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Hypersurfaces with planar lines of curvature in Euclidean Space

Hipersuperficies con líneas de curvatura planas en espacios euclideanos

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Resumen

En este trabajo, presentamos parametrizaciones explícitas de hipersuperficies parametrizadas por líneas de curvatura con aplicación de Gauss prescrita y caracterizamos las hipersuperficies con líneas de curvatura planas. Como aplicación obtenemos una clasificación de superficies isotérmicas con respecto a la tercera forma fundamental con dos líneas de curvatura planas. También, presentamos una clase de superficies con una familia de líneas de curvatura plana y generalizamos estos resultados para presentar clases de hipersuperficies con familias de líneas de curvatura planas.

Palabras clave. Congruencia de esferas, Invariantes de Laplace, líneas de curvatura, superficies isotérmicas.

Abstract

In this work, we present explicit parameterizations of hypersurfaces parameterized by lines of curvature with prescribed Gauss map and we characterize the hypersurfaces with planar curvature lines. As an application we obtain a classification of isothermic surfaces with respect to the third fundamental form with two planar curvature lines. Also, we present a class of surfaces with one family of planar curvature lines and generalize these results to present classes of hypersurfaces with families of planar curvature lines.

Keywords. Sphere congruence, Laplace invariants, lines of curvature, isothermic surfaces.

1. Introduction. A hypersurface M is said to be *Dupin* if each principal curvature is constant along its corresponding surface of curvature. The Dupin hypersurface M is said to be *proper* if the number g of distinct principal curvatures is constant on M. The class of Dupin hypersurfaces is invariant under Lie transformations [11]. Therefore, the classification of Dupin hypersurfaces is considered up to these transformations. Dupin hypersurfaces is a current research topic, we mention some for example [7], [9], [11]-[16], when the principal curvatures are all distinct, check that the curvature lines of a Dupin hypersurface are circles or straight lines, so the lines of curvature are planar curves. A generalization of Dupin hypersurfaces is to consider that all lines of curvature are planar curves, but it is weakly to study hypersurfaces with a certain amount of planar curvature lines.

Classify surfaces with planar curvature lines with a property additional geometry is a traditional and current research topic, for example according to Nitsche [10, § 175], minimal surfaces with planar lines of curvature in the Euclidian space was discovered by Bonnet and Enneper. In [4], the authors study surfaces with planar curvature lines and as a result, they stablish some characterization theorems for such surfaces. Moreover, they give a condition for a surface to be a surface of revolution. In [8], the authors study surfaces with planar lines of curvature in the framework of Laguerre geometry and provide explicit representation

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formulae for these surfaces in terms of a potencial function. As an application, explicitly integrate all L-minimal surfaces with planar curvature lines. In [5] Leite, determine the orthogonal systems of cycles (curves of constant geodesic curvature) on the hyperbolic plane, aiming at the classification of maximal surfaces with planar lines of curvature in the Minkowski space. Masal'tsev in [6] given a construction of surfaces in a three-dimensional Lobachevskii space $\mathbb{H}^3$ whose properties are similar to those of Joachimsthal surfaces in a three-dimensional Euclidean space; i.e., he gives a description of surfaces in $\mathbb{H}^3$ having one family of lines of curvature located in totally geodesic planes containing a common geodesic of $\mathbb{H}^3$. The author also proves that in $\mathbb{H}^3$, surfaces of constant mean curvature without umbilical points having one family of lines of curvature located in totally geodesic planes are surfaces of revolution (hyperbolic Delanoë surfaces). In [14], the authors given a characterization of hypersurfaces whose lines of curvature are planar. In [13], the authors study special classes of hypersurfaces parameterized by lines of curvature with some conditions on the Laplace invariants.

In this work, using [1] we present the parameterizations of hypersurfaces parameterized by lines of curvature with prescribed Gauss map. We obtain a new characterization of the hypersurfaces with lines of planar curvature which is equivalent to the result obtained in [14], as an application we obtain a classification of isothermic surfaces with respect to the third fundamental form with two planar curvature lines, this classification depends on certain holomorphic functions and two real functions of one variable. We present a class of surfaces with one family of planar curvature lines and we generalize these results to present classes of hypersurfaces with families of planar curvature lines. The results presented here are based on the theory of higher-dimensional Laplace invariants introduced by Kamran-Tenenblat [3].

