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An alternative approach to the power series method

Una forma alternativa para el método de series de Potencias

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Resumen

En este artículo consideramos un problema clásico, es decir el problema lineal de valor inicial de segundo orden no homogéneo con coeficientes analíticos. Se clasifica las posibles soluciones analíticas, dando criterios para la inexistencia de soluciones analíticas asi como para la existencia de soluciones analíticas múltiples. También se presenta una prueba alternativa para la convergencia del método de series de potencias, aplicando en puntos singulares irregulares .

Abstract

This article consider the classical problem of linear non-homogeneous second order Initial Value Problems with analytic coefficients. It classifies the possible kinds of analytic solutions, giving criteria for the non-existence of analytical solutions and for the existence of multiple analytic solutions. An alternative proof for the convergence of the power series method is given and it applies for some singular irregular points.

Keywords. ODE; Non-homogeneous; Initial Value Problem; Power Series; Strong Operator Convergence.

1. Introduction. The use of infinite series to solve differential equations is a classical technique for solving ordinary differential equations explicitly and is still used and taught to the present day. The method was used by Newton and Leibniz to integrate (solve) differential equations since the beginning of the XVIII century ([6], pg. 488-489).

The method gained importance with Euler from about 1750 on, whom sought solutions of the form

$$y = x^{\lambda}(A + Bx + Cx^2 + \cdots) \tag{1.1}$$

to several differential equations, many of them arising from physics problems.

Further development in power series solutions to ordinary differential equations occurred with works of Fuchs (e.g. [5]) and later with Frobenius (e.g. [4]). Both developed the theory and many related questions followed their efforts (for more on their developments [5]).

In the last century many problems in science, engineering and technology involve nonlinear complex phenomena. In many situations such problems deal with ordinary differential equations which are singular. Agarwal and O'Regan [1] state that there were not many results about such problems until the beginning of the 20th century. The situation has changed and currently there are several existence theorems proving the existence of weaker solutions to many problems. A motivation for this article is to prove the convergence of singular *irregular* Initial Value Problems (I.V.P.) from a classical point of view.

This paper studies the power series method for linear non-homogeneous second order I.V.P.s with analytic coefficients (2.1) and presents, in Theorem 5, a proof of convergence of the power series method based in the Theorem of the Strong Operator Convergence. This proof does not give, explicitly, the maximal radius of convergence, but the method works for some ordinary points, for some singular regular points and

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for some singular irregular points in the sense of the method of Frobenius. It is important to recall that most presentations on the method of Frobenius do not deal with singular irregular points.

Section 2 describes the proposed problem, states the necessary assumptions to apply the proposed method and details the restrictions on the initial conditions imposed by the method.

This study does not put the I.V.P. (2.1) in it's standard form because, in our method, the precise relation of the coefficients is critical to the proof of convergence. The recurrence relation (2.6) gives the coefficients of the solution and indicates the necessity of considering the first non-zero coefficients of the series expansions of the functions $a(x)$, $b(x)$ and $c(x)$ as in Definition 1. These κ-singular operators give some restrictions to weather it is possible or not to use the method. Further restrictions are found when the leading term of recurrence relation (2.6) is zero. These restrictions show that there are no analytic solutions to some of the studied I.V.P., as stated in Theorems 1, 3 and 4.

We call admissible and non-degenerate the problems that are not excluded by the fore mentioned restrictions and prove the convergence for all such problems. Some problems which are singular irregular in the sense of the Frobenius method fall into this category. For such problems there is a tendency that the radius of convergence of the solution is small, as stated in Remark 7 and Example 2. Further, Theorems 3 and 4 list the conditions for existence of multiple analytic solutions to the I.V.P. (2.1).

2. The problem and method proposed. We consider the linear non-homogeneous Initial Value Problem of the type

$$\begin{aligned} L(y) =& a(x)y'' + b(x)y' + c(x)y = g(x), \\ & y(x_0) = y_0, \qquad y'(x_0) = y_1, \end{aligned} \tag{2.1}$$

where a, b, c, g are real analytic functions in a neighborhood of x_0:

$$a(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, \qquad b(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n,$$
$$c(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n, \qquad g(x) = \sum_{n=0}^{\infty} g_n (x-x_0)^n.$$

Further, this article considers $a_n, b_n, c_n, g_n = 0$ for $n < 0$, $n \in \mathbb{Z}$.

In view of the Frobenius method, it would be natural to search for solutions of the form

$$y(x) = (x-x_0)^r \sum_{n=0}^{\infty} y_n (x-x_0)^n.$$

However for this non-homogeneous Initial Value Problem r must be integer if $g(x) \neq 0$. Further $y(x)$ only has finite values for $y(x_0)$ and $y'(x_0)$ if $r \geq 0$ and $r \in \mathbb{Z}$. These facts are the motivation to take $r = 0$ and restrict the search to analytic solutions.

Equation (2.2) combine coefficients of different orders and their relation is important in the following calculations. For this reason the following definition is presented.

Definition 1. *The operator L is said to be κ-singular at x_0 if there is $\kappa \in \mathbb{N}$, such that*

$$\begin{aligned} a_n =& 0, \ \forall n = -\infty, \dots, \kappa - 1; \\ b_n =& 0, \ \forall n = -\infty, \dots, \kappa - 2; \\ c_n =& 0, \ \forall n = -\infty, \dots, \kappa - 3; \end{aligned}$$

and at least one of a_κ, $b_{\kappa-1}$ or $c_{\kappa-2}$ is not zero.

Remark 1. *A κ-singular point for L with $a_\kappa \neq 0$ is a singular regular point in the sense of the Frobenius method if $g \equiv 0$.*

A κ-singular point for L with $a_\kappa = 0$ is a singular irregular point in the sense of the Frobenius method if $g \equiv 0$.

