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On a vibration problem of homogeneous string

Sobre un problema de vibración de cuerda homogénea

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Abstract

In this paper we study the existence and uniqueness of the weak solution of a mathematical model that describes the vibration of a string. This model is given by a wave equation with dynamic boundary conditions. Also, we show that this model is conservative but is not exponentially stable.

Keywords. Wave equation, Spectral analysis, Conservative system, Exponential stability.

Resumen

En este artículo estudiamos la existencia y unicidad de la solución débil de un modelo matemático que describe la vibración de una cuerda. Este modelo está dado por una ecuación de onda con condiciones de frontera dinámicas. También, mostramos que este modelo es conservativo pero no es exponencialmente estable.

Palabras clave. Ecuación de la onda, Análisis espectral, Sistema conservativo, Estabilidad exponencial.

1. Introduction. This paper is concerned with the initial-boundary value problem

$$
\begin{cases}
u_{tt} = u_{xx}, \ 0 < x < 1, \ t > 0, \\
u(0,t) = 0, \ t > 0. \\
Mu_{tt}(1,t) + Ku_x(1,t) = 0, \ t > 0. \\
u(x,0) = u_0(x), \ 0 < x < 1, \\
u_t(x,0) = v_0(x), \ 0 < x < 1, \\
u(1,0) = \eta \, (= u_0(1)), \\
u_t(1,0) = \mu \, (= v_0(1)).
\end{cases} \tag{1.1}
$$

where $u = u(x,t) \in \mathbb{R}$, $t \geq 0$ and the constants $K, M > 0$. This system represents the mathematical model of vibration problem of homogeneous string, where $u(x,t)$ is the magnitude of the string's displacement from a state of equilibrium at point x at an instant of time t, M is the mass localized in the point $x = 1$, K is the elastic modulo and we considered small vibrations of the string, without taking into account the external forces acting on the string. For the boundary conditions, we see that the end $x = 0$ of the string is fixed and in the end $x = 1$ is free and transports a localized mass M (for instance, see Bitsadze and Kalinichenko [4], Gulmamedov and Mamedov [6], Tikhonov and Samarskii [14]).

The boundary conditions which appear in (1.1) are usually called *dynamic boundary conditions*. Boundary value problems with dynamics boundary conditions have attracted interest of many authors in this last years, with practical motivations to stabilization and control of elastics structures (see Meurer and Kugi

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[10] and references therein). Nowadays the wave equation subject to dynamic boundary conditions is widely used in many applications. See Meurer and Kugi [10], Pellicer and Solà-Morales [12], [13] for some applications.

The aim of this work is to study the existence and uniqueness of weak solutions for (1.1) (see Theorem 2.3) along with the spectral analysis of operator $\mathbb{A}$ defined by (2.1). Also, we will show that the system (1.1) is conservative and not exponentially stable (see Proposition 3.3 and Theorem 3.4, respectively).

This paper is organized as follows. In Section 2, we study the existence and uniqueness of weak solutions of (1.1). The Section 3 is devoted to prove the conservation of energy, the semigroup is not exponentially stable and the spectral analysis of operator $\mathbb{A}$.

2. Functional setting and existence of solution. As in many problems with dynamic boundary conditions is appropriate to work in spaces whose elements are pairs of a function and its boundary value. Moreover, since we are writing a evolution equation of second order as a first order system our phase spaces will consist of pairs of such pairs.

Now, we give a functional setting for obtaining the existence and uniqueness of solutions for the Cauchy problem (1.1).

We want to write (1.1) as an abstract evolution equation, in a similar way as it is done by Grobbelaar-van in [5] (see also Pellicer and Solà-Morales [13]). For this, let us consider the following spaces:

$$\begin{aligned}\mathbb{X}_0 &= \{(u,a) \in L^2(0,1) \times \mathbb{C}\} = L^2(0,1) \times \mathbb{C},\\ \mathbb{X}_1 &= \{(u,a) \in H^1(0,1) \times \mathbb{C} : u(1) = a,\ u(0) = 0\},\end{aligned}$$

as a subspace of $H^1(0,1) \times \mathbb{C}$ and

$$\mathbb{X}_2 = \{(u,a) \in H^2(0,1) \times \mathbb{C} : u(1) = a,\ u(0) = 0\},$$

as a subspace of $H^2(0,1) \times \mathbb{C}$.