2. Preliminaries. Let Ω be an open subset of $I\!R^n$ and $u = (u_1, u_2, \cdots, u_n) \in \Omega$. Let $X : \Omega \subset I\!R^n \rightarrow I\!R^{n+1}$, $n \geq 3$, be a hypersurface parameterized by lines of curvature, with distinct principal curvatures λ_i, $1 \leq i \leq n$ and let $N : \Omega \subset I\!R^n \rightarrow I\!R^{n+1}$ be a unit normal vector field of X. Then

$$\langle X_{,i}, X_{,j}\rangle = \delta_{ij} g_{ii}\ , \ 1 \leq i,j \leq n\ , \tag{2.1}$$

$$N_{,i} = -\lambda_i X_{,i}\ , \tag{2.2}$$

where the subscript $_{,i}$ denotes the derivative with respect to u_i. Moreover,

$$X_{,ij} - \Gamma^i_{ij} X_{,i} - \Gamma^j_{ij} X_{,j} = 0\ , \ \ 1 \leq i \neq j \leq n\ , \tag{2.3}$$

where Γ^k_{ij} are the Christoffel symbols.
The Christoffel symbols in terms of the metric (2.1) are given by

$$\Gamma^k_{ij} = 0, \qquad \Gamma^i_{ii} = \frac{g_{ii,i}}{2g_{ii}}, \qquad \Gamma^j_{ii} = -\frac{g_{ii,j}}{2g_{jj}}, \qquad \Gamma^i_{ij} = \frac{g_{ii,j}}{2g_{ii}}, \tag{2.4}$$

where i, j, k are distinct.

We now consider the higher-dimensional Laplace invariants of the system of equations (2.3) (see [3]) for the definition of these invariants)

$$\begin{aligned} m_{ij} &= -\Gamma^i_{ij,i} + \Gamma^i_{ij}\Gamma^j_{ij}, \\ m_{ijk} &= \Gamma^i_{ij} - \Gamma^k_{kj}\ \ , \ \ k \neq i,j\ , \ \ 1 \leq k \leq n. \end{aligned} \tag{2.5}$$

In what follows we will consider these invariants as being associated with the hypersurface X parameterized by lines of curvatures.

The following Proposition characterizes the hypersurfaces parameterized by planar lines of curvature. This result was obtained in [14].

PROPOSITION 2.1. *Let $X : \Omega \subset I\!R^n \rightarrow I\!R^{n+1}$, be a hypersurface parameterized by lines of curvature, with distinct principal curvatures λ_i, $1 \leq i \leq n$.*
The lines of curvature $\alpha_i(u_i) = X(u_1^0, u_2^0, \cdots, u_{i-1}^0, u_i, u_{i+1}^0, \cdots, u_n^0)$, $1 \leq i \leq n$ are planar if and only if

$$\lambda_{i,i}\Gamma^i_{ij} + \lambda_i m_{ij} = 0,\ 1 \leq i \neq j \leq n. \tag{2.6}$$

In this paper the inner product $\langle , \rangle : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$ is defined by

$$\langle f, g\rangle = f_1 g_1 + f_2 g_2,\ \text{where}\ \ f = f_1 + i f_2,\ g = g_1 + i g_2,$$

are holomorphic functions.
In the computation we use the following properties: if $f, g : \mathbb{C} \to \mathbb{C}$ are holomorphic funtions of $z = u_1 + iu_2$, then

$$\langle f, g\rangle_{,1} = \langle f', g\rangle + \langle f, g'\rangle, \ \langle f, g\rangle_{,2} = \langle if', g\rangle + \langle f, ig'\rangle, \ \langle fg, h\rangle = \langle g, \ \overline{f}h\rangle. \tag{2.7}$$

In the next Theorem we present a way to obtain surfaces as envelope of a sphere congruence related to systems of hydrodynamic type, this parametrization generalizes the result obtained by Ferapontov [2]. This result was obtained in [1].

THEOREM 2.2. *Let M be an orientable hypersurface in $\mathbb{R}^{n+1}$ with Gauss-Kronecker curvature K non-zero everywhere and Gauss map $N \neq -e_{n+1}$. There exists a local orthogonal parametrization $Y : U \to \Pi$, where U is connected open subset of $\mathbb{R}^{n+1}$, $\Pi = \{(u_1, u_2, ..., u_{n+1}) \in \mathbb{R}^{n+1} : u_{n+1} = 0\}$ and differentiable function $h : U \to \mathbb{R}$, such that M is locally parametrized by*

$$X(u) = \left(Q - \frac{2R}{T}Y, -\frac{2R}{T}\right) \tag{2.8}$$

where $u = (u_1, ..., u_n) \in U \subset \mathbb{R}^n$,

$$T = 1 + |Y|^2, \ Q = \sum_{i=1}^{n} \frac{h_{,i}}{L_i} Y_{,i}, \ L_i = \langle Y_{,i}, Y_{,i}\rangle, \ R = \langle Q, Y\rangle - h. \tag{2.9}$$