Let us now assume that L is κ-singular at x_0 and that the problem has an analytic solution $y(x)$ around x_0. Thus eq. (2.1) can be differentiated as many times as required:

$$\begin{aligned} D_2(y) =& L(y) = ay'' + by' + cy = g \\ D_3(y) =& ay''' + [a' + b]y'' + [b' + c]y' + c'y = g', \\ D_4(y) =& ay^{(4)} + [2a' + b]y''' + [a'' + 2b' + c]y'' + [b'' + c']y' + c''y = g'', \\ & \vdots \end{aligned}$$

or in general

$$D_n(y) = \sum_{i=0}^{n-2} \binom{\text{n-2}}{\text{i}} a^{(i)} y^{(n-i)} + \binom{\text{n-2}}{\text{i}} b^{(i)} y^{(n-i-1)} + \binom{\text{n-2}}{\text{i}} c^{(i)} y^{(n-i-2)}$$
$$= \sum_{i=0}^{n} \left[\binom{\text{n-2}}{\text{i}} a^{(i)} + \binom{\text{n-2}}{\text{i-1}} b^{(i-1)} + \binom{\text{n-2}}{\text{i-2}} c^{(i-2)} \right] y^{(n-i)} = g^{(n-2)},$$

where we abuse of the notation by considering all the combinations outside Pascal's triangle as zero, and also $a^{(i)}$, $b^{(i)}$ and $c^{(i)}$ as zero, for $i < 0$. Applying this at $x = x_0$:

$$\sum_{i=0}^{n} [(n-i)(n-i-1)a_i + (n-i)b_{i-1} + c_{i-2}]\, y_{n-i} = g_{n-2}, \quad n \geq 2. \tag{2.2}$$

Remark 2. *For $\kappa > 0$, the left side of eq. (2.2) has all coefficients equal to zero for $n = 2, \ldots, \kappa - 1$. Thus they imply restrictions on function $g(x)$:*

$$g_0 = g_1 = \ldots = g_{\kappa-3} = 0.$$

Further, if $\kappa = 1$, eq. (2.2) gives, for $n = 2$:

$$c_0 y_0 + b_0 y_1 = g_0 \tag{2.3}$$

and this has three possible distinct implications:
- *If $c_0 = b_0 = 0$, then $g_0 = 0$.*
- *If $b_0 \neq 0$, then $y_1 = (g_0 - c_0 y_0)/b_0$.*
- *If $c_0 \neq 0$ and $b_0 = 0$ then $y_0 = g_0/c_0$.*

Thus it might be not possible to solve the I.V.P. for any initial condition.

Further, if $\kappa \geq 2$, for $n = \kappa$ and for $n = \kappa + 1$, eq. (2.2) gives:

$$c_{\kappa-2} y_0 = g_{\kappa-2} \tag{2.4}$$
$$c_{\kappa-1} y_0 + (b_{\kappa-1} + c_{\kappa-2}) y_1 = g_{\kappa-1} \tag{2.5}$$

Regarding y_0 and y_1 as the free variables in the system above, one gets three possible outcomes
- *If the equations are linear independent, then y_0 and y_1 are determined by the coefficients of $a(x), b(x), c(x)$ and $g(x)$, thus the I.V.P. can only be solved for these specific initial values. Further any small changes in y_0 and y_1 cause the I.V.P. not to have an analytic solution, showing an instability in the regularity of the solution.*
- *If $c_{\kappa-2} = c_{\kappa-1} = b_{\kappa-1} = 0$, then one gets two additional restrictions, $g_{\kappa-1} = g_{\kappa-2} = 0$, on the coefficients of $g(x)$.*
- *If the equations are linear dependent but not all coefficients are zero, then one of the following happens:*
  - *a) $c_{\kappa-2} = 0$, thus y_0 and y_1 are related by eq. (2.5), determining a line in $\mathbb{R}^2$ of admissible initial conditions (y_0, y_1);*
  - *b) $c_{\kappa-2} \neq 0$ and $b_{\kappa-1} = -c_{\kappa-2}$, thus the system will only admit solution if $c_{\kappa-1} g_{\kappa-2}/c_{\kappa-2} = g_{\kappa-1}$, case in which $y_0 = g_{\kappa-2}/c_{\kappa-2}$.*

Remark 2 gives several restrictions for I.V.P.s that can be solved using the eq. (2.2) recursively, such I.V.P.s will be called *admissible* hereafter:

Definition 2. *The I.V.P. (2.1) is said to be admissible at a κ-singular point x_0 if*

$$g_0 = g_1 = \ldots = g_{\kappa-3} = 0$$

and one of the following holds:

I *$\kappa = 0$;*

II *$\kappa = 1$ and:*
- *a) $c_0 = b_0 = 0$ and $g_0 = 0$.*
- *b) $b_0 \neq 0$ and $y_1 = (g_0 - c_0 y_0)/b_0$.*
- *c) $c_0 \neq 0$, $b_0 = 0$ and $y_0 = g_0/c_0$.*

III *$\kappa \geq 2$ and:*

a) $c_{\kappa-2}(b_{\kappa-1}+c_{\kappa-2})\neq 0$ *and*

$$y_0=\frac{g_{\kappa-2}}{c_{\kappa-2}},\qquad y_1=\frac{g_{\kappa-1}-c_{\kappa-1}g_{\kappa-2}/c_{\kappa-2}}{b_{\kappa-1}+c_{\kappa-2}}$$

b) $c_{\kappa-2}=c_{\kappa-1}=b_{\kappa-1}=0$ *and* $g_{\kappa-1}=g_{\kappa-2}=0$.
c) $c_{\kappa-2}=0$, $g_{\kappa-2}=0$ *and* $c_{\kappa-1}y_0+b_{\kappa-1}y_1=g_{\kappa-1}$.
d) $c_{\kappa-2}\neq 0$, $b_{\kappa-1}=-c_{\kappa-2}$, $c_{\kappa-1}g_{\kappa-2}/c_{\kappa-2}=g_{\kappa-1}$ *and* $y_0=g_{\kappa-2}/c_{\kappa-2}$.

Theorem 1. *If the I.V.P. (2.1) is not admissible at a κ-singular point x_0, then there is no analytic solution of the problem centered at x_0.*

Proof: If the I.V.P. (2.1) is not admissible at a κ-singular point x_0, then there is a contradiction to some of the restrictions in Remark 2; but these restrictions were derived from the hypothesis that there is an analytic solution to the I.V.P. (2.1) centered at x_0. □

Remark 3. *If the I.V.P. (2.1) is admissible at a κ-singular point x_0, then both sides of eq. (2.1) can be divided by $(x-x_0)^{\kappa-2}$, so that the only values of interest for κ are 0, 1 and 2.*

Further, being the I.V.P. admissible at a κ-singular point x_0, eq. (2.2) can be used recursively to calculate $y_{n-\kappa}$ in terms of $y_0,y_1,\dots,y_{n-\kappa-1}$:

$$\begin{aligned}&((n-\kappa)(n-1-\kappa)a_\kappa+(n-\kappa)b_{\kappa-1}+c_{\kappa-2})y_{n-\kappa}=\\ &g_{n-2}-\sum_{i=1}^{n-\kappa}\left[(n-\kappa-i)(n-1-\kappa-i)a_{i+\kappa}+(n-\kappa-i)b_{i+\kappa-1}+c_{i+\kappa-2}\right]y_{n-\kappa-i},\end{aligned}\tag{2.6}$$

as long as it does not hold:

$$(n-\kappa)(n-1-\kappa)a_\kappa+(n-\kappa)b_{\kappa-1}+c_{\kappa-2}=0.\tag{2.7}$$

If some natural $n-\kappa=d+1$ makes eq. (2.7) hold, then (2.6) is a restriction to the previously found coefficients $y_0,y_1,\dots,y_d$:

Theorem 2. *Let us suppose that the I.V.P. (2.1) is admissible and the coefficients $y_0,\dots,y_d$, $d\geq 1$, satisfy eq. (2.6) and that equation (2.7) holds for $n-\kappa=d+1$. If $y_0,\dots,y_d$ do not satisfy*

$$\sum_{i=1}^{d+1}\left[(d+1-i)(d-i)a_{i+\kappa}+(d+1-i)b_{i+\kappa-1}+c_{i+\kappa-2}\right]y_{d+1-i}=g_{d+\kappa-1}\tag{2.8}$$

then the I.V.P. (2.1) has no analytic solution.