In $\mathbb{X}_0$, a natural inner product is given by

$$\langle (u,a),(v,b)\rangle_{\mathbb{X}_0} = \int_0^1 u\bar{v}dx + \frac{M}{K}a\bar{b}$$

and in $\mathbb{X}_1$ we have

$$\langle (u,u(1)),(v,v(1))\rangle_{\mathbb{X}_1} = \int_0^1 u_x\overline{v_x}dx.$$

REMARK 1. *The inner products defined in $\mathbb{X}_0$ and $\mathbb{X}_1$ are equivalent to the usual inner products in $L^2(0,1) \times \mathbb{C}$ and $H^1(0,1) \times \mathbb{C}$ respectively.*

Now, we define the linear operator $\mathbb{A} : \mathscr{D}(\mathbb{A}) \subset \mathscr{H} \to \mathscr{H}$ where its domain, $\mathscr{D}(\mathbb{A})$, is

$$\mathscr{D}(\mathbb{A}) = \left\{ \begin{pmatrix} (u,u(1)) \\ (v,v(1)) \end{pmatrix} \in \mathbb{X}_1 \times \mathbb{X}_1 : u \in H^2(0,1) \right\} = \mathbb{X}_2 \times \mathbb{X}_1$$

and $\mathscr{H} = \mathbb{X}_1 \times \mathbb{X}_0$ is a Hilbert space with the inner product

$$\begin{aligned}\left\langle \begin{pmatrix} (u_1,u_1(1)) \\ (u_0,a_0) \end{pmatrix}, \begin{pmatrix} (v_1,v_1(1)) \\ (v_0,b_0) \end{pmatrix} \right\rangle_{\mathscr{H}} &= \langle (u_1,u_1(1)),(v_1,v_1(1))\rangle_{\mathbb{X}_1} + \langle (u_0,a_0),(v_0,b_0)\rangle_{\mathbb{X}_0}\\ &= \int_0^1 (u_1)_x\overline{(v_1)_x}dx + \int_0^1 u_0\overline{v_0}dx + \frac{M}{K}a_0\overline{b_0},\end{aligned}$$

for all $\begin{pmatrix} (u_1,u_1(1)) \\ (u_0,a_0) \end{pmatrix}, \begin{pmatrix} (v_1,v_1(1)) \\ (v_0,b_0) \end{pmatrix} \in \mathscr{H}$. With this, we have that if

$$\mathrm{U} = \begin{pmatrix} (u,u(1)) \\ (v,v(1)) \end{pmatrix} \in \mathscr{D}(\mathbb{A})$$

then we define $\mathbb{A}$ as

$$\mathbb{A}\mathrm{U} = \begin{pmatrix} (v, v(1)) \\ (u_{xx}, -\frac{K}{M}u_x(1)) \end{pmatrix}. \tag{2.1}$$

Thus, for

$$\mathrm{U} = \begin{pmatrix} (u, u(1)) \\ (u_t, u_t(1)) \end{pmatrix} \text{ and } \mathrm{U}(0) = \mathrm{U}_0 = \begin{pmatrix} (u_0, \eta) \\ (v_0, \mu) \end{pmatrix},$$

the equation (1.1) can be written as the abstract evolution equation in $\mathscr{H}$

$$\begin{cases} \dfrac{d}{dt}\mathrm{U} = \mathbb{A}\mathrm{U}, & t > 0 \\ \mathrm{U}(0) = \mathrm{U}_0. \end{cases} \tag{2.2}$$

With this in mind, we obtain the following results.

LEMMA 2.1. *The operator $\mathbb{A}$ is skew−symmetric in $\mathscr{H}$, that is, $\langle \mathbb{A}\mathrm{U}, \mathrm{V}\rangle_{\mathscr{H}} = -\langle \mathrm{U}, \mathbb{A}\mathrm{V}\rangle_{\mathscr{H}}$, for each $\mathrm{U}, \mathrm{V} \in \mathscr{D}(\mathbb{A})$.*

Proof: Given any $\mathrm{U} = \begin{pmatrix} (u, u(1)) \\ (v, v(1)) \end{pmatrix}$, $\mathrm{V} = \begin{pmatrix} (w, w(1)) \\ (z, z(1)) \end{pmatrix} \in \mathscr{D}(\mathbb{A})$, we obtain

$$\begin{aligned} \langle \mathbb{A}\mathrm{U}, \mathrm{V}\rangle_{\mathscr{H}} &= \langle (v, v(1)), (w, w(1))\rangle_{\mathbb{X}_1} + \langle (u_{xx}, -\frac{K}{M}u_x(1)), (z, z(1))\rangle_{\mathbb{X}_0} \\ &= \int_0^1 v_x \overline{w_x} dx + \int_0^1 u_{xx}\overline{z} dx - u_x(1)\overline{z(1)} \\ &= \int_0^1 v_x \overline{w_x} dx - \int_0^1 u_x \overline{z_x} dx = -\langle \mathrm{U}, \mathbb{A}\mathrm{V}\rangle_{\mathscr{H}}. \end{aligned}$$

□

Next, we show that the operator $\mathbb{A}$ is an infinitesimal generator.