The Gauss map is given by

$$N(u) = \frac{1}{1 + |Y|^2}(2Y, 1 - |Y|^2). \tag{2.10}$$

The Weingartem matrix W is given by

$$W = 2(TV - 2RI)^{-1}, \tag{2.11}$$

where I is identity matrix and the matrix $V = (V_{ij})$ is defined by

$$V_{ij} = \frac{1}{L_j}\left(h_{,ij} - \sum_{l=1}^{n} \widetilde{\Gamma}_{ij}^{l} h_{,l}\right) \tag{2.12}$$

$$\widetilde{\Gamma}_{ii}^{i} = \frac{L_{i,i}}{2L_i}, \qquad \widetilde{\Gamma}_{ij}^{i} = \frac{L_{i,j}}{2L_i} = -\frac{L_j}{L_i}\widetilde{\Gamma}_{ii}^{j}, \quad i \neq j. \tag{2.13}$$

The regularity condition of X is given by

$$P = det(TV - 2RI) \neq 0. \tag{2.14}$$

Moreover, the coefficients of the first, second and third fundamental form of X are given by

$$a_{ii} = \frac{L_i}{T^2}A_i^2 + \sum_{k\neq i}^{n}(TV_{ik})^2 L_k, \ a_{ij} = -\frac{L_j}{T}V_{ij}[A_i + A_j] + \sum_{k\neq i,j}^{n} V_{ik}V_{jk}L_k, \ 1 \leq i \neq j \leq n, \tag{2.15}$$

$$b_{ii} = \frac{2L_i}{T^2}A_i, \ b_{ij} = -\frac{2}{T}L_j V_{ij}, \ 1 \leq i \neq j \leq n, \ A_i = 2R - TV_{ii}, \tag{2.16}$$

$$\langle N_{,i}, N_{,i}\rangle = \frac{4}{T^2}L_i, \ \langle N_{,i}, N_{,j}\rangle = 0, \ 1 \leq i \neq j \leq n. \tag{2.17}$$

Conversely, let $Y : U \to \Pi$ be an orthogonal parametrization of Π, where U is a connected open subset of $\mathbb{R}^n$ and a differentiable function $h : U \to \mathbb{R}$. Then (2.8) define an immersion in $\mathbb{R}^{n+1}$ with Gauss-Kronecker curvature non-zero, Gauss map given by (2.10) and (2.11)-(2.17) are satisfied.

3. Hipersurfaces with planar curvature lines. In this section we characterizes hypersurfaces parameterized by lines of curvature with prescribed Gauss map.

THEOREM 3.1. *Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface as in Theorem 2.2. Then M is parameterized by lines of curvature if and only if*

$$h_{,ij} - \sum_{l=1}^{n} \widetilde{\Gamma}_{ij}^{l} h_{,l} = 0,\ 1 \le i \ne j \le n. \tag{3.1}$$

The coefficients of the first and the second fundamental form of X are given by

$$a_{ii} = \frac{L_i}{T^2} A_i^2,\ \ a_{ij} = 0,\ \ 1 \le i \ne j \le n,$$

$$b_{ii} = \frac{2L_i}{T^2} A_i,\ \ b_{ij} = 0,\ \ 1 \le i \ne j \le n,\ \ A_i = 2R - TV_{ii}.$$

Besides, the principal curvatures are given by

$$\lambda_i = \frac{2}{2R - TV_{ii}},\ 1 \le i \le n. \tag{3.2}$$

Proof: The result is a direct consequence of expressions (2.11), (2.12), (2.15) and (2.16). □

LEMMA 3.2. *Let $X : \Omega \subset I\!R^n \to I\!R^{n+1}$, be a hypersurface parametrized by lines of curvature, with Gauss map N, Gauss-Kronecker curvature non-zero everywhere and with distinct principal curvatures $\lambda_i,\ 1 \le i \le n$. Then the Laplace invariants $\widehat{m}_{ij}$ of N satisfy*

$$\widehat{m}_{ij} = \frac{\lambda_j}{\lambda_i^2}\left[\lambda_{i,i}\Gamma_{ij}^i + \lambda_i m_{ij}\right],\ 1 \le i \ne j \le n. \tag{3.3}$$