Proof: This follows from the fact that the coefficients of an analytic solution of (2.1) must satisfy (2.6) and thus the equation (2.8), which has no unknowns. □

On the other hand, if

$$(d+1)da_\kappa+(d+1)b_{\kappa-1}+c_{\kappa-2}=0$$

and the coefficients $y_0,\dots,y_d$ satisfy eq. (2.8), then the coefficient of y_{d+1} in eq. (2.6) is zero for $n-\kappa=d+1$. So that y_{d+1} is not determined by $y_0,\dots,y_d$ and could be could be chosen to assume any value. And y_{n-k} depends on the chosen value y_{d+1} for $n-k>d+1$.

Example 1. *The I.V.P.*

$$\begin{aligned}&x^2y''-2xy'+(2+x^2)y=0\\ &y(0)=\alpha,\ \ y'(0)=\beta,\end{aligned}\tag{2.9}$$

is 2-singular at $x_0=0$ and falls into category III, d) of definition 2 and thus is admissible only if $y(0)=0$. Further eq. (2.7) holds for $n=4$ and the restriction (2.8) is satisfied by $y_0=0$ and $y_1=\beta$. In this fashion we can assign any value for $y_{n-\kappa}=y_2=\delta$.

Taking $y_1=\beta=0$ we find the following solutions using the recurrence relations for the choices $y_2=\delta=1$ (blue), $y_2=\delta=2$ (red), $y_2=\delta=3$ (green). We draw attention to the fact that this problem is solved by the analytic function $y(x)=Cx\sin(x)$, for any constant C.

Taking $y_1=\beta=1$ we find the following solutions using the recurrence relations for the choices $y_2=\delta=0$ (blue), $y_2=\delta=1$ (red), $y_2=\delta=2$ (green).

Theorem 3. *Let us suppose that the I.V.P. (2.1) is admissible and that there is $d\in\mathbb{N}$, $d\geq 1$, such that the coefficients $y_0,\dots,y_d$, satisfy eq. (2.6) for any $1\leq n-\kappa\leq d$ and that $n-\kappa=d+1$ is the only natural solution to equation (2.7). If $y_0,\dots,y_d$ do satisfy eq. (2.8), then the I.V.P. (2.1) has infinitely many*

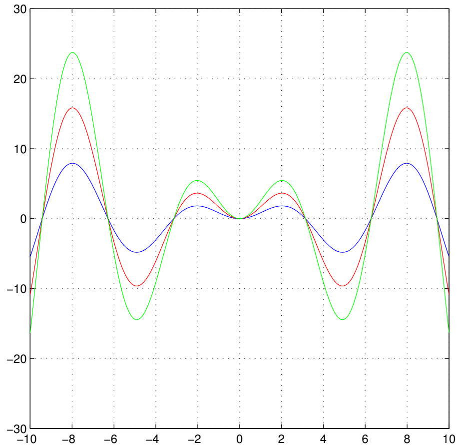

FIGURE 2.1. *Solutions to* $x^2y'' - 2xy' + (2 + x^2)y = 0,\ y(0) = 0,\ y'(0) = 0.$

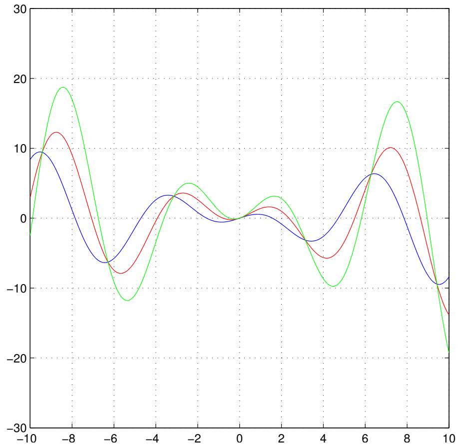

FIGURE 2.2. *Solutions to* $x^2y'' - 2xy' + (2 + x^2)y = 0,\ y(0) = 0,\ y'(0) = 1.$

analytic solutions in a neighborhood of x_0*, which are determined by* $y_0, \ldots, y_d$ *and a chosen constant* y_{d+1} *through the recurrence relation* (2.6)*.*

Proof: The method described above implies that the coefficient must be given in this fashion. That the series obtained converges in a neighborhood of x_0 is the subject of the next section (Theorem 5). □

Remark 4. *One sees immediately that eq.* (2.7) *is quadratic in* $n - \kappa$*, thus it can have up to two roots. If the smaller root* $d + 1$ *is natural and satisfies the conditions on Theorem 3 and* $e + 1 \in \mathbb{N}$ *is the second root, with* $e > d$*, then* $y_0, y_1, \ldots, y_e$ *are subject to the restriction*

$$\sum_{i=1}^{e+1} \left[(e + 1 - i)(e - i)a_{i+\kappa} + (e + 1 - i)b_{i+\kappa-1} + c_{i+\kappa-2}\right] y_{e+1-i} = g_{e+\kappa-1}. \tag{2.10}$$

This equation is linear $y_0, y_1, \ldots, y_e$ *in and thus there are three possible outcomes:*

1. *Equation* (2.10) *does not hold, independently of the choosen value for* y_{d+1}*. This means that there are no analytic solutions to the I.V.P.* (2.1)*.*
2. *There is a unique value for* y_{d+1} *which makes eq.* (2.10) *hold. This means that there are infinitely many analytic solutions (their convergence with some positive radius of convergence is shown in the next section) to the I.V.P.* (2.1)*, which depend on the variable* y_{e+1}*.*
3. *Equation* (2.10) *holds for any* $y_{d+1} \in \mathbb{R}$*. This means that there are infinitely many analytic solutions (their convergence with some positive radius of convergence is shown in the next section) to the I.V.P.* (2.1)*, which depend on two free constants* y_{d+1} *and* y_{e+1}*.*

