1} + 2\tilde{k}_{2,1} + \frac{(\tilde{k}_1 - \tilde{k}_2)\tilde{k}_{2,11}}{\tilde{k}_{2,1}} \right) (x_1)Y(x_1, x_2^0) + (\tilde{k}_2 - \tilde{k}_1)(x_1)Y_{,11}(x_1, x_2^0)$$

$$G_2(x_2) = \left((\tilde{k}_2 - \tilde{k}_1) e^{\int \frac{\tilde{k}_{1,1}}{\tilde{k}_2 - \tilde{k}_1} dx_1} \right) (x_1^0)Y(x_1^0, x_2)$$

$$G_3(x_3) = (a_1 x_3 + b_1, a_2 x_3 + b_2, a_3 x_3 + b_3),$$

where

$$(a_1, a_2, a_3) = \left(\frac{\tilde{k}_2}{\tilde{k}_1 - \tilde{k}_2} e^{-\int \frac{\tilde{k}_{1,1}}{\tilde{k}_2 - \tilde{k}_1} dx_1} \right) (x_1^0) \left[\left(\int \frac{G_1(x_1)(\tilde{k}_1 - \tilde{k}_2) e^{-\int \frac{\tilde{k}_{1,1}}{\tilde{k}_2 - \tilde{k}_1} dx_1}}{\tilde{k}_{2,1}} dx_1 \right) (x_1^0) \right. \\ \left. + G_2(x_2^0) \right] - \tilde{N}(x_1^0, x_2^0) - \left(\int \frac{G_1(x_1)\tilde{k}_2}{\tilde{k}_{2,1}} dx_1 \right) (x_1^0).$$

We observed that this class of hypersurfaces has nonconstant Möbius curvature and they are not Dupin.

4. Conclusions. From the results obtained in this work we can make the following conclusion: The fact of considering hypersurfaces parametrized by lines of curvature in \mathbb{R}^4 with Laplace invariants $m_{ji} = m_{ki} = 0$, $m_{jik} \neq 0$ for i, j, k distinct fixed, allows us to find the same representation obtained in [15], therefore, this work generalizes the results obtained in [15], i.e., the hypersurfaces of Dupin studied in [15] are contained in this class of hypersurfaces.

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