Proof: From (2.2) and (2.3) we obtain

$$N_{,ij} - \widehat{\Gamma}_{ij}^i N_{,i} - \widehat{\Gamma}_{ij}^j N_{,j} = 0, \tag{3.4}$$

where $\widehat{\Gamma}_{ij}^i = \dfrac{\lambda_j}{\lambda_i}\Gamma_{ij}^i,\ 1 \le i \ne j \le n$. The Laplace invariants for the equation (3.4) are given by $\widehat{m}_{ij} = -\widehat{\Gamma}_{ij,i}^i + \widehat{\Gamma}_{ij}^i \widehat{\Gamma}_{ij}^j$, by differentiation we get

$$\widehat{m}_{ij} = \frac{\lambda_j}{\lambda_i}\Gamma_{ij}^i\Gamma_{ij}^j + \frac{\lambda_j}{\lambda_i^2}\Gamma_{ij}^i\lambda_{i,i} - \frac{\lambda_j}{\lambda_i}\Gamma_{ij,i}^i,$$

hence, the result it follows from (2.5). □

THEOREM 3.3. *Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface as in Theorem 3.1. Then The lines of curvature of X corresponding to u_i are planar curves if and only if*

$$\widetilde{m}_{ij} = -\widetilde{\Gamma}_{ij,i}^i + \widetilde{\Gamma}_{ij}^i \widetilde{\Gamma}_{ji}^j = 0$$

where $\widetilde{\Gamma}_{ij}^i$ are the Christoffel symbols of Y.

Proof: From Proposition 2.1, the lines of curvature of X corresponding to u_i are planar curves if and only if $\lambda_{i,i}\Gamma_{ij}^i + \lambda_i m_{ij} = 0,\ \ 1 \le i \ne j \le n$. On the other hand, it follows from (2.9) that

$$\widehat{\Gamma}_{ij}^i = \widetilde{\Gamma}_{ij}^i - \frac{T_{,j}}{T},\ \ 1 \le i \ne j \le n,$$

by using this relation in the expression of $\widehat{m}_{ij}$, after simplifications one has that $\widehat{m}_{ij} = \widetilde{m}_{ij}$, hence, by Lemma 3.2, it follows the result. □

4. Isothermic surfaces with planar curvature lines. In this section classify a class of isothermic surfaces parametrized by planar curvature lines.

LEMMA 4.1. *Let $Y = (g, 0)$ be an orthogonal parametrization of $\Pi = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : u_3 = 0\}$ where $g : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Then* $\widetilde{m}_{12} = 0$ *if and only if g is given by*

$$g(z) = \frac{z_1 z + z_2}{z_3 z + z_4} \quad or \quad g(z) = \frac{z_1 e^{\sqrt{-2c}z} + z_2}{z_3 e^{\sqrt{-2c}z} + z_4}, \; z_1 z_4 - z_2 z_3 \neq 0. \tag{4.1}$$

where $z_i \in \mathbb{C}$. Moreover, in this case $\widetilde{m}_{12} = 0$ if and only if $\widetilde{m}_{21} = 0$.
Proof: Using (2.5) and (2.13) one has

$$\widetilde{m}_{12} = 0 \text{ if and only if } -|g'|_{,21}|g'| + 2|g'|_{,1}|g'|_{,2} = 0. \tag{4.2}$$

This equation is equivalent to

$$\left\langle 1, i\left[\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2\right]\right\rangle = 0. \tag{4.3}$$

Hence, it follows

$$\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 = c \Longleftrightarrow \left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2 = c, \; c = \text{constant}. \tag{4.4}$$

Considering $c = 0$ and $c \neq 0$ we get that the solutions of (4.4) are given by (4.1). □
As a consequence of Theorem 2.2 we obtain the following result.

THEOREM 4.2. *Let S be an orientable isothermic surface parameterized by planar curvature lines in $\mathbb{R}^3$ with Gaussian curvature K non-zero everywhere, Gauss map $N \neq (0, 0, -1)$ and $\Pi = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : u_3 = 0\}$. Then there exist real functions $f_i(u_i), f_j(u_j)$, a holomorphic function $g : U \to \Pi$ given in (4.1) where U is a connected open subset of $\mathbb{R}^2$ such that S is locally parameterized by*

$$X(u) = (Q, 0) - \frac{2R}{T}(g, 1) \tag{4.5}$$

where $u = (u_1, u_2) \in U \subset \mathbb{R}^2, |g'| \neq 0,$

$$T = 1 + |g|^2, \;\; Q = \frac{g'}{|g'|^2}\nabla h, \;\; R = \left\langle \nabla h \, , \frac{g}{g'}\right\rangle - h, \tag{4.6}$$