THEOREM 2.2. *The operator $\mathbb{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $\{e^{\mathbb{A}t} : t \geq 0\}$ in $\mathscr{H}$.*

Proof: First, following the ideas in Lemma 2.1, we can see that $\mathbb{A}$ is a dissipative operator in $\mathscr{H}$ (see Pazy [11, Definition 4.1]). Also, we can obtain that $R(I - \mathbb{A}) = \mathscr{H}$. The operator $\mathbb{A}$ is densely defined, that is, $\overline{\mathscr{D}(\mathbb{A})} = \mathscr{H}$, since $\mathscr{H}$ is reflexive and using Pazy [11, Theorem 4.6]. Finally, using Lumer-Phillips theorem (for instance, see Pazy [11, Theorem 4.3]) the result follows. □

Now, we show the existence and uniqueness of the solutions for the problem (2.2) (equivalently, for problem (1.1)), in terms of C_0-semigroups, following the results given by Brézis in [1, 2] and Grobbelaar-van in [5] for problems more general using the theory of $B-$evolution.

THEOREM 2.3. *Assume that $\big((u_0, \eta), (v_0, \mu)\big) \in \mathbb{X}_2 \times \mathbb{X}_1$, then the problem (1.1) has a unique weak solution*

$$(u, \eta) \in C([0, \infty), \mathbb{X}_2) \cap C^1([0, \infty), \mathbb{X}_1) \cap C^2([0, \infty), \mathbb{X}_0). \tag{2.3}$$

Moreover, if $\big((u_0, \eta), (v_0, \mu)\big) \in \mathbb{X}_1 \times \mathbb{X}_0$, then the weak solution

$$(u, \eta) \in C([0, \infty), \mathbb{X}_1) \cap C^1([0, \infty), \mathbb{X}_0). \tag{2.4}$$

Proof: The problem (1.1) is equivalent to the problem (2.2) with $\mathrm{U}_0 = \big((u_0, \eta), (v_0, \mu)\big) \in \mathscr{D}(\mathbb{A})$. We know from Theorem 2.2 that $\mathbb{A}$ is infinitesimal generator of a C_0-semigroup contractions in $\mathscr{H}$, then we have an unique solution (see Pazy [1, Theorem 7.4])

$$\mathrm{U} \in C([0, \infty); \mathscr{D}(\mathbb{A})) \cap C^1([0, \infty); \mathscr{H}).$$

Thus,

$$\begin{aligned} &\big((u, u(1), (u_t, u_t(1))\big) \in C([0, \infty); \mathbb{X}_2 \times \mathbb{X}_1) \cap C^1([0, \infty); \mathbb{X}_1 \times \mathbb{X}_0) \\ \Rightarrow\ &(u, u(1)) \in C([0, \infty), \mathbb{X}_2) \cap C^1([0, \infty), \mathbb{X}_1) \cap C^2([0, \infty), \mathbb{X}_0). \end{aligned}$$

This proves the first part of the theorem.

On the other hand, since the operator $\mathbb{A}$ is densely defined in $\mathscr{H}$, we have that $\operatorname{int}(\mathscr{D}(\mathbb{A})) \neq \emptyset$. To show this, we argue by contradiction assuming that $\operatorname{int}(\mathscr{D}(\mathbb{A})) = \emptyset$. Thus,

$$\mathscr{H} = (\operatorname{int}(\mathscr{D}(\mathbb{A})))^c = \operatorname{int}(\mathscr{D}(\mathbb{A})^c) = (\overline{\mathscr{D}(\mathbb{A})})^c.$$

The latter equality implies that $\overline{\mathscr{D}(\mathbb{A})} = \emptyset$ and arrive at a contradiction. Therewith, if $\mathrm{U}_0 \in \mathscr{H} = \mathbb{X}_1 \times \mathbb{X}_0$, then there is an unique solution $\mathrm{U} \in C([0,\infty), \mathscr{H})$ by using of Brézis [2, Theorem 3.3]. Hence,

$$\begin{aligned}&\big((u, u(1), (u_t, u_t(1))\big) \in C([0,\infty), \mathbb{X}_1 \times \mathbb{X}_0)\\ &\qquad \Rightarrow (u, u(1)) \in C([0,\infty), \mathbb{X}_1) \cap C^1([0,\infty), \mathbb{X}_0).\end{aligned}$$

This shows the second part of the theorem. □

3. Spectral analysis and exponential stability. In this section, we study the eigenvalues of operator $\mathbb{A}$ and also show that the semigroup $T(t) = e^{\mathbb{A}t}$ is not exponentially stable. For this, we recall the following definition.