Theorem 4. *Let us suppose that the I.V.P.* (2.1) *is admissible and that there is* $d \in \mathbb{N}$*,* $d \geq 1$*, such that the coefficients* $y_0, \ldots, y_d$*, satisfy eq.* (2.6) *for any* $1 \leq n-\kappa \leq d$ *and that* $n-\kappa = d+1$ *and* $n-\kappa = e+1$*,* *with $d < e$ are the natural solutions of equation (2.7). Further, if $y_0, \dots, y_d$ do satisfy*

$$\sum_{i=1}^{d+1} \left[(d+1-i)(d-i)a_{i+\kappa} + (d+1-i)b_{i+\kappa-1} + c_{i+\kappa-2}\right] y_{d+1-i} = g_{d+\kappa-1} \tag{2.11}$$

then let y_{d+1} be some arbitrary constant and eq. (2.6) determines y_n for $n = d+1, \dots, e$. Then condition

$$\sum_{i=1}^{e+1} \left[(e+1-i)(e-i)a_{i+\kappa} + (e+1-i)b_{i+\kappa-1} + c_{i+\kappa-2}\right] y_{e+1-i} = g_{e+\kappa-1}, \tag{2.12}$$

has three possible outcomes:

a) It is not possible to solve; thus the I.V.P. (2.1) has no analytic solution.

b) There is a unique value y_{d+1} which solves equation (2.12), thus there are infinitely many analytic solutions to the I.V.P. (2.1) in a neighborhood of x_0, which are given by the different choices in y_{e+1} through the recurrence relation (2.6).

c) The eq. (2.12) holds for any y_{d+1}, thus there are infinitely many analytic solutions to the I.V.P. (2.1) in a neighborhood of x_0, which given by the different choices in the constants y_{d+1} and y_{e+1} through the recurrence relation (2.6).

Proof: The method described above implies that the coefficient must be given in this fashion. That the series obtained converges in a neighborhood of x_0 is the subject of the next section (Theorem 5). □

We shall exclude the cases that do not allow analytic solutions.

Definition 3. *Let the I.V.P. (2.1) be admissible at a κ-singular point x_0 and $y_0, y_1, \dots, y_d$ satisfy eq. (2.6) for $n - \kappa = 1, \dots, d$. We say that the I.V.P. (2.1) is d-degenerate if*

$$(d+1)da_\kappa + (d+1)b_{\kappa-1} + c_{\kappa-2} = 0$$

and

$$\sum_{i=1}^{d+1} \left[(d+1-i)(d-i)a_{\kappa+i} + (d+1-i)b_{\kappa+i-1} + c_{\kappa+i-2}\right] y_{d+1-i} \neq g_{d+\kappa-1}. \tag{2.13}$$

Further we say that the I.V.P. (2.1) is non-degenerate if it is not d-degenerate for any $d \in \mathbb{N}$.

Theorems 3 and 4 show that d-degenerate operators have no analytic solutions.

3. Analytic solutions. In order to complete our proof of the convergence of the power series method, it will be necessary to change the variable in problem (2.1):

Remark 5. *A function $y : (x_0 - \alpha, x_0 + \alpha) \to \mathbb{R}$ solves the I.V.P. (2.1) if, and only if, the function $\tilde{y} : (\rho x_0 - \rho\alpha, \rho x_0 + \rho\alpha) \to \mathbb{R}$, defined as $\tilde{y}(x) = y(x/\rho)$ solves the I.V.P.*

$$\begin{aligned} \tilde{a}(x)\tilde{y}''(x) + \tilde{b}(x)\tilde{y}'(x) + \tilde{c}(x)\tilde{y}(x) &= \tilde{g}(x), \\ \tilde{y}(\tilde{x}_0) = y_0, \qquad \tilde{y}'(\tilde{x}_0) &= y_1/\rho, \end{aligned} \tag{3.1}$$

with $\tilde{x}_0 = \rho x_0$ and

$$\tilde{a}(x) = \rho^2 a(x/\rho) = \rho^2 \sum \frac{a_n}{\rho^n}(x - \rho x_0)^n, \qquad \tilde{b}(x) = \rho b(x/\rho) = \rho \sum \frac{b_n}{\rho^n}(x - \rho x_0)^n,$$
$$\tilde{c}(x) = c(x/\rho) = \sum \frac{c_n}{\rho^n}(x - \rho x_0)^n, \qquad \tilde{g}(x) = g(x/\rho) = \sum \frac{g_n}{\rho^n}(x - \rho x_0)^n.$$

Note that the coefficients of the power series (at $\tilde{x}_0$) of $\tilde{a}(x)$, $\tilde{b}(x)$, $\tilde{c}(x)$ and $\tilde{g}(x)$ are respectively

$$\tilde{a}_n(x) = \frac{a_n}{\rho^{n-2}}, \quad \tilde{b}_n(x) = \frac{b_n}{\rho^{n-1}}, \quad \tilde{c}_n(x) = \frac{c_n}{\rho^n}, \quad \tilde{g}_n(x) = \frac{g_n}{\rho^n}.$$

Further problem (2.1) is κ-singular and admissible at x_0 if, and only if, problem (3.1) is κ-singular and admissible at $\tilde{x}_0 = \rho x_0$.

Remark 6. *We shall prove in Theorem 5 that, for ρ big enough, the power series method obtained for the rescaled problem converges in $(\rho x_0 - 1, \rho x_0 + 1)$ and thus is a solution $\tilde{y}(x)$ to I.V.P. (3.1) in this neighborhood. This implies, by remark 5, that $y(x) = \tilde{y}(x/\rho)$ solves the I.V.P. (2.1) and it's convergence radius is at least $1/\rho$.*

Before we proceed to solving I.V.P. (3.1) we choose the rescaling variable ρ in order to ensure small value to some quantities:

Lemma 1. *Let $\tilde{a}, a, \tilde{b}, b, \tilde{c}, c, \tilde{g}, g$ be as in remark 5. Then there are $\rho > 0$ and $n_0 \in \mathbb{N}$ such that $n > n_0$ implies*

$$\sum_{i=1}^{n-\kappa}\left|\frac{-(n-\kappa-i)(n-1-\kappa-i)\tilde{a}_{i+\kappa}-(n-\kappa-i)\tilde{b}_{i+\kappa-1}-\tilde{c}_{i+\kappa-2}}{(n-\kappa)(n-1-\kappa)\tilde{a}_{\kappa}+(n-\kappa)\tilde{b}_{\kappa-1}+\tilde{c}_{\kappa-2}}\right|<\epsilon$$

and

$$\left|\frac{\tilde{g}_{n-2}}{(n-\kappa)(n-\kappa-1)\tilde{a}_{\kappa}+(n-\kappa)\tilde{b}_{\kappa-1}+\tilde{c}_{\kappa-2}}\right|<\epsilon.$$