where

$$h = |g'| \left[f_i(u_i) + f_j(u_j)\right]. \tag{4.7}$$

The Gauss map is given by

$$N(u) = \frac{1}{1 + |g|^2}\left(2g, 1 - |g|^2\right). \tag{4.8}$$

Moreover, the coefficients of the first and the second fundamental form of X are given by

$$a_{11} = \frac{|g'|^2}{T^2}A_1^2, \;\; a_{12} = 0, \;\; a_{22} = \frac{|g'|^2}{T^2}A_2^2 \tag{4.9}$$

$$b_{11} = \frac{2|g'|^2}{T^2}A_1, \;\; b_{12} = 0, \;\; b_{22} = \frac{2|g'|^2}{T^2}A_2, \;\; A_i = 2R - TV_{ii}, \; i = 1, 2, \tag{4.10}$$

where

$$\begin{aligned} V_{11} &= \frac{1}{|g'|^2}\left[h_{,11} - \langle \frac{g''}{g'}, \nabla h\rangle\right], \\ V_{22} &= \frac{1}{|g'|^2}\left[h_{,22} + \langle \frac{g''}{g'}, \nabla h\rangle\right]. \end{aligned} \tag{4.11}$$

Moreover, the Weingarten matrix W of X is given by (2.11) and the matrix $V = (V_{ij})$ is defined by (4.11). The regularity condition of X is $P \neq 0$ where P is given by (2.14). The third fundamental form is determined by

$$\langle N_{,i}, N_{,i}\rangle = \frac{4}{T^2}|g'|^2, \; i = 1, 2, \;\; \langle N_{,1}, N_{,2}\rangle = 0. \tag{4.12}$$

Conversely, let be a holomorphic function $g : U \to \Pi$ given by (4.1), where U is a connected open subset of $\mathbb{R}^2$ and a differentiable function $h : U \to \mathbb{R}$. Then (4.5) define an isothermic surface parameterized by planar curvature lines in $\mathbb{R}^3$ with Gaussian curvature non-zero, Gauss map given by (4.8) and (4.6)-(4.12) are satisfied.

Proof: From Theorem 2.2, for $n = 2$, since the Gaussian curvature K is non-zero the Gauss map define a regular immersion in the sphere, thus, the third fundamental form is positive defined and we can take conformal parameters, i.e. $Y = (g, 0)$, where g is a holomorphic function, in this case we get $L_i = |g'|^2$, utilizing this expression in (2.8), (2.9) and (2.10) it follows (4.5), (4.6) and (4.8). Also, $\widetilde{\Gamma}^i_{ii} = \frac{|g'|_{,i}}{|g'|}$, $\widetilde{\Gamma}^i_{ij} = \frac{|g'|_{,j}}{|g'|} = -\widetilde{\Gamma}^j_{ii}$, $1 \leq i \neq j \leq 2$, by using these expressions in (2.15), (2.16), (2.12) and (2.17) we get (4.9)-(4.12). We observe that the coefficients of the first fundamental form satisfy $a_{11}a_{22} - a_{12}^2 = |g'|^4 P^2$, thus, the regularity conditions of X is $P \neq 0$. On the other hand, since X is parametrized by planar curvature lines, it follows from Theorem 3.1 and Theorem 3.3 for $n = 2$, that

$$h_{,ij} - \widetilde{\Gamma}^i_{ij} h_{,i} - \widetilde{\Gamma}^j_{ij} h_{,j} = 0, \; \widetilde{m}_{,ij} = 0, \; 1 \leq i \neq j \leq 2.$$

The solutions of this equation is given by

$$h = e^{\int \widetilde{\Gamma}^i_{ij} du_j} \left[\int e^{-\int \widetilde{\Gamma}^i_{ij} du_j + \int \widetilde{\Gamma}^j_{ji} du_i} \tilde{f}_j(u_j) du_j + f_i(u_i) \right], \; 1 \leq i \neq j \leq 2, \tag{4.13}$$

and by using the expressions of $\widetilde{\Gamma}^i_{ij}$ the function h can be rewritten as (4.7). Since $\widetilde{m}_{ij} = 0$ one has by Lemma 4.1 that g is given by (4.1).

The converse is straightforward calculation and it follows from fundamental theorem of the surfaces in $\mathbb{R}^3$. □

Some graphs of isothermic surfaces with planar curvature lines.