DEFINITION 3.1. *A C_0-semigroup $\{T(t) : t \geq 0\}$ on a Hilbert space H is exponentially stable if there exist positive constants M and C such that*

$$\|T(t)\|_{\mathscr{L}(H)} \leq Me^{-Ct}, \quad \forall t \geq 0. \tag{3.1}$$

The following theorem is due to Huang [7].

LEMMA 3.2. *A C_0-semigroup $\{e^{tA} : t \geq 0\}$ on a Hilbert space H is exponentially stable if and only if*

$$\sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} < 0 \tag{3.2}$$

and

$$\sup_{\operatorname{Re}\lambda \geq 0} \|(\lambda - A)^{-1}\|_{\mathscr{L}(H)} < \infty \tag{3.3}$$

hold. Here, $\sigma(A)$ stands for the spectrum A.

On the other hand, the square of the norm defined by scalar product $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ can be seen as the total physical energy (see Pellicer and Solà-Morales [13]), $E : \mathscr{H} \to \mathbb{R}$ of the system (1.1), as

$$E(t) := E(\mathrm{U}(t)) = \frac{1}{2}\int_0^1 |u_x|^2 dx + \frac{1}{2}\int_0^1 |u_t|^2 dx + \frac{M}{2K}|u_t(1)|^2. \tag{3.4}$$

PROPOSITION 3.3. *The system (1.1) is conservative, that is, $E(t) = E(0)$, for all $t \geq 0$.*

Proof: First, we recall that $\operatorname{Re} z = \frac{z+\overline{z}}{2}$. Thus, using the chain rule, we obtain

$$\begin{aligned}\frac{d}{dt}\Big(\frac{1}{2}\|u_x\|^2_{L^2(0,1)}\Big) &= \frac{\langle u_{xt}, u_x\rangle_{L^2(0,1)} + \overline{\langle u_{xt}, u_x\rangle}_{L^2(0,1)}}{2} = \operatorname{Re}\Big[\int_0^1 u_{xt}\overline{u}_x dx\Big]\\ &= \operatorname{Re}\Big[\overline{u}_x(1)u_t(1) - \int_0^1 \overline{u}_{xx}u_t dx\Big] = \operatorname{Re}\Big[-\frac{M}{K}\overline{u}_{tt}(1)u_t(1) - \int_0^1 \overline{u}_{xx}u_t dx\Big]\\ &= -\frac{d}{dt}\frac{M}{2K}|u_t(1)|^2 - \operatorname{Re}\Big[\int_0^1 \overline{u}_{xx}u_t dx\Big]\end{aligned} \tag{3.5}$$

and

$$\frac{d}{dt}\Big(\frac{1}{2}\|u_t\|^2_{L^2(0,1)}\Big) = \operatorname{Re}\Big[\int_0^1 \overline{u}_{tt}u_t dx\Big]. \tag{3.6}$$

Now, using the equalities (3.5) and (3.6) along with the fact that $\overline{u}_{tt} = \overline{u}_{xx}$, we obtain

$$\frac{dE(t)}{dt} = 0, \tag{3.7}$$

where the funcional $E(\cdot)$ is defined by (3.4). Therefore, we deduce that $E(t) = E(0)$. □

REMARK 2. *If we evaluate the energy functional with the solutions of the type* $u(x,t) = e^{\lambda t}\phi(x)$, *where* $\lambda \in \mathbb{C}$, *we have*

$$\frac{dE(e^{\lambda t}\phi(x))}{dt} = (2\mathrm{Re}\,\lambda)e^{(2\mathrm{Re}\,\lambda)t}E_0, \tag{3.8}$$

where

$$E_0 = \frac{1}{2}\int_0^1 |\phi_x|^2 dx + \frac{1}{2}\int_0^1 |\lambda\phi|^2 dx + \frac{M}{2K}|\lambda|^2|\phi(1)|^2 > 0. \tag{3.9}$$

Hence, using (3.7), we get that $\mathrm{Re}\,\lambda = 0$.

Now we are in conditions to prove the main result of this section:

THEOREM 3.4. *Let* (u, η) *be the solution of (1.1) with initial data in* $\mathbb{X}_1 \times \mathbb{X}_0$. *Then the semigroup* $\{e^{t\mathbb{A}} : t \geq 0\}$ *is not exponentially stable.*

Proof: Since the system (1.1) is conservative, $\mathrm{Re}\,\lambda = 0$, then we can see that the condition (3.2) in Lemma 3.2 is not satisfied. □

Moreover, we obtain the following result.