Proof: By one side, if n is sufficiently large, it holds

$$|(n-\kappa)(n-1-\kappa)a_{\kappa}+(n-\kappa)b_{\kappa-1}+c_{\kappa-2}|>|a_{\kappa}|+|b_{\kappa-1}|+|c_{\kappa-2}|>0.$$

So that

$$\begin{aligned}
&\sum_{i=1}^{n-\kappa}\left|\frac{-(n-\kappa-i)(n-1-\kappa-i)\tilde{a}_{i+\kappa}-(n-\kappa-i)\tilde{b}_{i+\kappa-1}-\tilde{c}_{i+\kappa-2}}{(n-\kappa)(n-1-\kappa)\tilde{a}_{\kappa}+(n-\kappa)\tilde{b}_{\kappa-1}+\tilde{c}_{\kappa-2}}\right|\\
&=\sum_{i=1}^{n-\kappa}\left|\frac{-(n-\kappa-i)(n-1-\kappa-i)\frac{a_{i+\kappa}}{\rho^{i+\kappa-2}}-(n-\kappa-i)\frac{b_{i+\kappa-1}}{\rho^{i+\kappa-2}}-\frac{c_{i+\kappa-2}}{\rho^{i+\kappa-2}}}{(n-\kappa)(n-1-\kappa)\frac{a_{\kappa}}{\rho^{\kappa-2}}+(n-\kappa)\frac{b_{\kappa-1}}{\rho^{\kappa-2}}+\frac{c_{\kappa-2}}{\rho^{\kappa-2}}}\right|\\
&\leq\sum_{i=1}^{n-\kappa}\left|\frac{-(n-\kappa-i)(n-1-\kappa-i)a_{i+\kappa}-(n-\kappa-i)b_{i+\kappa-1}-c_{i+\kappa-2}}{(|a_{\kappa}|+|b_{\kappa-1}|+|c_{\kappa-2}|)\rho^{i}}\right|\\
&\leq\sum_{i=1}^{\infty}\left|\frac{-(n-\kappa-i)(n-1-\kappa-i)a_{i+\kappa}-(n-\kappa-i)b_{i+\kappa-1}-c_{i+\kappa-2}}{(|a_{\kappa}|+|b_{\kappa-1}|+|c_{\kappa-2}|)\rho^{i}}\right|<\epsilon\in\mathbb{R}
\end{aligned}$$

for any ρ sufficiently big, because the last sum can be seen as a constant times the power series

$$\sum_{i=1}^{\infty}\left|-(n-\kappa-i)(n-1-\kappa-i)a_{i+\kappa}-(n-\kappa-i)b_{i+\kappa-1}-c_{i+\kappa-2}\right|(x-x_0)^i,$$

applied at $x = 1/\rho + x_0$. This power series has the coefficient of $(x - x_0)^0$ equal to zero and converges for x with $|x - x_0| < \min\{\rho_a, \rho_b, \rho_c\}$, where ρ_a, ρ_b and ρ_c are the radii of convergence of $a(x)$, $b(x)$ and $c(x)$, respectively. Further, $n_0 \in \mathbb{N}$ and ρ can be made big enough so that it also holds

$$\left|\frac{\tilde{g}_{n-2}}{(n-\kappa)(n-\kappa-1)\tilde{a}_{\kappa}+(n-\kappa)\tilde{b}_{\kappa-1}+\tilde{c}_{\kappa-2}}\right|<\epsilon,$$

for any $n > n_0$, because g is analytic. □

Remark 7. *Do note that these estimates include some I.V.P. which are singular irregular in the sense of the method of Frobenius*[1] *at x_0 and our proof of the convergence (Theorem 5) also holds for such cases. The I.V.P. (2.1) is singular irregular exactly when it is κ-singular with $a_\kappa = 0$.*

Although we prove convergence for some singular irregular cases, for these cases the estimates in Lemma 1 can have a second degree polynomial in the numerator, but only a first degree (or a constant) polynomial in the denominator. Thus it is necessary that $\rho > 1$ for the sum to be convergent. This means we can only guarantee that solutions at singular irregular points converge in a neighborhood of x_0 with radius $1/\rho < 1$. We expect such solutions to be convergent only in a small neighborhood of x_0.

Example 2. *The I.V.P.*

$$x^5y''+2x^2y'-6xy=x^6e^{-x^2},\quad y(0)=\alpha,\ \ y'(0)=\beta,$$

is 3-singular in $x_0 = 0$ and falls into category III, a) of definition 2 and thus is admissible only if $y(0) = y'(0) = 0$. Further eq. (2.7) holds for $n = 6$ and the restriction (2.8) is satisfied. In this fashion we can assign any value for $y_3 = \delta$.

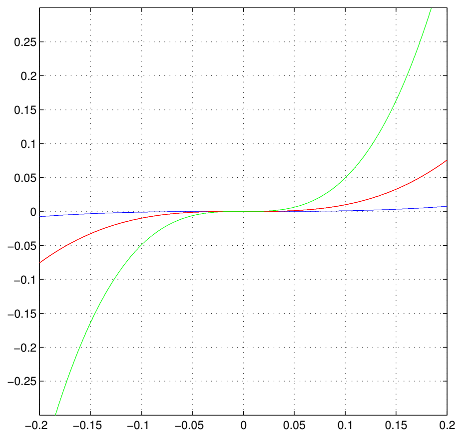

FIGURE 3.1. *Solutions to the singular irregular I.V.P.* $x^5y'' + 2x^2y' - 6xy = x^6e^{-x^2}$, $y(0) = 0$, $y'(0) = 0$.

Figure 3.1 shows plots of solutions with $y_3 = \delta = 1$ *(blue),* $y_3 = \delta = 10$ *(red),* $y_3 = \delta = 50$ *(green), which only converge numerically in a small neighborhood of* x_0.

Now we proceed to the proof of the convergence of the power series method for the I.V.P. (3.1) in some neighborhood of x_0 under the hypothesis of Lemma 1.

Let L be a non-degenerate κ-singular operator. Further let $t \in \mathbb{N}$ be big enough so that $n - \kappa > t$ implies

$$(n-\kappa)(n-1-\kappa)a_\kappa + (n-\kappa)b_{\kappa-1} + c_{\kappa-2} \neq 0,$$

and $y_0, y_1, \ldots, y_t$ satisfy eq. (2.6) for $n - \kappa = 2, \ldots, t$. Then eq. (2.6) can be used recursively to obtain $y_{n-\kappa}$ in terms of $y_0, y_1, \ldots y_t$.