1. For the function $g(z) = \dfrac{z_1 z + z_2}{z_3 z + z_4}$

i) $z_1 = -1 - \frac{1}{2}i, \; z_2 = 1 + i, \; z_3 = 0, \; z_4 = 1, \; f_1(u_1) = u_1^2 + u_1 + 1, \; f_2(u_2) = u_2^2 + u_2 - 1$

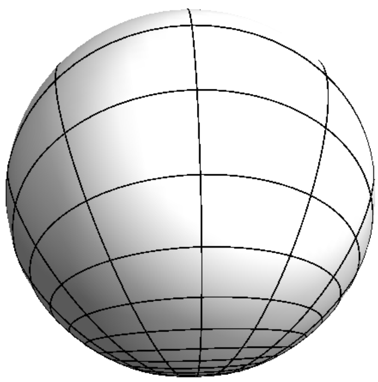

FIGURE 4.1.

ii) $z_1 = 1 - \frac{1}{2}i, \; z_2 = \frac{2}{3} + i, \; z_3 = 1 - i, \; z_4 = 1 - 2i, \; f_1(u_1) = u_1 - 3, \; f_2(u_2) = u_2^2 + 3u_2$

2. For the function $g(z) = \dfrac{z_1 e^{\sqrt{-2c}z} + z_2}{z_3 e^{\sqrt{-2c}z} + z_4}$

iii) $z_1 = 1, \; z_2 = z_3 = 0, \; z_4 = 1, c = -\frac{1}{8}, \; f_1(u_1) = e^{u_1}, \; f_2(u_2) = \cos\left(\frac{u_2}{2}\right)$

iv) $z_1 = 1, \; z_2 = z_3 = 0, \; z_4 = 1, \; c = -\frac{1}{3}, \; f_1(u_1) = e^{u_1}, \; f_2(u_2) = \cos\left(\frac{u_2}{2}\right)$

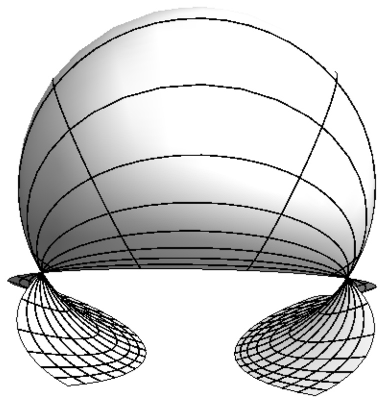

FIGURE 4.2.

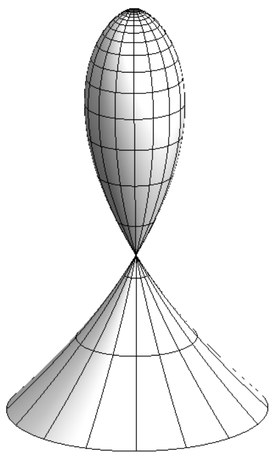

FIGURE 4.3.

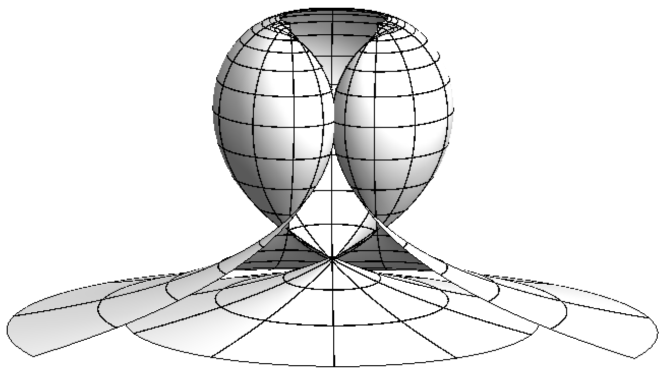

FIGURE 4.4.

v) $z_1 = 1 + i,\ z_2 = z_3 = -1 + i,\ z_4 = 1 + i, c = -\frac{1}{2},\ f_1(u_1) = \sin u_1,\ f_2(u_2) = \cos u_2$
vi) $z_1 = -\pi,\ z_2 = z_3 = 0,\ z_4 = \frac{1}{3},\ c = -1,\ f_1(u_1) = \sin u_1,\ f_2(u_2) = 1$

THEOREM 4.3. *Under the same conditions of Theorem 2.2, considering* $n = 2k$, $u = (z_1, \cdots, z_r, \cdots, z_n)$ *where* $z_r = (u_{2r-1}, u_{2r}),\ r = 1, \cdots, n$,

$$Y(u) = (g_1(z_1), \cdots, g_r(z_r), \cdots, g_n(z_n)),\ \ h(u) = \sum_{r=1}^{k} h^r(z_r)$$

with $h^r(z_r) = |g_r'(z_r)|[f_{2r-1}(u_{2r-1}) + f_{2r}(u_{2r})]$ *and* $g_r(z_r)$ *is a function given (4.1). Then the hypersurface* X *defined by (2.8) is parameterized by planar curvature lines.*