LEMMA 3.5.

- **(i)** *The operator* $\mathbb{A}$ *is a densely defined discrete operator in* $\mathscr{H}$, *hence the spectrum* $\sigma(\mathbb{A})$ *consists entirely of eigenvalues. Moreover, the set of generalized eigenfunctions of* $\mathbb{A}$ *is complete in* $\mathscr{H}$.
- **(ii)** *For any* $0 \neq \lambda = i\tau \in \sigma(\mathbb{A})$, *there is only one associated (linearly independent) eigenfunction which takes the form* $\mathrm{F} = \big((\phi, \phi(1), \lambda(\phi, \phi(1))\big)$, *where*

$$\phi(x) = \sin \tau x. \tag{3.10}$$

Proof: The item $(\mathbf{i})$ follows directly of Pellicer and Solà-Morales [13, Lemma 3.1] when $\varepsilon = \frac{K}{M}$ and $r = 0$. Finally, let us show the item $(\mathbf{ii})$. We can see that $\lambda = i\tau \in \sigma(\mathbb{A})$ if and only if $(\lambda - \mathbb{A})U = 0$ for all $\mathrm{U} = \begin{pmatrix} (\phi, \phi(1)) \\ (\psi, \psi(1)) \end{pmatrix} \in \mathscr{D}(\mathbb{A})$ if and only if there is $\phi(x) \neq 0$ and $\psi(x) = \lambda\phi(x)$ satisfying

$$\begin{cases} \phi'' + \tau^2\phi = 0, \ 0 < x < 1, \\ \phi(0) = 0, \\ \tau^2\phi(1) = \frac{K}{M}\phi'(1). \end{cases} \tag{3.11}$$

The solution of problem (3.11) is given by $\phi(x) = \sin \tau x$, and the characteristic equation is

$$\tau \sin \tau = \frac{K}{M}\cos \tau. \tag{3.12}$$

Also, we can see that the associated eigenfunction is $\mathrm{F} = \begin{pmatrix} (\phi, \phi(1)) \\ \lambda(\phi, \phi(1)) \end{pmatrix}$. □

DEFINITION 3.6. *A basis* $\{x_n\}$ *for a Banach space* X *is unconditional if, for every* x *in* X, *the expansion* $\sum_n \alpha_n x_n$ *for* x *in terms of the basis is unconditionally convergent. A basis for a Banach space is conditional if it is not unconditional.*

In the following result we get an asymptotic expansion for the eigenvalues.

LEMMA 3.7.

- (i) *There exists a family of eigenvalues* $\{\lambda_n, \bar{\lambda}_n\}$, $\lambda_n = i\tau_n$ *of* $\mathbb{A}$ *with the following asymptotic expression:*

$$\tau_n = n\pi + O(n^{-1}). \tag{3.13}$$

(**ii**) *Let k_0 be an arbitrary fixed nonnegative integer. Then, the system* $\{\sqrt{2}\sin n\pi x + O(n^{-1})\}$ $(n = 1, 2 \ldots ; n \neq k_0)$ *is an unconditional basis for* $L^2(0, 1)$.

Proof: First, note that the problem (3.11) with spectral parameter τ^2 coincides with the spectral problem (0.1)-(0.3) given in Kerimov and Mirzoev [9] for the case that $q(x) \equiv 0$, $b_0 = 1 = a_1$, $d_0 = 0 = c_1$, $b_1 = 0$, $d_1 = \frac{K}{M}$ and $\lambda = \tau^2$. In Binding et al. [3, Corollaries 3.6 and 5.4] was proven that $\tau_n^2 = (n\pi)^2 + O(1)$ for n sufficiently large, since $c_1 = 0 = d_0$. Hence,

$$\tau_n = \sqrt{(n\pi)^2[1 + O(n^{-2})]} = n\pi\sqrt{1 + O(n^{-2})} = n\pi + O(n^{-1}),$$

where we have used the Taylor series expansion and item (**i**) is shown.

Finally, following Kerimov and Mirzoev [9, Theorem 2.1] with $c_1 = 0 = d_0$, we have that the system $\{\sqrt{2}\sin n\pi x + O(n^{-1})\}_{n\in\mathbb{N}, n\neq k_0}$ is an unconditional basis for $L^2(0, 1)$ where $k_0 \in \mathbb{N}$ is arbitrary fixed. Therefore, item (**ii**) have been shown. □

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