In order to express recurrence relation (2.6) in a more compact way, let us denote

$$\zeta_\kappa^{i,n} := \frac{-(n-\kappa-i)(n-1-\kappa-i)a_{i+\kappa} - (n-\kappa-i)b_{i+\kappa-1} - c_{i+\kappa-2}}{(n-\kappa)(n-1-\kappa)a_\kappa + (n-\kappa)b_{\kappa-1} + c_{\kappa-2}}, \tag{3.2}$$

$$\gamma_\kappa^n := \frac{g_{n-2}}{(n-\kappa)(n-1-\kappa)a_\kappa + (n-\kappa)b_{\kappa-1} + c_{\kappa-2}}, \tag{3.3}$$

With this notation eq. (2.6) can be written as

$$\begin{aligned} y_{n-\kappa} =& \gamma_\kappa^n + \sum_{i=1}^{n-\kappa} \zeta_\kappa^{i,n} y_{n-\kappa-i} = \gamma_\kappa^n + \zeta_\kappa^{1,n} y_{n-\kappa-1} + \sum_{i=2}^{n-\kappa} \zeta_\kappa^{i,n} y_{n-\kappa-i} \\ =& \gamma_\kappa^n + \zeta_\kappa^{1,n}\left(\gamma_\kappa^{n-1} + \sum_{i=1}^{n-\kappa-1} \zeta_\kappa^{i,n-1} y_{n-1-\kappa-i}\right) + \sum_{i=2}^{n-\kappa} \zeta_\kappa^{i,n} y_{n-\kappa-i} \\ =& \gamma_\kappa^n + \zeta_\kappa^{1,n}\gamma_\kappa^{n-1} + \zeta_\kappa^{1,n} \sum_{i=1}^{n-\kappa-1} \zeta_\kappa^{i,n-1} y_{n-1-\kappa-i} + \sum_{i=1}^{n-\kappa-1} \zeta_\kappa^{i+1,n} y_{n-\kappa-1-i} \end{aligned}$$

$$y_{n-\kappa} = \gamma_\kappa^n + \zeta_\kappa^{1,n}\gamma_\kappa^{n-1} + \sum_{i=1}^{n-\kappa-1} (\zeta_\kappa^{1,n}\zeta_\kappa^{i,n-1} + \zeta_\kappa^{i+1,n}) y_{n-\kappa-i-1}, \tag{3.4}$$

if $n - \kappa - 1 > t$.

As the recurrence relation (2.6) is affine, it is possible to construct a family of matrices to calculate $y_{n-\kappa}$ in terms of $y_0, \ldots, y_{n-\kappa-1}$. In fact we use the family of $(n-\kappa+1) \times (n-\kappa+2)$ matrices:

$$\tilde{A}_\kappa^n = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & \gamma_\kappa^n \\ 0 & 1 & 0 & 0 & \cdots & 0 & \zeta_\kappa^{n-\kappa,n} \\ 0 & 0 & 1 & 0 & \cdots & 0 & \zeta_\kappa^{n-\kappa-1,n} \\ 0 & 0 & 0 & 1 & \cdots & 0 & \zeta_\kappa^{n-\kappa-2,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & \zeta_\kappa^{1,n} \end{pmatrix}$$

[1] See Remark 1.

Lemma 2. *Let $y_0, y_1, \ldots, y_t$ satisfy eq.(2.6) for $\max\{0, 2-\kappa\} < n-\kappa < t$, let $d-\kappa > t$, and suppose that $y_{d-\kappa}$ satisfy eq. (2.6), i.e.*

$$y_{d-\kappa} = \gamma_\kappa^d + \sum_{i=1}^{d-\kappa} \zeta_\kappa^{i,d} y_{d-\kappa-i}$$

and, for $n > d$, that $y_{n-\kappa}$ can be written as an affine function of $y_0, y_1, \ldots, y_{d-\kappa}$:

$$y_{n-\kappa} = z + \sum_{i=0}^{d-\kappa} x_i y_{d-\kappa-i},$$

with $z, x_0, x_1, \ldots, x_{d-\kappa} \in \mathbb{R}$. Further let us write

$$\mathbf{x} = (z,\ x_{d-\kappa},\ \ldots,\ x_1,\ x_0)^T.$$

Then $y_{n-\kappa}$ can also be written as an affine function of $y_0, y_1, \ldots, y_{d-\kappa-1}$:

$$y_{n-\kappa} = \tilde{z} + \sum_{i=1}^{d-\kappa} \tilde{x}_i y_{d-\kappa-i},$$

with

$$\tilde{z} = (\tilde{A}_\kappa^d \mathbf{x})_1 \text{ and } \tilde{x}_i = (\tilde{A}_\kappa^d \mathbf{x})_{d-\kappa+2-i} \text{ for } 1 \le i \le d-\kappa.$$

Proof: By hypothesis

$$\begin{aligned} y_{n-\kappa} =& z + \sum_{i=0}^{d-\kappa} x_i y_{d-\kappa-i} = z + x_0 y_{d-\kappa} + \sum_{i=1}^{d-\kappa} x_i y_{d-\kappa-i} \\ =& z + x_0 \left(\gamma_\kappa^d + \sum_{i=1}^{d-\kappa} \zeta_\kappa^{i,d} y_{d-\kappa-i} \right) + \sum_{i=1}^{d-\kappa} x_i y_{d-\kappa-i} \\ =& (z + x_0 \gamma_\kappa^d) + \sum_{i=1}^{d-\kappa} (x_i + x_0 \zeta_\kappa^{i,d}) y_{d-\kappa-i} \end{aligned}$$

On the other hand

$$(\tilde{A}_\kappa^d \mathbf{x})_1 = z + x_0 \gamma_\kappa^d$$

and

$$(\tilde{A}_\kappa^d \mathbf{x})_{d-\kappa+2-i} = x_i + x_0 \zeta_\kappa^{i,d}.$$

□

Remark 8. *In order to apply these matrices recursively and find $y_{n-\kappa}$ in terms of a fixed set of given entries $y_0, \ldots, y_t$, for some $t \ge 2$ we need them all to have compatible dimensions, and we do this by filling them with zeros and thus obtaining a family of linear operators $A_\kappa^n : \ell^1 \to \ell^1$ defined by:*

$$A_\kappa^n = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & \gamma_\kappa^n & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots & 0 & \zeta_\kappa^{n-\kappa,n} & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots & 0 & \zeta_\kappa^{n-\kappa-1,n} & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots & 0 & \zeta_\kappa^{n-\kappa-2,n} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ldots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \zeta_\kappa^{1,n} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ldots \end{pmatrix}.$$

Remark 9. *Using this notation it follows that the coefficients of* $1, y_0, y_1, \ldots, y_{n-\kappa-2}$ *in equation* (3.4) *are, respectively, the first* $n-\kappa$ *entries of the vector*

$$\mathbf{x} = A_\kappa^{n-1} \begin{pmatrix} \gamma_\kappa^n \\ \zeta_\kappa^{n-\kappa,n} \\ \zeta_\kappa^{n-\kappa-1,n} \\ \zeta_\kappa^{n-\kappa-2,n} \\ \vdots \\ \zeta_\kappa^{1,n} \\ 0 \\ \vdots \end{pmatrix},$$

and $y_{n-\kappa} = \mathbf{x} \cdot (1, y_0, y_1, \ldots, y_{n-\kappa-2}, 0, \ldots)$.