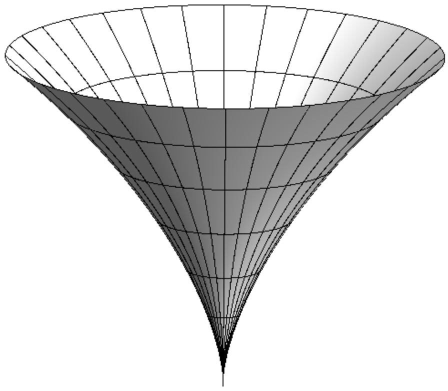

FIGURE 4.5.

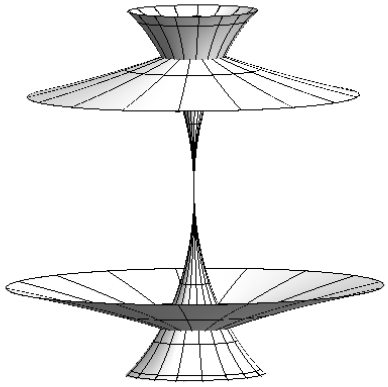

FIGURE 4.6.

Proof: Differentiating Y with respect to u_{2r-1} and u_{2r} we have

$$Y_{,2r-1} = (0, \cdots, g_r', \cdots, 0), \;\; Y_{,2r} = (0, \cdots, ig_r', \cdots, 0), \; r = 1, \cdots, n,$$

thus, Y is an orthogonal parametrization and $L_{2r-1} = L_{2r} = |g_r'|^2$.
Consequently,

$$\widetilde{\Gamma}^{2r-1}_{2r-1\,2r} = \frac{\langle g_r', ig_r''\rangle}{|g_r'|^2}, \;\; \widetilde{\Gamma}^{2r}_{2r-1\,2r} = \frac{\langle g_r', g_r''\rangle}{|g_r'|^2}$$

Now, we will show that X given by (2.8) is parameterized by lines of curvature, by differentiation we get

$$h_{,2r-1} = h^r_{,2r-1} = \frac{\langle g_r', g_r''\rangle}{|g_r'|}[f_{2r-1} + f_{2r}] + |g_r'|f_{2r-1}',$$

$$h_{,2r} = h^r_{,2r} = \frac{\langle g_r', ig_r''\rangle}{|g_r'|}[f_{2r-1} + f_{2r}] + |g_r'|f_{2r}',$$

$$h_{,2r-1\,2r} = |g_r'|\left[(f_{2r-1} + f_{2r})\left\langle 1, i\left(\frac{g_r'''}{g_r'} - \frac{1}{2}\left(\frac{g_r''}{g_r'}\right)^2\right)\right\rangle + \left\langle 1, i\frac{g_r''}{g_r'}\right\rangle f_{2r-1}' + \left\langle 1, \frac{g_r''}{g_r'}\right\rangle f_{2r}'\right].$$

By using these expressions joint with (2.7) and (4.4) we obtain

$$h_{,2r-1\,2r} - \widetilde{\Gamma}^{2r-1}_{2r-1\,2r}h_{,2r-1} - \widetilde{\Gamma}^{2r}_{2r-1\,2r}h_{,2r} = 0, \;\; r = 1, \cdots, n.$$

Therefore, $V_{2r-1\;2r} = 0, \; r = 1, \cdots, n$ and from Theorem 3.1 X is parameterized by lines of curvature. It remains to show that these lines of curvature are planar, i.e. we will show that $\widetilde{m}_{2r-1\;2r} = 0$ and by Theorem 3.3 it follows the result. In fact, by definition of Laplace invariants we have

$$\widetilde{m}_{2r-1\,2r} = -\left(\widetilde{\Gamma}^{2r-1}_{2r-1\,2r}\right)_{,2r-1} + \widetilde{\Gamma}^{2r-1}_{2r-1\,2r}\widetilde{\Gamma}^{2r}_{2r-1\,2r},$$

since, $\left(\widetilde{\Gamma}^{2r-1}_{2r-1\,2r}\right)_{,2r-1} = \left\langle 1, i\left(\frac{g_r'''}{g_r'} - \left(\frac{g_r''}{g_r'}\right)^2\right)\right\rangle$, it follows

$$\begin{aligned}\widetilde{m}_{2r-1\,2r} &= -\left\langle 1, i\left(\frac{g_r'''}{g_r'} - \left(\frac{g_r''}{g_r'}\right)^2\right)\right\rangle + \frac{\langle g_r', g_r''\rangle\langle g_r', ig_r''\rangle}{|g_r'|^4},\\ &= -\left\langle 1, i\left(\frac{g_r'''}{g_r'} - \frac{3}{2}\left(\frac{g_r''}{g_r'}\right)^2\right)\right\rangle,\\ &= -\left\langle 1, i\left(\left(\frac{g_r''}{g_r'}\right)' - \frac{1}{2}\left(\frac{g_r''}{g_r'}\right)^2\right)\right\rangle.\end{aligned}$$

Therefore, $\widetilde{m}_{2r-1\,2r} = 0$, because g_r is a function that satisfy (4.4) and the proof is complete. □

PROPOSITION 4.4. *Under the same conditions of Theorem 2.2, for $n = 2$, considering $Y = \alpha(u_1) + u_2 n(u_1)$, where α is a planar regular curve, $n(u_1)$ is the unit normal vector of α. Then X defined by (2.8) is parameterized by lines of curvature and with one family of planar curvature lines if and only if*

$$h(u_1, u_2) = \int f_1(u_1)(1 - u_2 k_\alpha)|\alpha'|du_1 + f_2(u_2), \tag{4.14}$$

where k_α is the curvature of α.
Proof: Differentiating Y with respect to u_1 and u_2

$$Y_{,1} = (1 - k_\alpha u_2)\alpha',\ Y_{,2} = n(u_1),$$

We observe that Y is an orthogonal parametrization.
Hence, $L_1 = (1 - k_\alpha u_2)^2|\alpha'|^2$ and $L_2 = 1$, consequently, $\widetilde{\Gamma}^1_{12} = -\dfrac{k_\alpha}{1 - k_\alpha u_2}$, $\widetilde{\Gamma}^2_{12} = 0$.
From Theorem 3.1 for $n = 2$, X is parameterized by lines of curvature if and only if the function h satisfy $h_{,12} - \widetilde{\Gamma}^1_{12}h_{,1} - \widetilde{\Gamma}^2_{12}h_{,2} = 0$, whose solution is given by (4.13), by using the Christoffel symbols obtained above we obtain the expression (4.14).
On the other hand, $\widetilde{m}_{21} = 0$ and from Theorem 3.3 for $n = 2$, the line of curvature of X corresponding to u_2 is planar. Note that $\widetilde{m}_{12} = \dfrac{k_\alpha'}{(1 - k_\alpha u_2)^2}$, hence, $\widetilde{m}_{12} = 0$ if and only if α has constant curvature. □

THEOREM 4.5. *Under the same conditions of Theorem 2.2, for $n = 2k$, $u = (z_1, \cdots, z_r, \cdots, z_n)$ where $z_r = (u_{2r-1}, u_{2r})$, $r = 1, \cdots, n$. Considering $Y(u) = (Y_1(z_1), \cdots, Y_r(z_r), \cdots, Y_n(z_n))$, $h(u) = \sum_{r=1}^{n} h_r(z_r)$ with $Y_r(z_r) = \alpha_r(u_{2r-1}) + u_{2r}n_r(u_{2r-1})$, α_r is a planar regular curve, $n_r(u_{2r-1})$ is the unit normal vector of α_r and $h(z_r) = \int f_{2r-1}(u_{2r-1})(1 - u_{2r}k_{\alpha_r})|\alpha_r'|du_{2r-1} + f_{2r}(u_{2r})$, k_{α_r} is the curvature of α_r. Then X defined by (2.8) is parameterized by lines of curvature with k planar curvature lines.*
Proof: The proof it follows from Proposition 4.4. □

5. Conclusions. From the results obtained in this work we can make the following conclusions: Study the planar curvature lines of a hypersurface $X \subset \mathbb{R}^{n+1}$ satisfying the conditions of Theorem 3.1 corresponds to study the Laplace invariants $\widehat{m}_{ij}$ of the Gauss map, which in this case must be identically zero. All isothermic surface with respect to the third fundamental form satisfying the conditions of Theorem 2.2, with two planar curvature lines can be written in terms of a holomorphic function and two real functions of one variable. From a planar regular curve and its unit normal vector we can obtain a class of surfaces with one family of planar curvature lines and we can generalize these results to obtain classes of hypersurfaces with families of k planar curvature lines. Finally, this work can be used to classify hypersurfaces with planar curvature lines that satisfy an additional geometric or analytical property with prescribed Gauss map, future works in this direction we will be presenting.

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