Further $\mathbf{x}$ *is just the* $n-\kappa+2$ *column of the product* $A_\kappa^{n-1}A_\kappa^n$. *Which means that if eq.* (2.6) *holds for* $y_{d-\kappa}$ *to any* $n-\kappa > d-\kappa > t > \kappa$, *then* $y_{n-\kappa}$ *can be calculated in terms of* $y_0, y_1, \ldots y_t$, *by applying these linear operators successively. More precisely*

$$y_{n-\kappa} = A_\kappa^{t+\kappa+1} A_\kappa^{t+\kappa+2} \cdots A_\kappa^{n-1} \begin{pmatrix} \gamma_\kappa^n \\ \zeta_\kappa^{n-\kappa,n} \\ \zeta_\kappa^{n-\kappa-1,n} \\ \zeta_\kappa^{n-\kappa-2,n} \\ \vdots \\ \zeta_\kappa^{1,n} \\ 0 \\ \vdots \end{pmatrix} \cdot (1, y_0, y_1, \ldots, y_t, 0, \ldots).$$

This motivates the following definition:

Definition 4. *Let* $n \geq t+\kappa+1$ *be a natural number and* T_κ^n *be the linear operator on* ℓ^1 *defined by*

$$T_\kappa^n := A_\kappa^{t+\kappa+1} A_\kappa^{t+\kappa+2} \cdots A_\kappa^{n-1} A_\kappa^n.$$

Let us now turn the attention to how this sequence of operators behaves as n grows.

Lemma 3. *The matrices representing* T_κ^n *are of the form*

$$T_\kappa^n = \begin{pmatrix} 1 & 0 & \cdots & 0 & \phi_0^1 & \phi_0^2 & \cdots & \phi_0^{n-\kappa-t} & 0 & \cdots \\ 0 & 1 & \cdots & 0 & \phi_1^1 & \phi_1^2 & \cdots & \phi_1^{n-\kappa-t} & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 1 & \phi_{t+1}^1 & \phi_{t+1}^2 & \cdots & \phi_{t+1}^{n-\kappa-t} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots \end{pmatrix}.$$

Further

$$T_\kappa^{n+1} = T_\kappa^n A_\kappa^{n+1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & \phi_0^1 & \cdots & \phi_0^{n-\kappa-t} & \phi_0^{n-\kappa-t+1} & 0 & \cdots \\ 0 & 1 & \cdots & 0 & \phi_1^1 & \cdots & \phi_1^{n-\kappa-t} & \phi_1^{n-\kappa-t+1} & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 1 & \phi_{t+1}^1 & \cdots & \phi_{t+1}^{n-\kappa-t} & \phi_{t+1}^{n-\kappa-t+1} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots \end{pmatrix},$$

i. e. the first $n-\kappa+2$ *columns of* T_κ^{n+1} *are identical to the first* $n-\kappa+2$ *columns of* T_κ^n *and the only other non-zero column is the* $(n-\kappa+3)$*-th column.*

Proof: This follows, by induction, from the fact that the first $n-\kappa+2$ columns of A_κ^{n+1} are just the same as the first $n-\kappa+2$ columns of the identity and the only other nonzero column is the $n-\kappa+3$ column. □

Remark 10. *From Remark* 9 *it follows that*

$$y_{n-\kappa} = \phi_0^{n-\kappa-t} + y_0\phi_1^{n-\kappa-t} + y_1\phi_2^{n-\kappa-t} + \cdots + y_t\phi_{t+1}^{n-\kappa-t}. \tag{3.5}$$

This means the convergence of the power series $y = \sum y_n(x-x_0)^n$ is related to the convergence of the sequence of operators $T^n_\kappa \in B(\ell^1, \ell^1)$.

Theorem 5. *Consider a non-degenerate, admissible κ-singular I.V.P. of the form (3.1), and let $\rho \in \mathbb{R}$ and $n > n_0 \in \mathbb{N}$ big enough to make the estimates in Lemma 1 hold for some $\epsilon < 1/2$. Additionally let $t \in \mathbb{N}$ be big enough so that equation (2.7) does not hold for $n - \kappa > t$. Further let $y_0, y_1, \ldots, y_t$ be the first $t+1$ coefficients given by (2.6) in terms of y_0 and y_1 and possibly two arbitrary constants y_{d+1} and y_{e+1} as in Theorems 3 and 4. Then the coefficients $y_{n-\kappa}$ given by eq. (3.5) define a power series*

$$\sum_{n=0}^{\infty} y_n(x-x_0)^n,$$

which converges for all x with $|x - x_0| < 1$ and solves the I.V.P. (2.1).

Proof: Let the constants $\zeta^{i,n}_\kappa$ and γ^n_κ be defined as in eqs. (3.2) and (3.3). Further let the family of linear operators $T^n_\kappa \in B(\ell^1, \ell^1)$ be defined as in Def. 4. We show that the sequence of operators T^{n+1}_κ converges to some operator $T \in B(\ell^1, \ell^1)$; and do so by showing that $\|T^n_\kappa\|$ is bounded and $T^n_\kappa x$ is Cauchy in ℓ^1 for any $x \in \ell^1$ (Theorem of the Strong Operator Convergence in [7]).

Indeed, as the estimates in Lemma 1 hold, for $n > n_0$

$$\sum_{i=1}^{n} |\zeta^{i,n}_\kappa| < \epsilon < \frac{1}{2}$$

and

$$|\gamma^n_\kappa| < \epsilon < \frac{1}{2}.$$

Therefore for $n \geq n_0$ and any $\mathbf{x} \in \ell^1$, $\mathbf{x} = (x_1, x_2, x_3, \ldots)$, with $\|x\|_1 = 1$, it holds

$$\|A^n_\kappa(x)\|_1 \leq \sum_{i=1}^{n-\kappa+1} |x_i| + |x_{n-\kappa+2}| \left(|\gamma^n_\kappa| + \sum_{i=1}^{n-\kappa} |\zeta^{i,n}_\kappa| \right) \leq \sum_{i=1}^{n-\kappa+2} |x_i| \leq 1.$$

Thus $\|A^n_\kappa\|_1 \leq 1$ for all $n \geq n_0$ and

$$\|T^n_\kappa\|_1 \leq \left\| \prod_{i=t+1}^{n_0} A^i_\kappa \right\|_1 \prod_{i=n_0+1}^{n} \|A^i_\kappa\|_1 \leq \left\| \prod_{i=t+1}^{n_0} A^i_\kappa \right\|_1 .$$

Further, if we denote $\mathbf{x}^n = (0, 0, \ldots, 0, x_n, x_{n+1}, x_{n+2}, \ldots)$ and $\mathbf{I}_n$ for the operator in $B(\ell^1, \ell^1)$ that act as the identity in the first n entries of $\mathbf{x}$ and as zero for the other entries, it is possible to show that $T^n_\kappa(\mathbf{x})$ is a Cauchy sequence:

$$\begin{aligned}
\|T^{n+p}_\kappa(x) - T^n_\kappa(x)\|_1 &\leq \left\| \prod_{i=t+1}^{n_0} A^i_\kappa \right\|_1 \left\| \left(\prod_{i=n+1}^{n+p} A^i_\kappa - \mathbb{I}_{n-\kappa+2} \right) (\mathbf{x}) \right\|_1 \\
&\leq \left\| \prod_{i=t+1}^{n_0} A^i_\kappa \right\|_1 \left\| \left(\prod_{i=n+1}^{n+p} A^i_\kappa \right) (\mathbf{x}^{n-\kappa+3}) \right\|_1 \\
&\leq \left\| \prod_{i=t+1}^{n_0} A^i_\kappa \right\|_1 \left\| \mathbf{x}^{n-\kappa+3} \right\|_1 < \delta \ll 1
\end{aligned}$$

for n big enough, because $\mathbf{x}$ is in ℓ^1.

Then there is a bounded linear operator $T \in B(\ell^1, \ell^1)$ such that $T^n_\kappa \to T$. Beyond this,

$$T = \begin{pmatrix}
1 & 0 & \cdots & 0 & \phi^1_0 & \phi^2_0 & \cdots & \phi^{n-\kappa-t}_0 & \cdots \\
0 & 1 & \cdots & 0 & \phi^1_1 & \phi^2_1 & \cdots & \phi^{n-\kappa-t}_1 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\
0 & 0 & 0 & 1 & \phi^1_{t+1} & \phi^2_{t+1} & \cdots & \phi^{n-\kappa-t}_{t+1} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots
\end{pmatrix},$$

because A^n_κ preserves the first $n-\kappa+1$ columns.

In particular, T is defined for any sequence of the form $\mathbf{x} = (0, 1, (x-x_0), (x-x_0)^2, (x-x_0)^3, \dots)$, with $|x - x_0| < 1$, and (3.5) implies

$$\begin{aligned}
y(x) = &\sum_{n=0}^{\infty} y_n (x-x_0)^n = \sum_{n=\kappa}^{\infty} y_{n-\kappa}(x-x_0)^{n-\kappa} \\
=&y_0 + y_1(x-x_0) + \cdots + y_t(x-x_0)^t + \sum_{n=t+1+\kappa}^{\infty} y_{n-\kappa}(x-x_0)^{n-\kappa} \\
=&\sum_{n=t+\kappa+1}^{\infty} \phi_0^{n-\kappa-t}(x-x_0)^{n-\kappa} + y_0\left(1 + \sum_{n=t+\kappa+1}^{\infty} \phi_1^{n-\kappa-t}(x-x_0)^{n-\kappa}\right) \\
&+ y_1\left((x-x_0) + \sum_{n=t+\kappa+1}^{\infty} \phi_2^{n-\kappa-t}(x-x_0)^{n-\kappa}\right) + \dots \\
&+ y_t\left((x-x_0)^t + \sum_{n=t+\kappa+1}^{\infty} \phi_{t+1}^{n-t}(x-x_0)^{n-\kappa}\right) \\
=&(T(\mathbf{x}))_1 + y_0(T(\mathbf{x}))_2 + \cdots + y_t(T(\mathbf{x}))_{t+2}
\end{aligned}$$

The convergence of $T(\mathbf{x})$ implies that the calculations above make sense and the restriction $|x-x_0| < 1$ implies that the radius of convergence is at least[2] 1. □

Remark 11. *The I.V.P. (2.1) has a unique analytic solution, no solutions at all, or multiple analytic solutions exactly when the I.V.P. (3.1) has a unique analytic solution, no solutions at all, or multiple analytic solutions. Beyond this, the radius of convergence of y for the non rescaled problem is at least $1/\rho$, where ρ is a constant that makes the estimates in Lemma 1 hold.*

4. Summary. By searching analytic solutions to the I.V.P. 2.1, we found that the index of the first non-zero coefficients of $a(x)$, $b(x)$ and $c(x)$ are important to solve the problem and name the index κ, for which one of the a_κ, $b_{\kappa-1}$ or $c_{\kappa-2}$ is not zero, as in Definition 1. Writing a recurrence relation (2.6) for $y_{n-\kappa}$ we found several restrictions to the existence of analytical solutions to the I.V.P. 2.1 and called the problems that do not fall into these restrictions admissible, as in Definition 2.

Further the coefficient $(n-\kappa)(n-1-\kappa)a_\kappa + (n-\kappa)b_{\kappa-1} + c_{\kappa-2}$ of $y_{n-\kappa}$ in eq. (2.6) is a second degree polynomial in $n-\kappa$ and thus has roots. If those roots are natural, they imply further restrictions on the I.V.P. 2.1. If the restrictions are not satisfied, then there are no analytic solutions, as stated in Theorems 1, 2, 3 and 4.

The I.V.P.s that satisfy these restrictions can have a unique solution or multiple analytic solutions. Criteria for the existence of these multiple analytic solutions are given in Theorems 3 and 4, and illustrated in Example 1. Further, we prove the convergence of the power series defined by eq. (2.6) using the Theorem of the Strong Operator Convergence in Theorem 5. Our Theorem is an alternative to the Theorem of Fuchs for I.V.P.s in which both work. Although not all singular regular I.V.P.s in the sense of the Frobenius method are included in our Theorem, our theorem works for some singular irregular I.V.P.s, which would be the most important aspect of this work. Example 3.1 illustrates the solution of a singular irregular problem.

The present work can be developed in a few directions:

1. To consider solutions to the singular irregular second order linear non-homogeneous O.D.E. (instead of the I.V.P.), which would force the inclusion of other forms of solutions (not only the analytical).
2. To consider the n-th order I.V.P., for Theorem 5 does not rely directly on the second order assumption.
3. To a method of solving recurrence relations with non-constant coefficients and proving their convergence.

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